Lecture I Group action.

Let $X$ be a non-empty set.
The set of all bijections from $X$ to $X$ can be realized as a Group:
Denote $B(X) = \{ f : X \to X \mid f \text{ is bijective} \}$.

If $f, g \in B(X)$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & X \arrow[r, "g"] & X \end{tikzcd}$$

On $B(X)$ we have an operation $(f,g) \mapsto g \circ f$ composition $f$ with $g$. Note that $Id_X(x) = x$, $x \in X$, is bijection and hence $Id_X \in B(X)$.

If $f, g, h \in B(X)$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend right, "(g\circ f)"'] \arrow[rrr, bend left=50, "(h\circ g)\circ f"] \arrow[rrr, bend right=60, "h\circ(g\circ f)"'] & X \arrow[r, "g"] \arrow[rr, bend left, "(h\circ g)"] & X \arrow[r, "h"] & X
\end{tikzcd}$$

" $(h\circ(g\circ f))(x) = h(g(f(x))) = ((h\circ g)\circ f)(x)$ "

The composition is associative.

If $f : X \longrightarrow X$ is a bijection

then $f^{-1} : X \longrightarrow X$ is again bijection

$f\circ f^{-1} = Id_X = f^{-1}\circ f.$

$Id_X \circ f = f = f\circ Id_X.$

Hence $(B(X), \circ)$ is a group.

Recall: A Group $G$ is pair $(G, \cdot)$

where $G$ is a non-empty set

and $\cdot : G \times G \longrightarrow G$ is a binary operation

$$(g_1, g_2) \longmapsto g_1 \cdot g_2$$

which satisfies

1) $\exists$ an element $e$ (identity of the group $G$) s.t

$$g \cdot e = g = e \cdot g.$$

2) "$\cdot$" is associative i.e., $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$

$\forall\ g_1, g_2, g_3 \in G$.

3) For every $g \in G$ $\exists$ $g^{-1} \in G$ satisfying

$$g \cdot g^{-1} = e = g^{-1} \cdot g. \text{ (Existence of inverse).}$$

More over if $g_1 \cdot g_2 = g_2 \cdot g_1$ $\forall$ $g_1$ and $g_2 \in G$

Then we say that $G$ is a commutative group.

Examples:

1) $X = \{1\}$.

$B(X) = \{1d_X\}$ is the trivial group.

2) $X = \{1,2\}$.

$B(X) = \{1d_X, (12)\}$ $\qquad (12)(1) = 2$
$(12)(2) = 1$.

$B(X)$ is a group with 2-elements. $B(X) \simeq \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$

3) $X = \{1,2,3\}$.

$B(X) = \{\underset{e}{\underset{\parallel}{1d_X}}, (12), (13), (32), (123), (132)\}$.

$(12)(1) = 2 \qquad (12)(2) = 1 \qquad (12)(3) = 3$

$(13)(1) = 3 \qquad (13)(2) = 2 \qquad (13)(3) = 1$.

$B(X)$ is not commutative

$(12)\cdot(13)(1) = 3$

$(12)\cdot(13)(2) = 1$, $\qquad (1\,3\,2) = (12)(13)$

$(12)\cdot(13)(3) = 2$

$(13)(12)(1) = 2$

$(13)(12)(2) = 3$

$(13)(12)(3) = 1.$

$$(13)(12) = (123)$$

$\therefore \; (12)(13) \neq (13)(12)$

— x — x —

Let $G$ be a group and $X$ be a non-empty set.

DFN: $G$ is said to act on $X$ if $\exists$ a map

$$G \times X \longrightarrow X$$
$$(g, x) \longmapsto g \cdot x$$

Satisfying 1) $e \cdot x = x \quad \forall \, x \in X$

where $e \in G$ is the identity element

2) for all $g_1, g_2 \in G$ and $x \in X$

$$g_1 \cdot (g_2 \cdot x) = \underset{\substack{\uparrow \\ \text{Group mult.}}}{(g_1 \cdot g_2)} \cdot x$$

---

Remark:

$B(X)$ acts naturally on $X$

$$B(X) \times X \longrightarrow X$$

1) $(f, x) \longmapsto f(x)$

$(Id_X, x) \longmapsto Id_X(x) = x.$

2) $f_1, f_2 \in B(X)$

$$f_1(f_2(x)) = (f_1 \circ f_2)(x)$$

**Lemma:** Let $G$ be group acting on a set $X$

Then the map $g \longmapsto \{x \mapsto g\cdot x\} \in B(X)$

is a group homomorphism from $G$ to $B(X)$.

—x—

**Recall:** If $G_1$, and $G_2$ are two groups

then a homomorphism $f : G_1 \longrightarrow G_2$ is map

satisfying 1) $f(e_{G_1}) = e_{G_2}$, $f(\underset{\substack{\uparrow \\ \text{mult} \\ \text{in } G_1}}{g_1 \cdot g_2}) = \underset{\substack{\uparrow \\ \text{mult.} \\ \text{in } G_2}}{f(g_1) \cdot f(g_2)}$.

—x—

**Proof of Lemma:** Let $\varphi : G \longrightarrow B(X)$ be the map

Given by the action of $G$ on $X$

i.e. $\varphi(g) := \{x \mapsto g\cdot x\}$

$\varphi(g)$ is a bijection: $\varphi(g^{-1}) := x \longmapsto g^{-1}\cdot x$.

$$x \xmapsto{g} g\cdot x \xmapsto{g^{-1}} g^{-1}\cdot(g\cdot x) = (g^{-1}\cdot g)\cdot x$$
$$= e\cdot x$$
$$= x$$

$\varphi(g)$ is a bijection and its inverse is

$\varphi(g^{-1}) := \varphi(g)^{-1}$

$\varphi(e) = 1d_X$

$\varphi(g_1\cdot g_2) = \varphi(g_1)\circ\varphi(g_2)$ – This follows from the property of action : $g_1(g_2\cdot x) = (g_1\cdot g_2)(x)$ .

—x—

Whenever a group $G$ acts on a set $X$ we naturally get a homomorphism of $G \to B(X)$.

Conversly if $\rho : G \to B(X)$ is a homomorphism then we get an action of $G$ on $X$ :

$$G\times X \longrightarrow X$$
$$(g, x) \longmapsto \underset{\substack{\uparrow \\ B(X)}}{\rho(g)}(x) =: g\cdot x .$$

Since $\rho$ is a homo implies

$$e.x = x \quad \forall\, x \in X$$

$$(g_1 \cdot g_2)(x) = g_1(g_2(x)).$$

—x—

A Group $G$ acting on a Set $X$ is equivalent to a homomorphism (of groups) $G \longrightarrow B(X)$

—x—

<u>Cor</u>: If $G$ is a group then $G$ can be relaized as a subgroup of $B(G)$.

<u>proof</u>: $G$ acts naturally on $G$:

The mult $G \times G \longrightarrow G \quad (g_1, g_2) \mapsto g_1 \cdot g_2$ is action:

For $g_1 \in G$ $\; L_{g_1} : g \mapsto g_1 \cdot g$ is a bijection of $G$ and it is easy to check $g \mapsto L_g$ gives a homomorphism $L_G : G \to B(G)$ of of Groups and $L_G : G \longrightarrow B(G)$ is an injective homomorphism.

Recall : 1) A homo $\varphi : G_1 \longrightarrow G_2$ of groups is said to be injective if $\varphi(g) = e_{G_2} \Rightarrow g = e_{G_1}$.

2) A Subset $H$ of $G$ ($G$ is a group) is called subgroup of $G$ if $e_G \in H$ and a) $h_1, h_2 \in H \Rightarrow \underset{\uparrow}{h_1 \cdot h_2} \in H$

Group operation on $G$

b) $h_1 \in H \Rightarrow h_1^{-1} \in H$

3) If $\varphi : G_1 \longrightarrow G_2$ is an injective homomorphism then $\varphi(G_1)$ is a subgroup of $G_2$ which can be identified with $G_1$.

<u>Corollary</u> (Cayley's Theorem)

Every group is a subgroup of the group of bijections of a set.