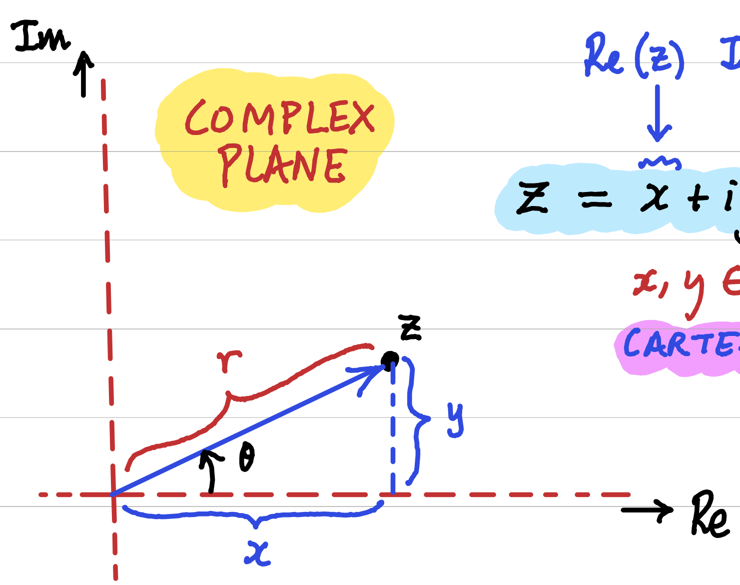
IMSc Facets (July 3, 2026) "Visualizing plane curves / $\mathbb{C}$"

Plan:
1. Basics of complex numbers ~15 min
2. Complex n^{th} roots ~30 min
3. Example $x^3+y^2-1=0$ over $\mathbb{C}$ ~45 min

① BASICS OF COMPLEX NUMBERS

Question: How many number systems do you know?

NAME		INTUITIVE MEANING
Natural numbers $\mathbb{N}$	–	counting
Integers $\mathbb{Z}$	–	accounting (debt, profit/loss)
Rational numbers $\mathbb{Q}$	–	dividing into equal pieces
Real numbers $\mathbb{R}$	–	length ($\sqrt{2}$), area (π)
Complex numbers $\mathbb{C}$	–	??



$$z = \overset{\mathrm{Re}(z)}{x} + i\overset{\mathrm{Im}(z)}{y} = \overset{|z|}{r}(\cos\overset{\arg(z)}{\theta} + i\sin\overset{\arg(z)}{\theta})$$

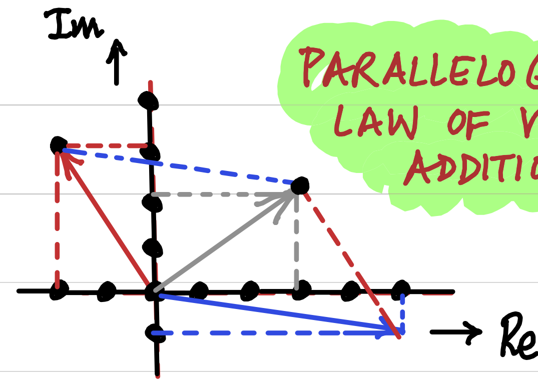
$x, y \in \mathbb{R}$ — CARTESIAN

$r \geq 0,\ \theta \in \mathbb{R}/2\pi\mathbb{Z}$ — POLAR

Rules of algebra in $\mathbb{C}$
- rules from $\mathbb{R}$
- $i^2 = -1$

Adding complex numbers

$$(-2+3i)+(5-i) \overset{?}{=} 3+2i$$



PARALLELOGRAM LAW OF VECTOR ADDITION

Multiplying complex numbers

Question: How to write multiplication in $\mathbb{C}$ in polar coords?

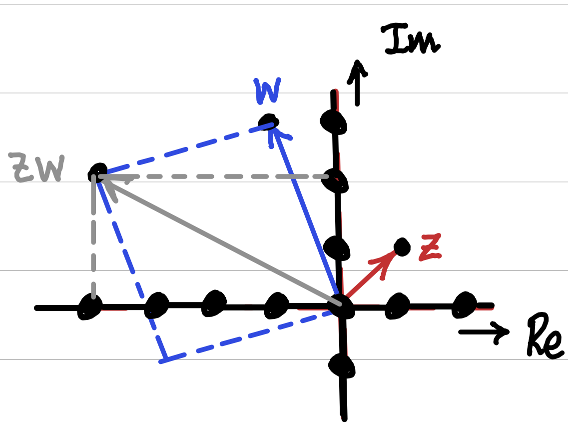
$$r(\cos\theta + i\sin\theta)\cdot r'(\cos\theta' + i\sin\theta')$$

$$= rr'\big((\cos\theta\cos\theta' - \sin\theta\sin\theta') + i(\sin\theta\cos\theta' + \cos\theta\sin\theta')\big)$$

$$= \underbrace{rr'}_{\text{modulus multiplied}}\big(\cos(\underbrace{\theta+\theta'}_{\text{arguments added}}) + i\sin(\underbrace{\theta+\theta'}_{\text{arguments added}})\big)$$

Suggests notation $re^{i\theta} = r(\cos\theta + i\sin\theta)$

Question: How to interpret this geometrically?
What does zw look like in the figure below?



$z = 1+i = \sqrt{2}e^{i\pi/4}$

$w = -1+3i$

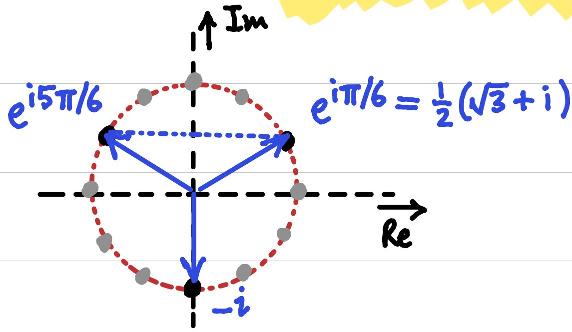
$zw = -4+2i$

MULTIPLICATION BY $re^{i\theta}$ = ROTATE BY r, SCALE BY θ

Special case (!): $i^2 = -1$

② COMPLEX n^{th} ROOTS

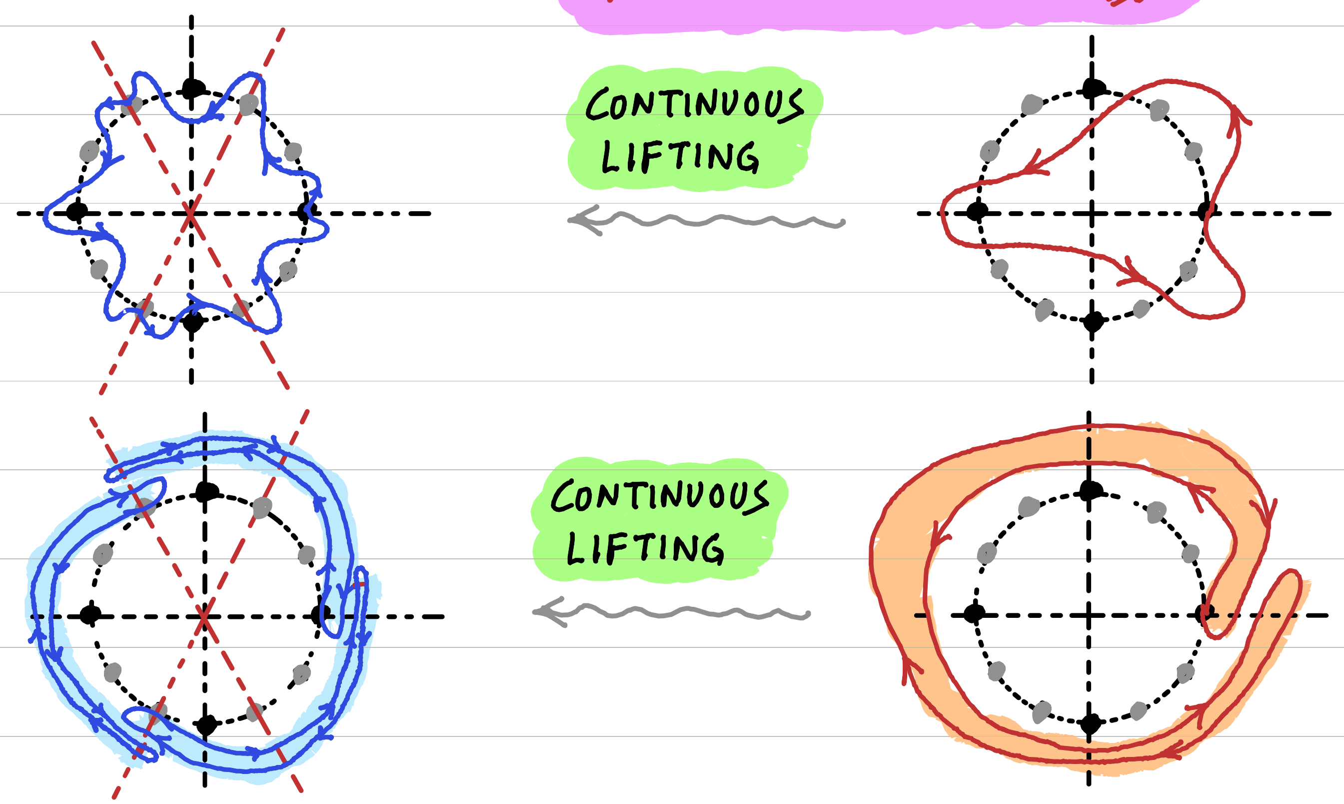
Question: Draw (& compute) a cube root of i in $\mathbb{C}$. Is it the only one?
How about n^{th} root of some general $z \in \mathbb{C}$? $z \neq 0$ / $z = 0$



> Any $z \in \mathbb{C} \setminus \{0\}$ has exactly n distinct n^{th} roots; ratio of any two is a power of $\zeta_n = e^{i\,2\pi/n}$.

Question: Can we choose an n^{th} root for all numbers in $\mathbb{C}\setminus\{0\}$ such that the resulting function $\sqrt[n]{\cdot} : \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$ is CONTINUOUS?

How about on $\mathbb{C} \setminus e^{i\theta}\mathbb{R}_{\geq 0}$?



w-plane
$w = \sqrt[3]{z}$

z-plane
$z = w^3$

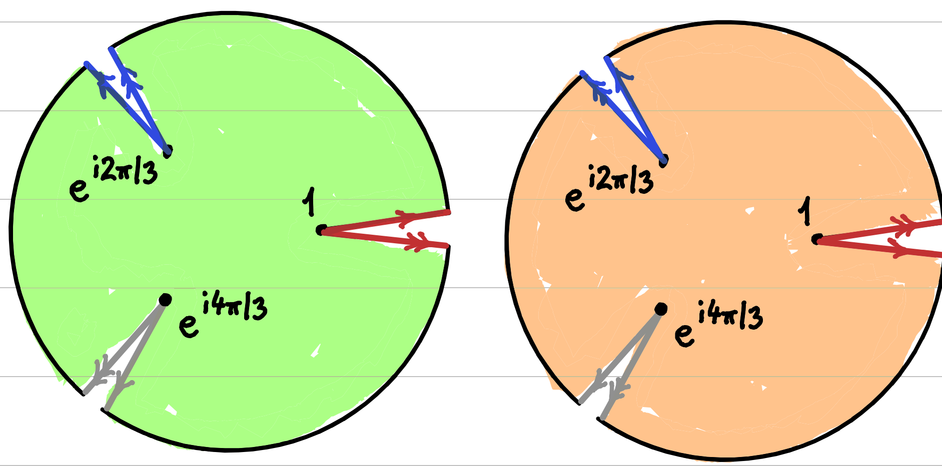
③ EXAMPLE $x^3+y^2-1=0$ OVER $\mathbb{C}$

For simplicity, we look at the portion with $|x| \leq \overset{\text{large}}{R}$.

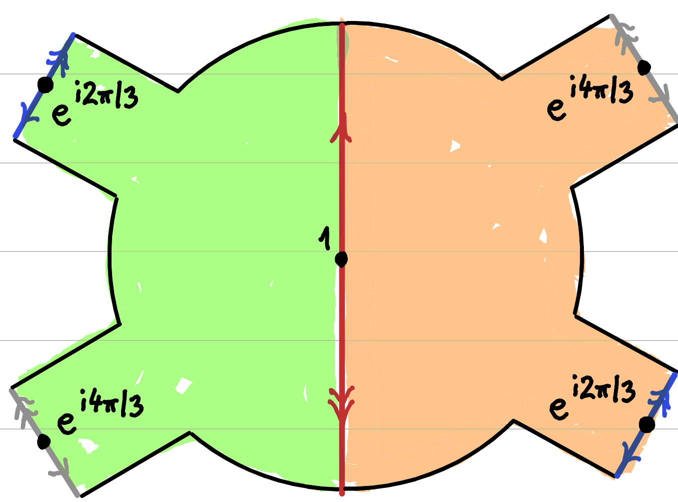
IDEA: Make slits in $\mathbb{C}$ so that $\sqrt{1-x^3}$ exists & is continuous.

Two branches of $y=\sqrt{1-x^3}$ shown as the green & orange sheets.

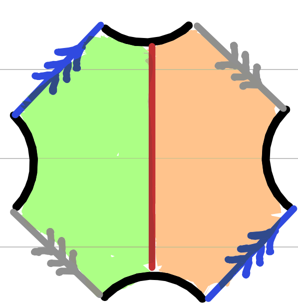
Step 1



Step 2



Step 3



Conclusion