Geodesics on Surfaces

Day One

May 22, 2025

FACETS @ IMSc Chennai

Vijay Ravikumar

Azim Premji University, Bengaluru

Prologue

the journey of a hungry ant

Prologue

the journey of a hungry ant

Ants are surprisingly adept at finding the shortest path to their destination — in this case a sweet laddu.

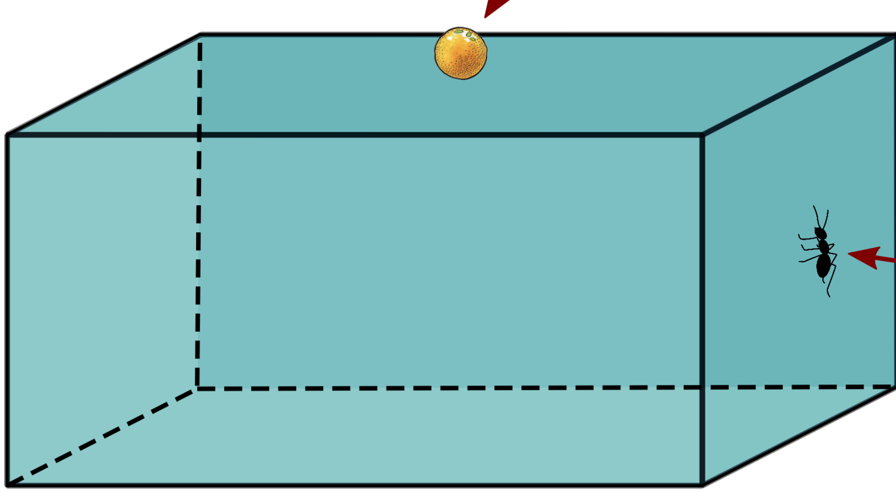
Prologue

the journey of a hungry ant

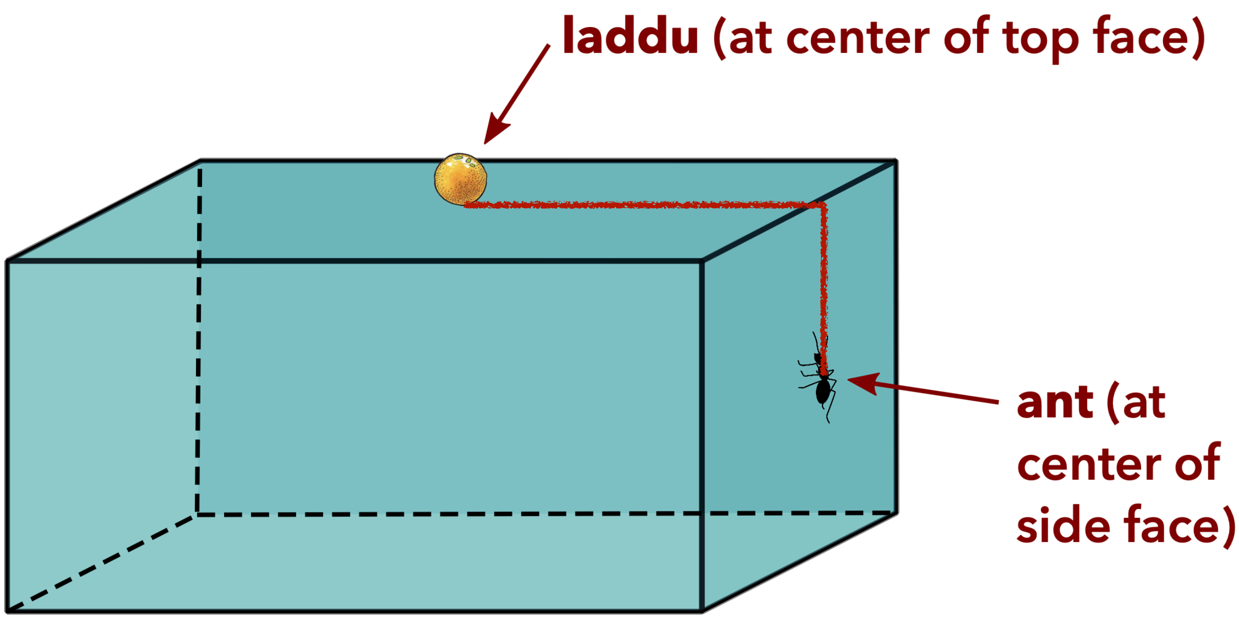
Ants are surprisingly adept at finding the shortest path to their destination — in this case a sweet laddu.

Let's try to determine the shortest path in a few cases.

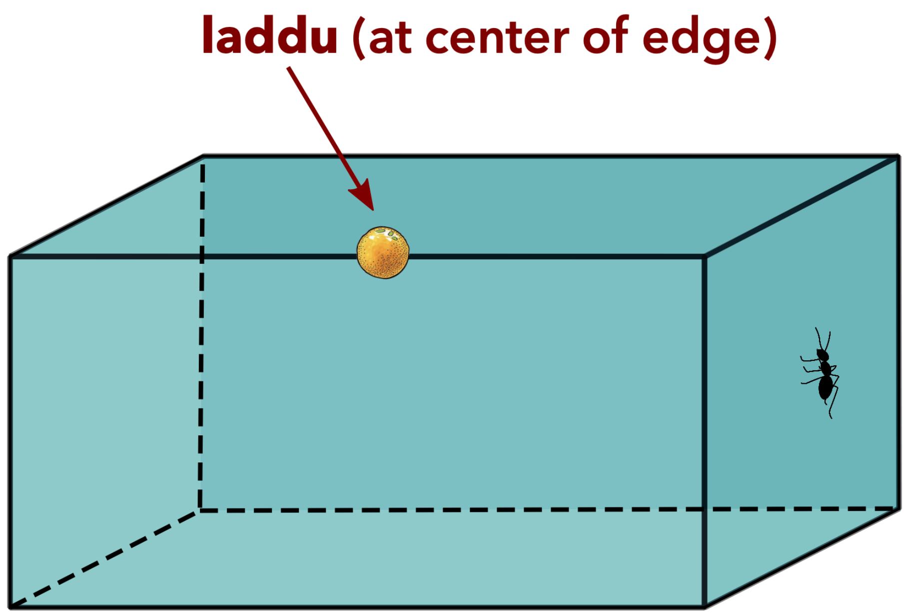
What is the shortest path from the ant to the laddu?



What is the shortest path from the ant to the laddu?

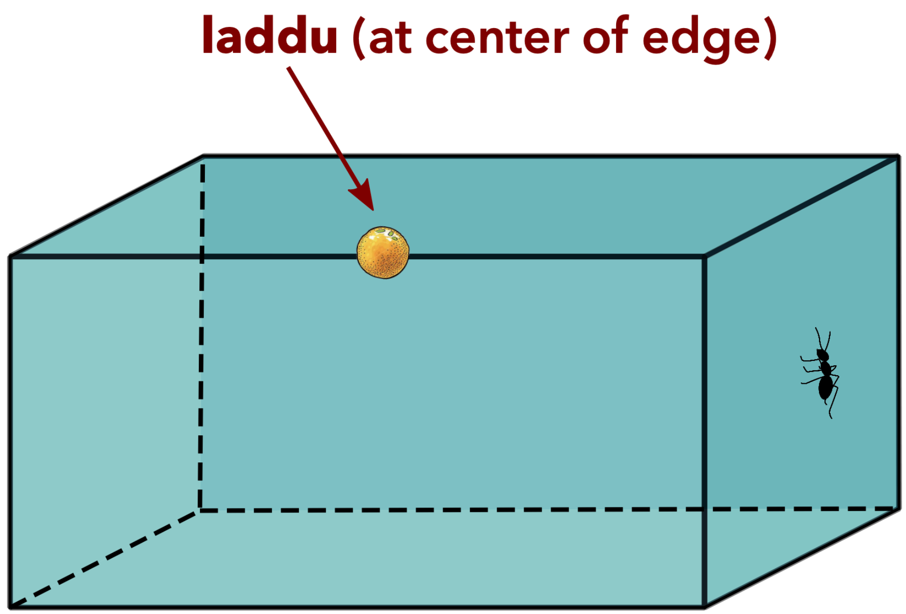


What about now? Can you find the shortest path?



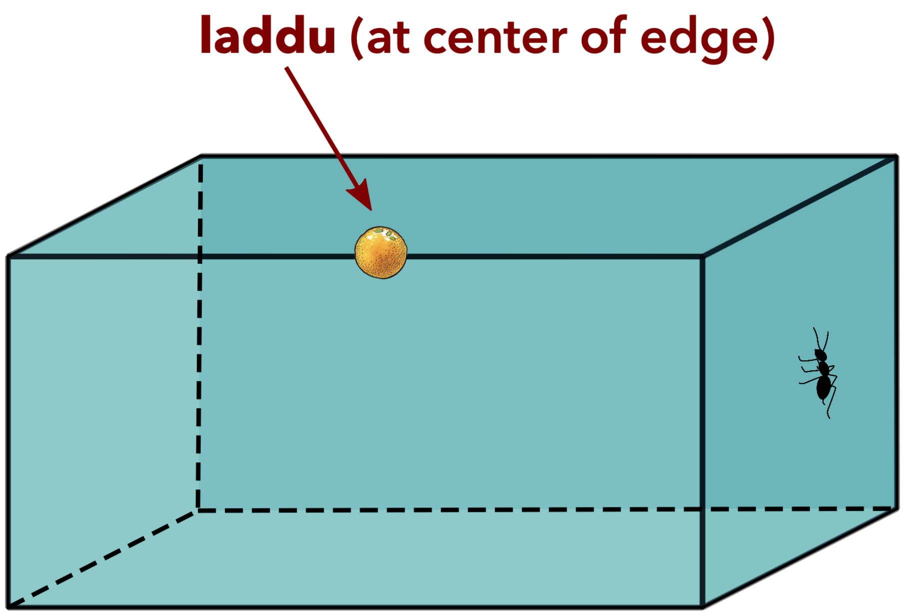
What about now? Can you find the shortest path?

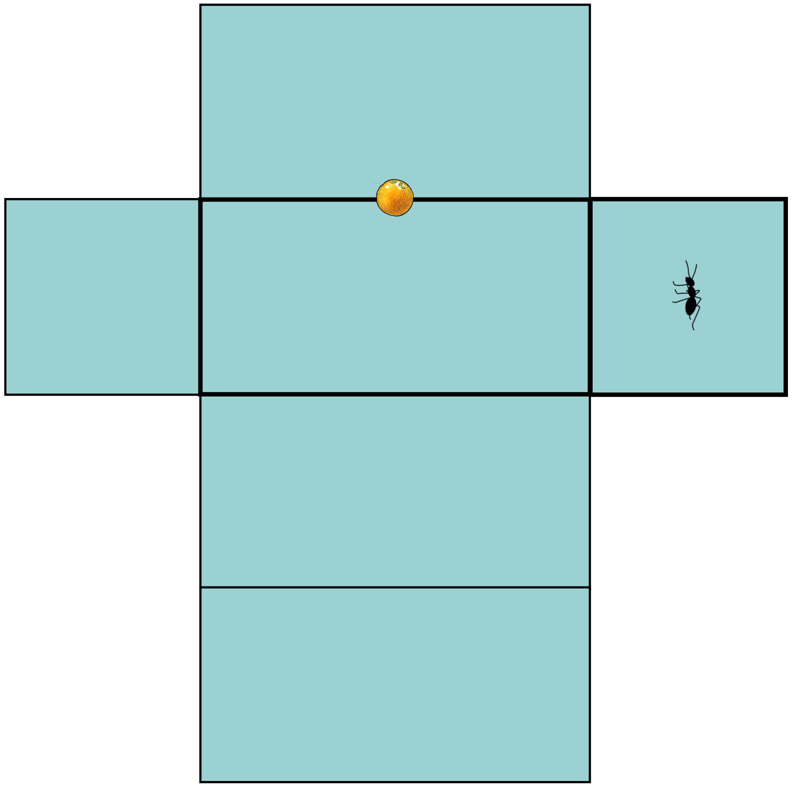
Hint: Imagine *unfolding* the box!



What about now? Can you find the shortest path?

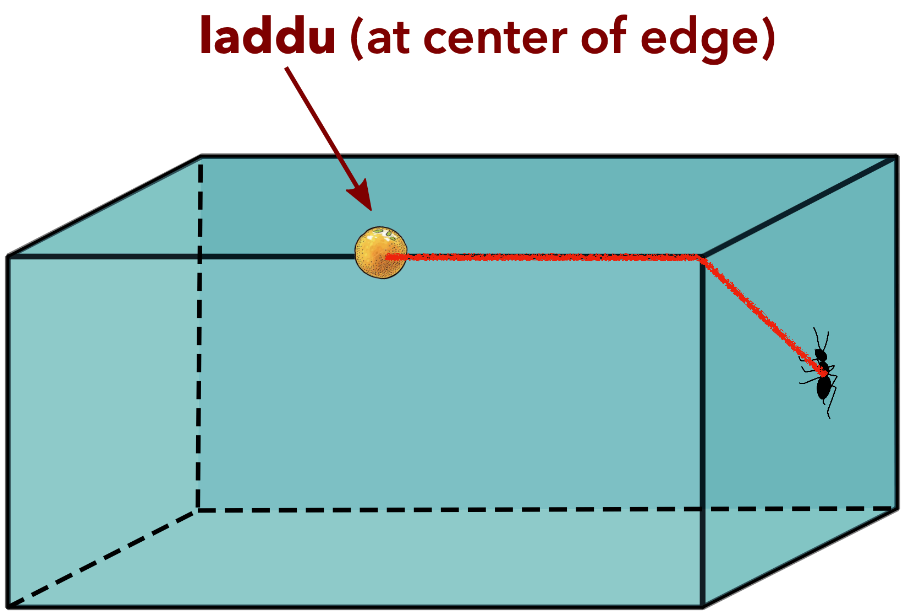
Hint: Imagine *unfolding* the box!

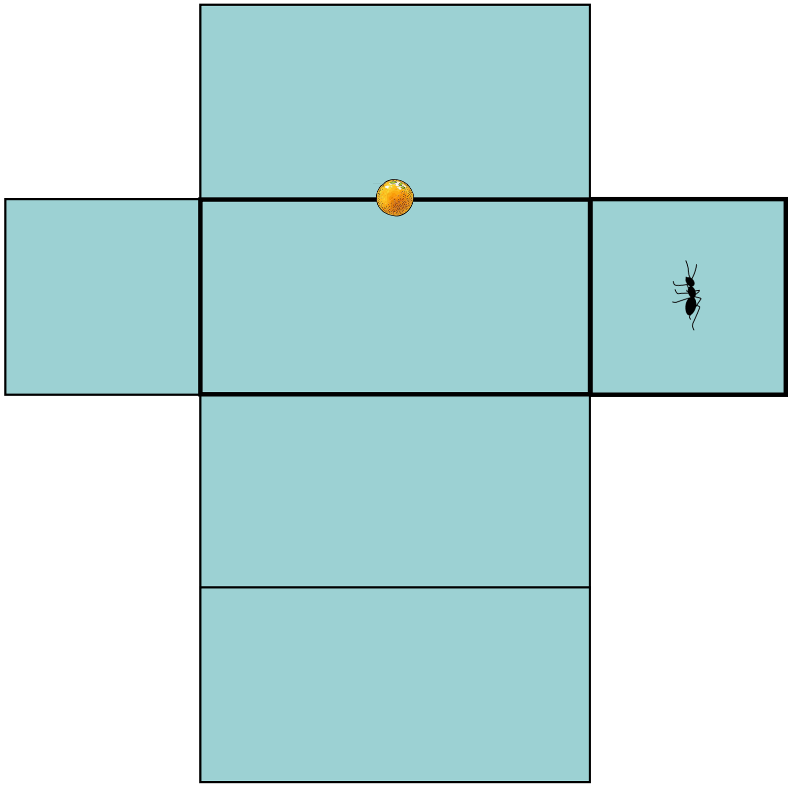




What about now? Can you find the shortest path?

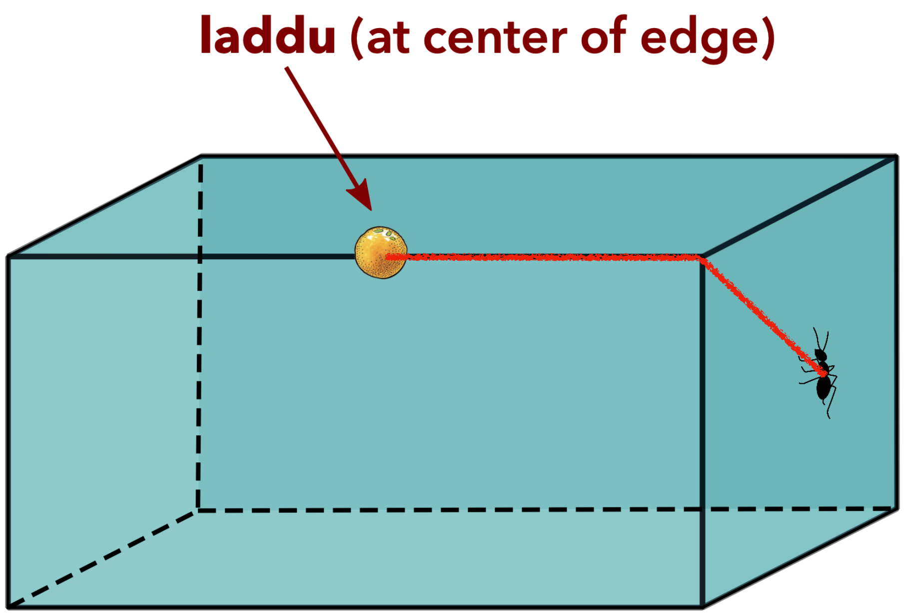
Hint: Imagine *unfolding* the box!

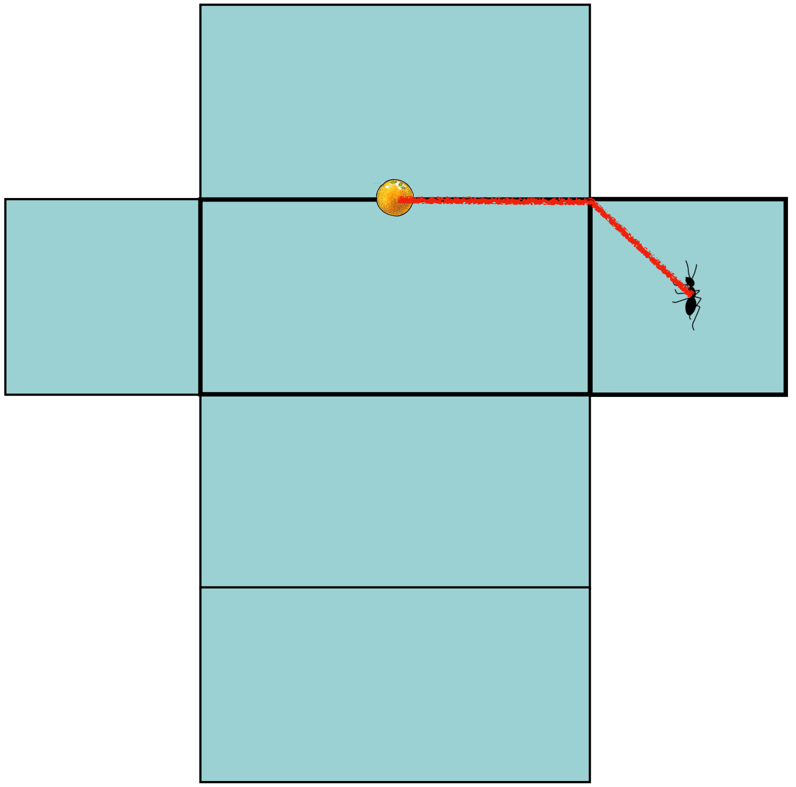




What about now? Can you find the shortest path?

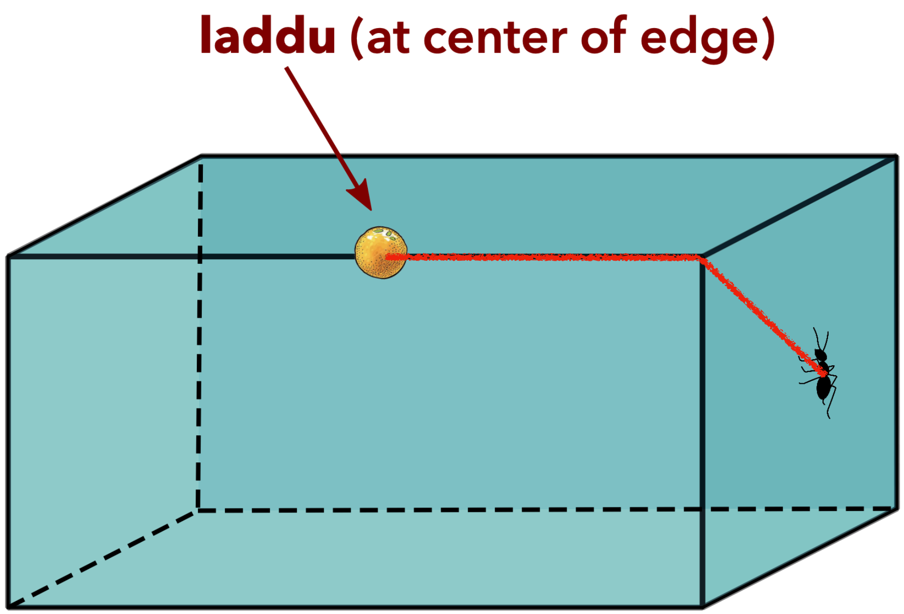
Hint: Imagine *unfolding* the box!

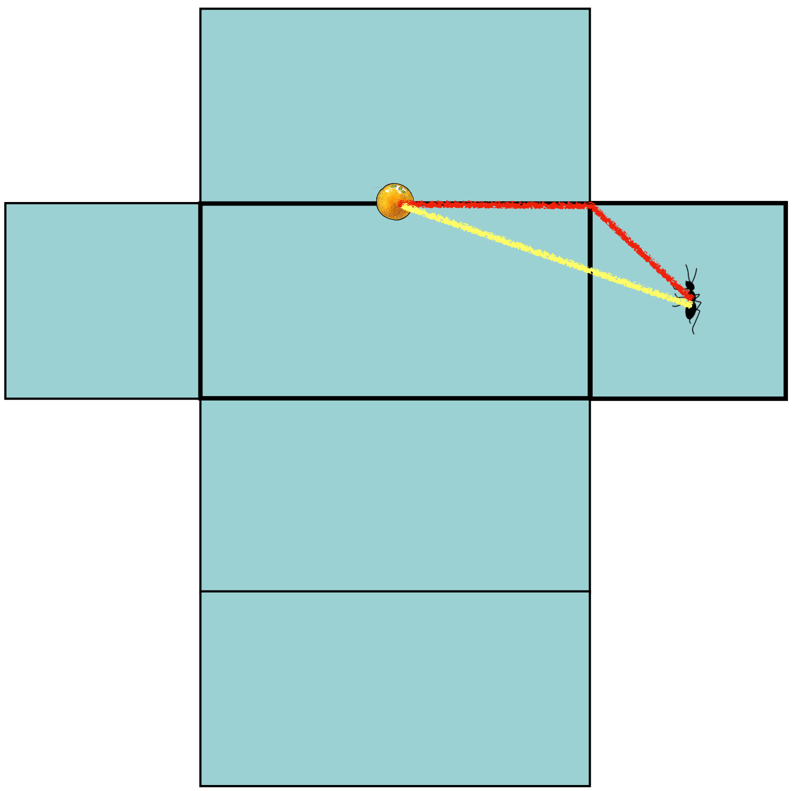




What about now? Can you find the shortest path?

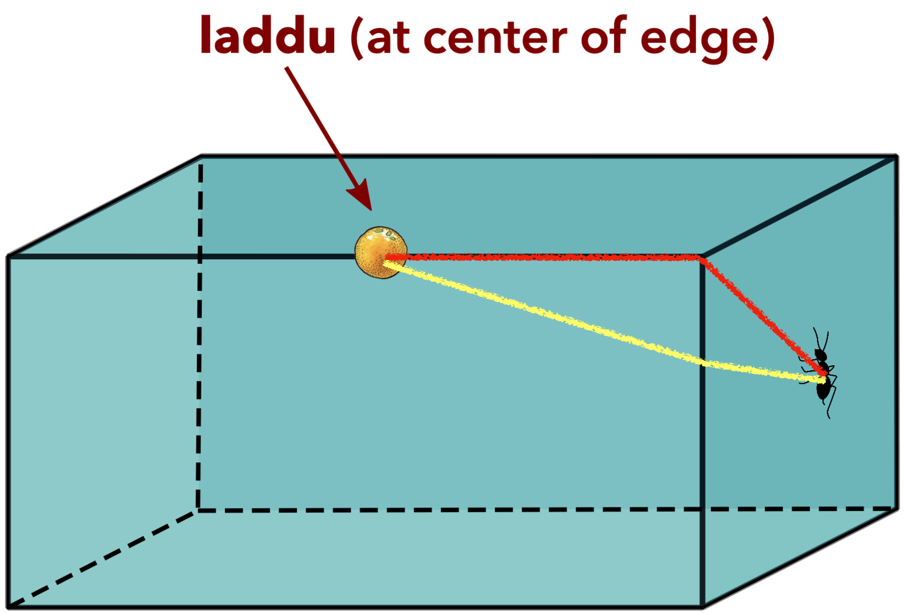
Hint: Imagine *unfolding* the box!

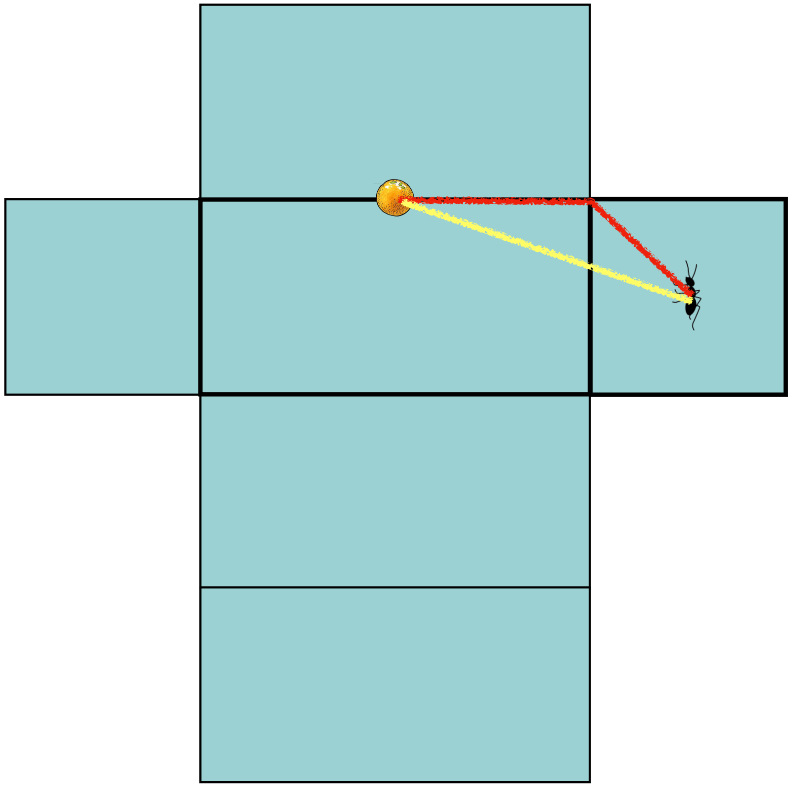




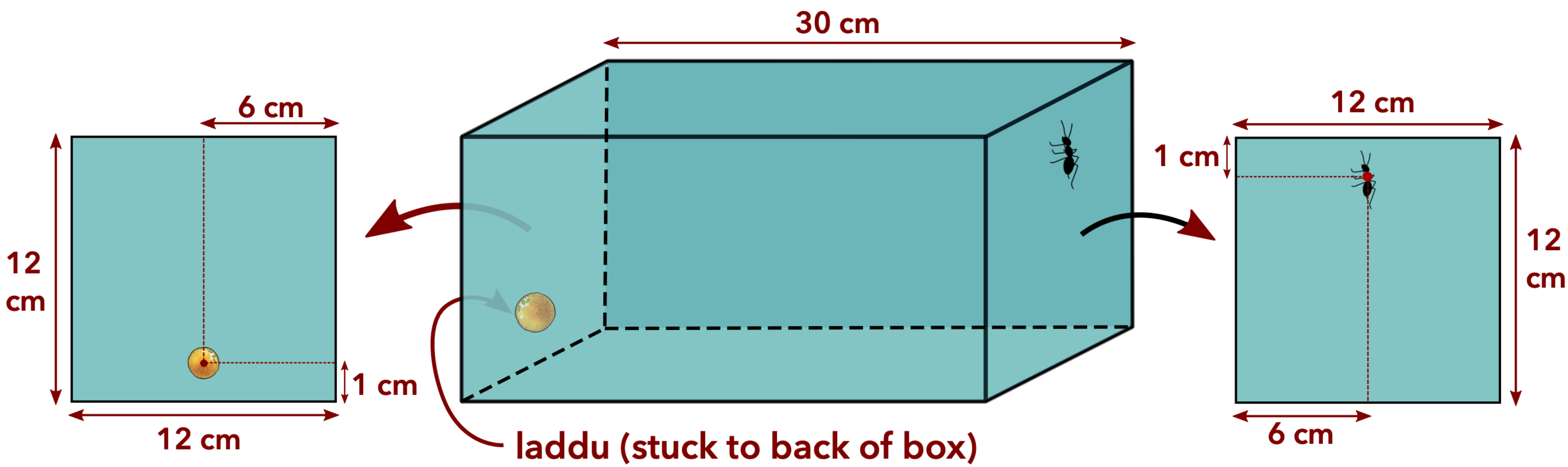
What about now? Can you find the shortest path?

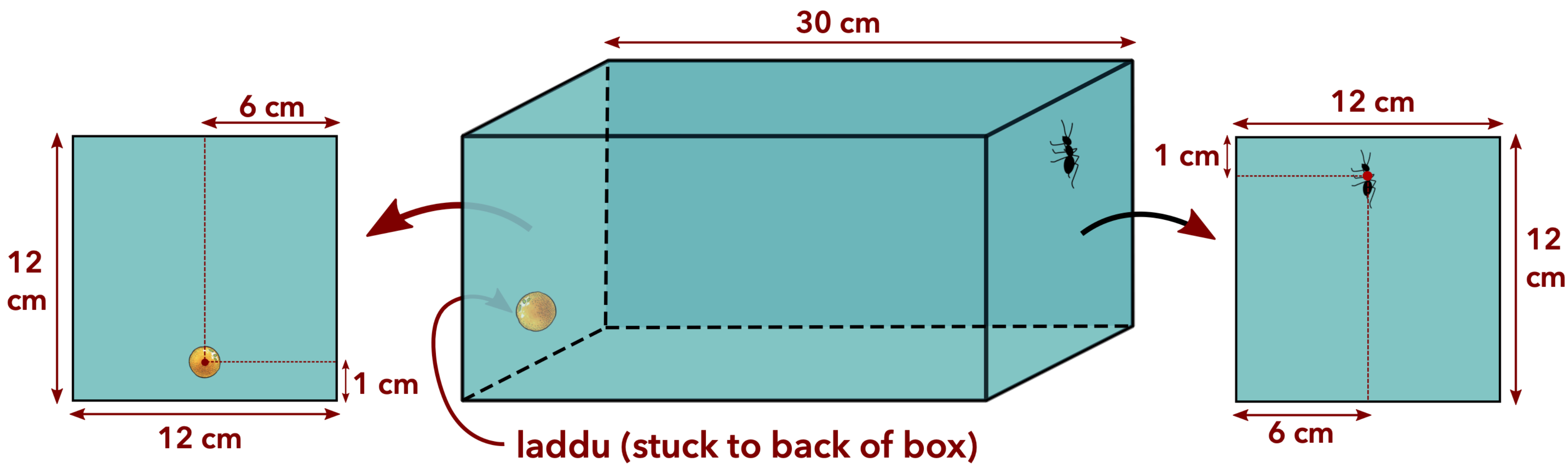
Hint: Imagine *unfolding* the box!



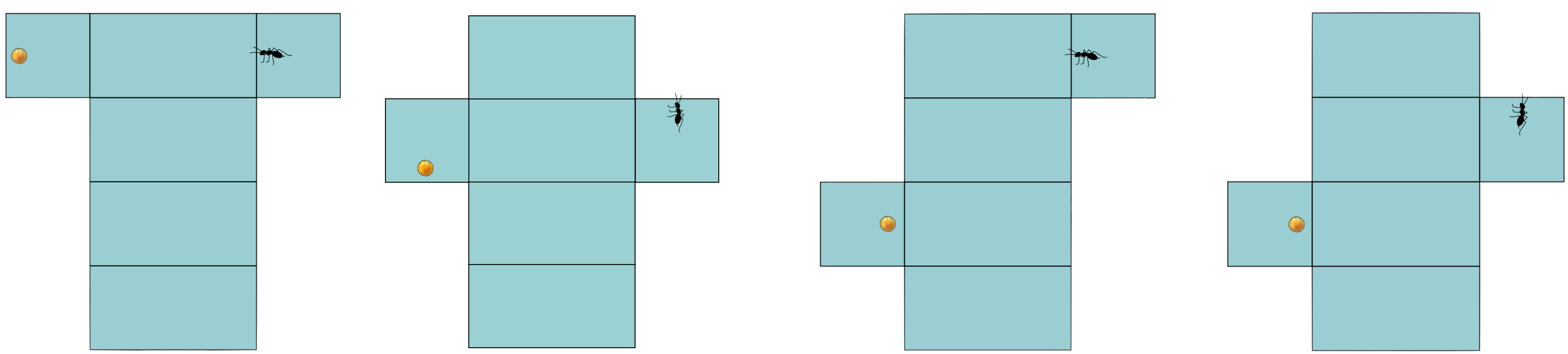


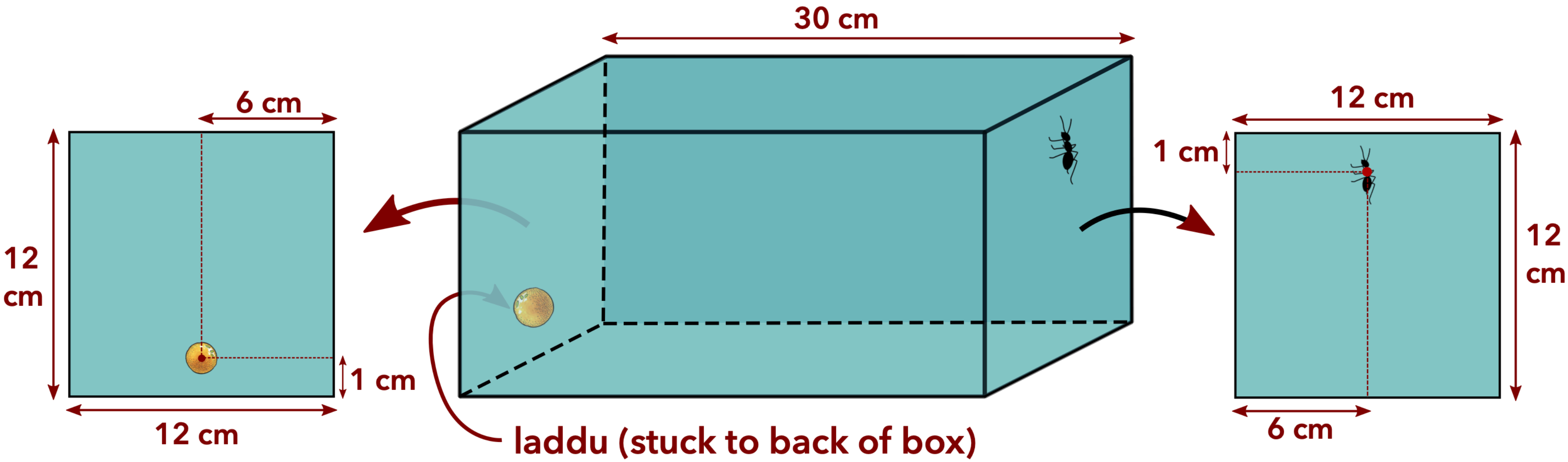
Here's a trickier situation. Find the shortest path. Consider different ways of unfolding the box.



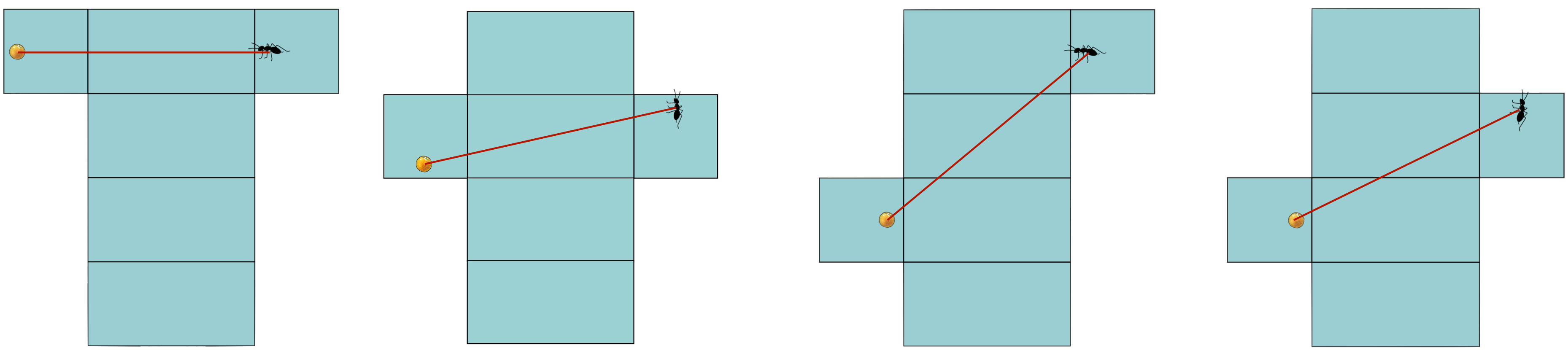


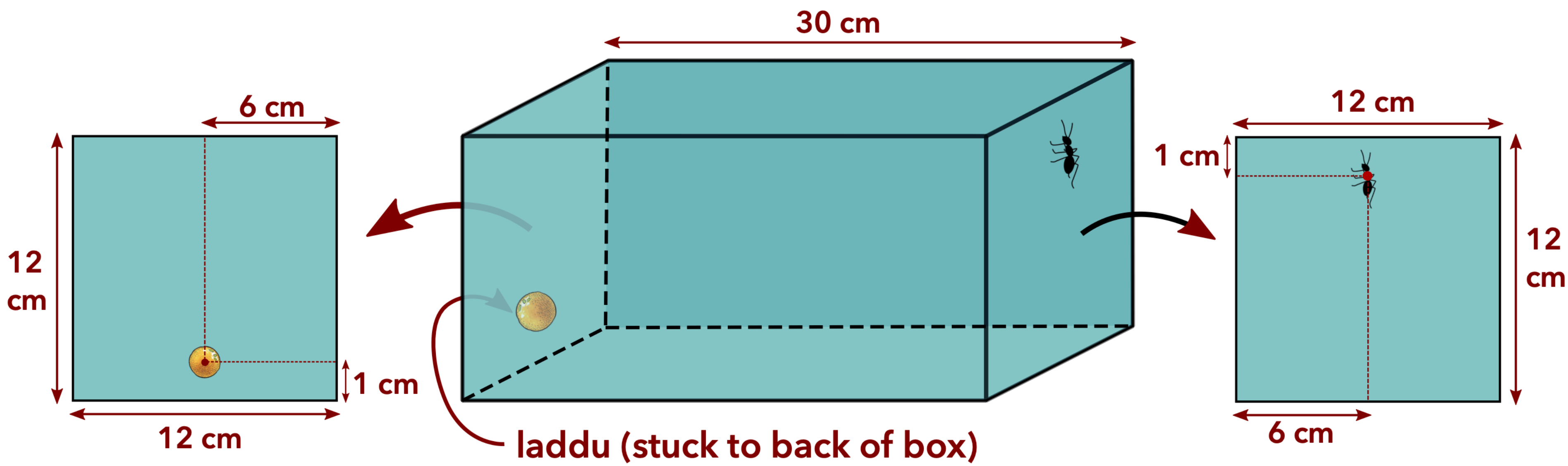
Hint: Here are a few possible *unfoldings* of the box:



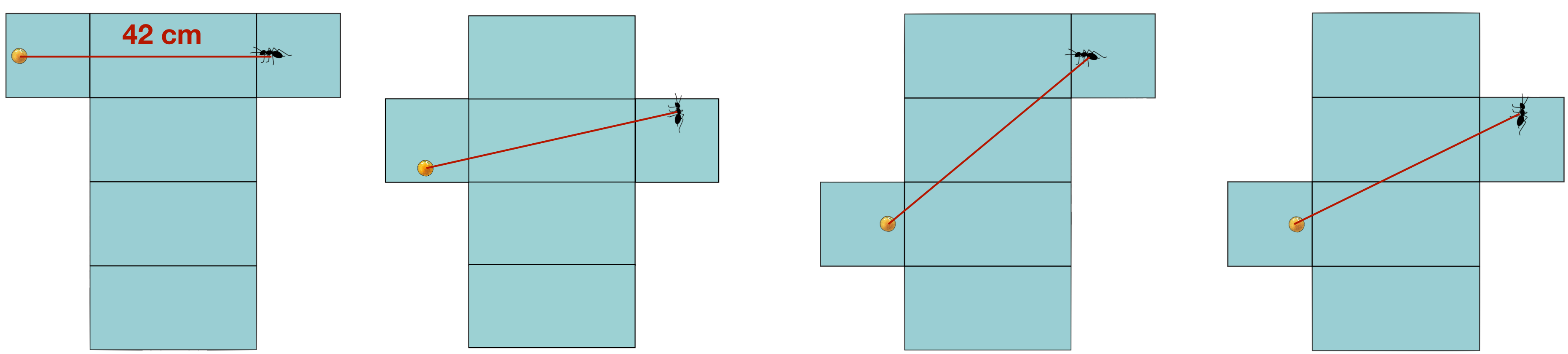


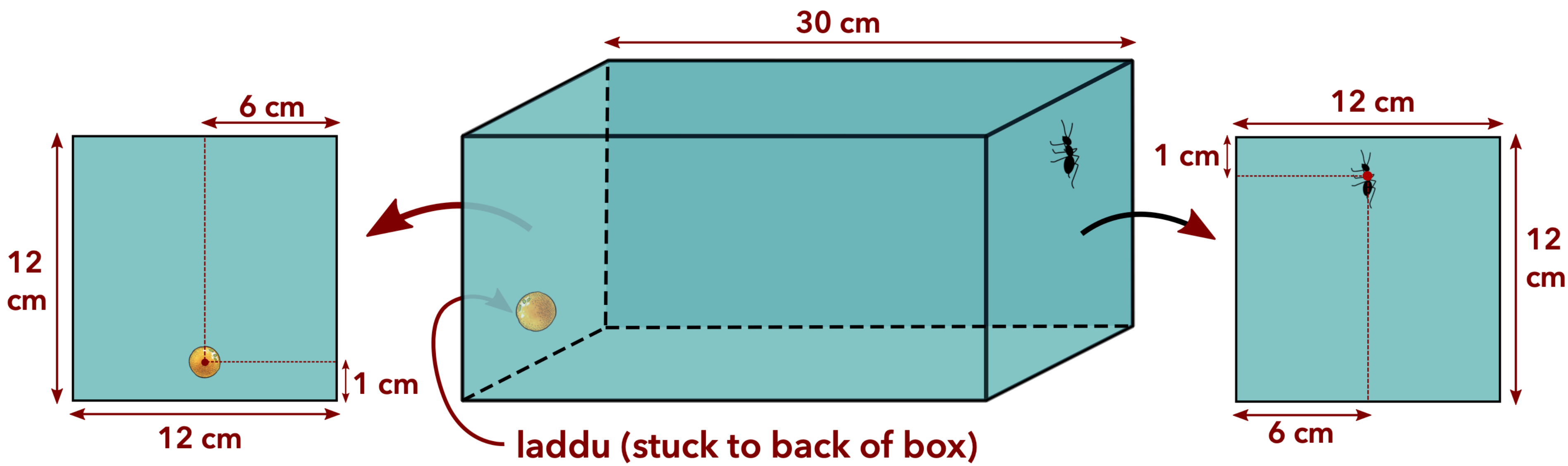
Hint: Here are a few possible *unfoldings* of the box:



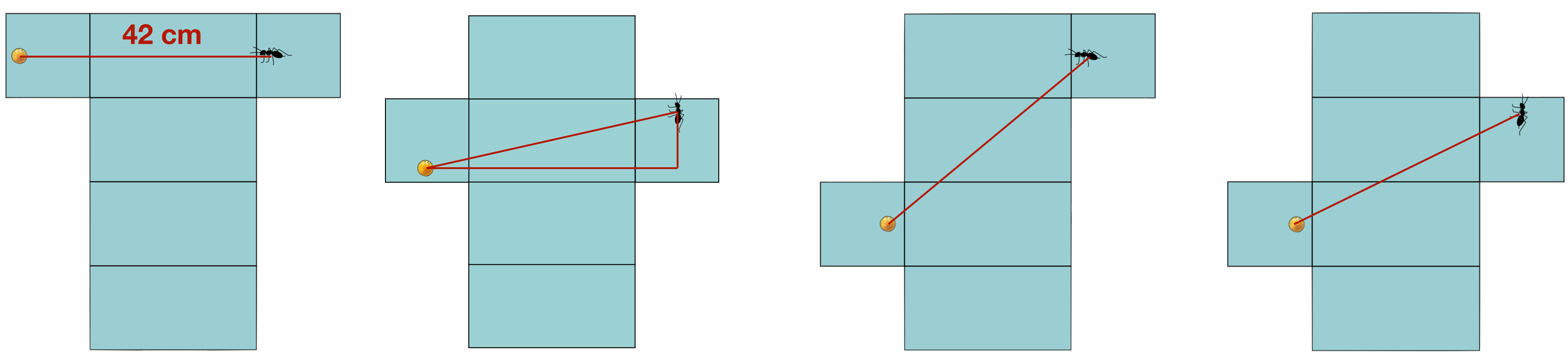


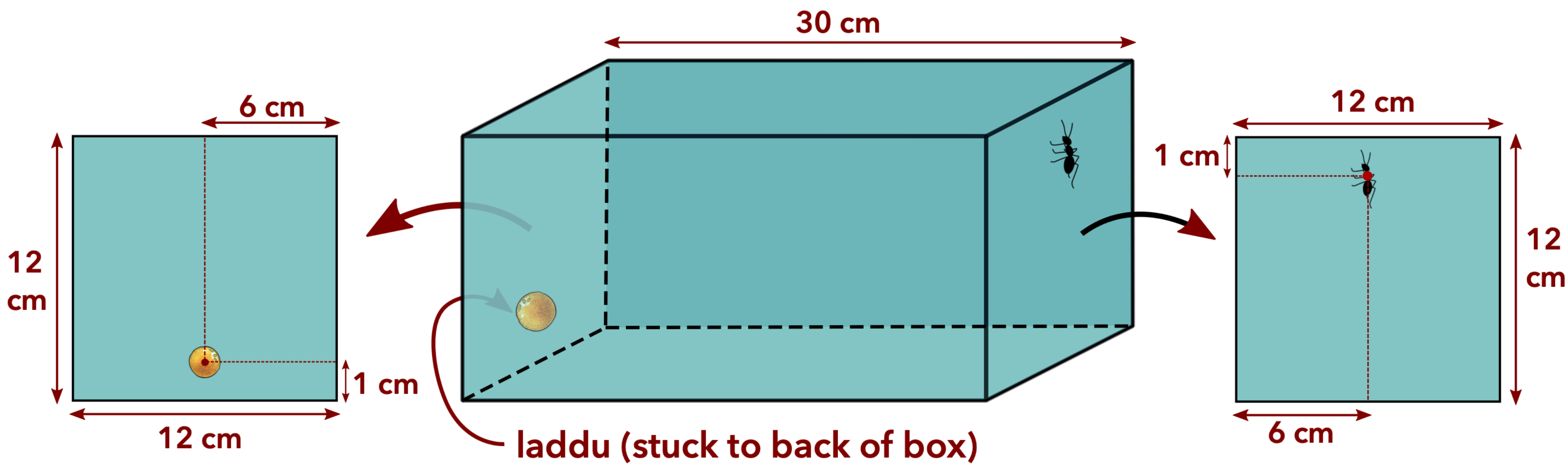
Hint: Here are a few possible *unfoldings* of the box:



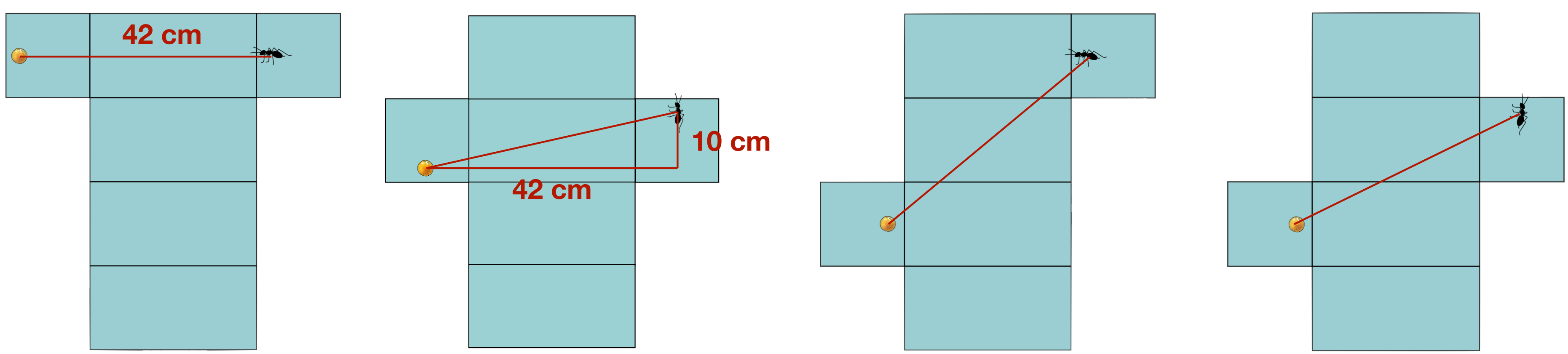


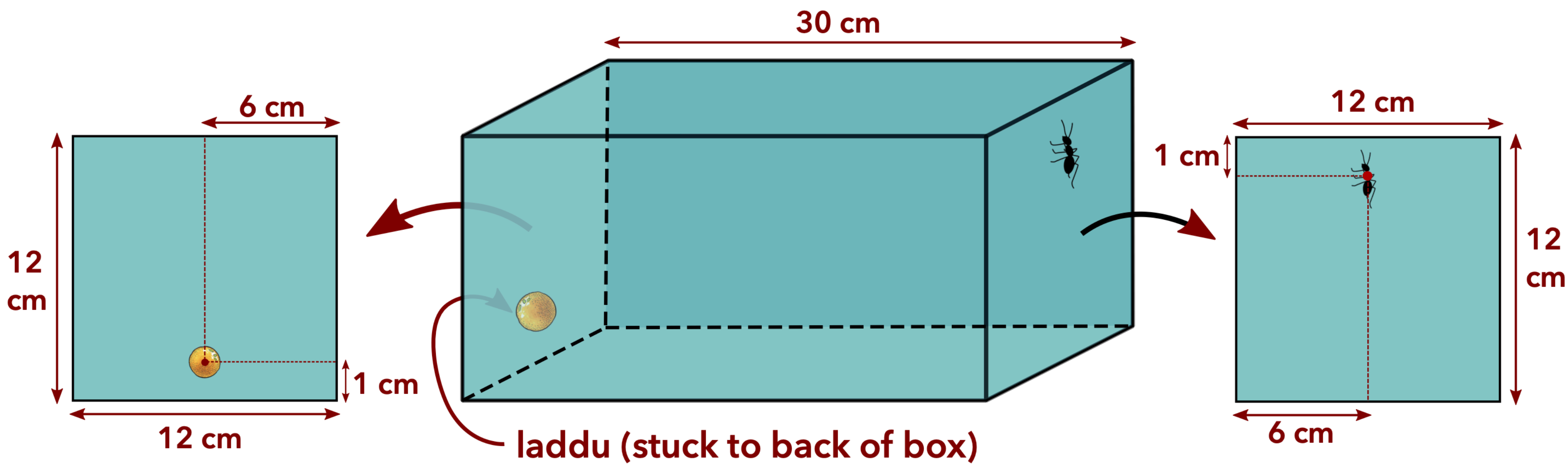
Hint: Here are a few possible *unfoldings* of the box:



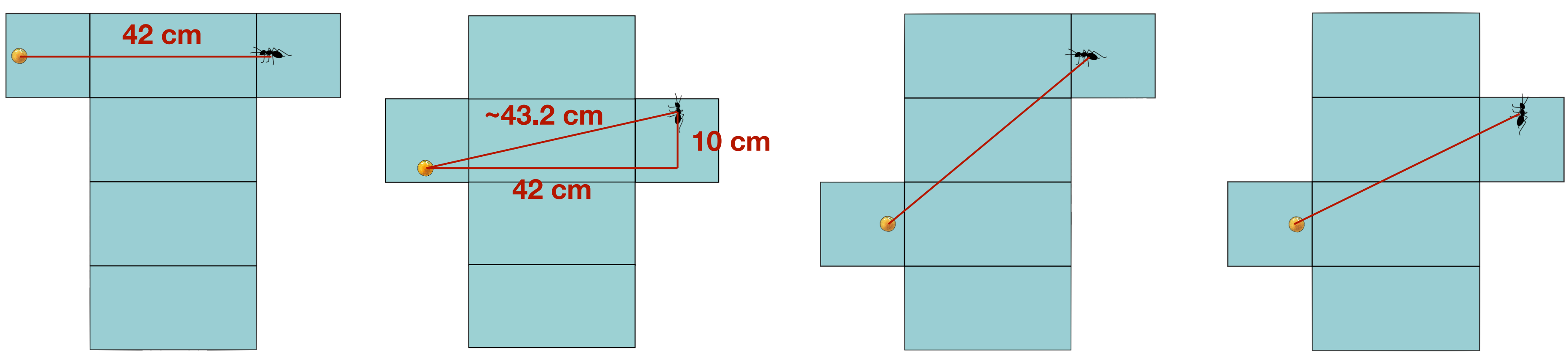


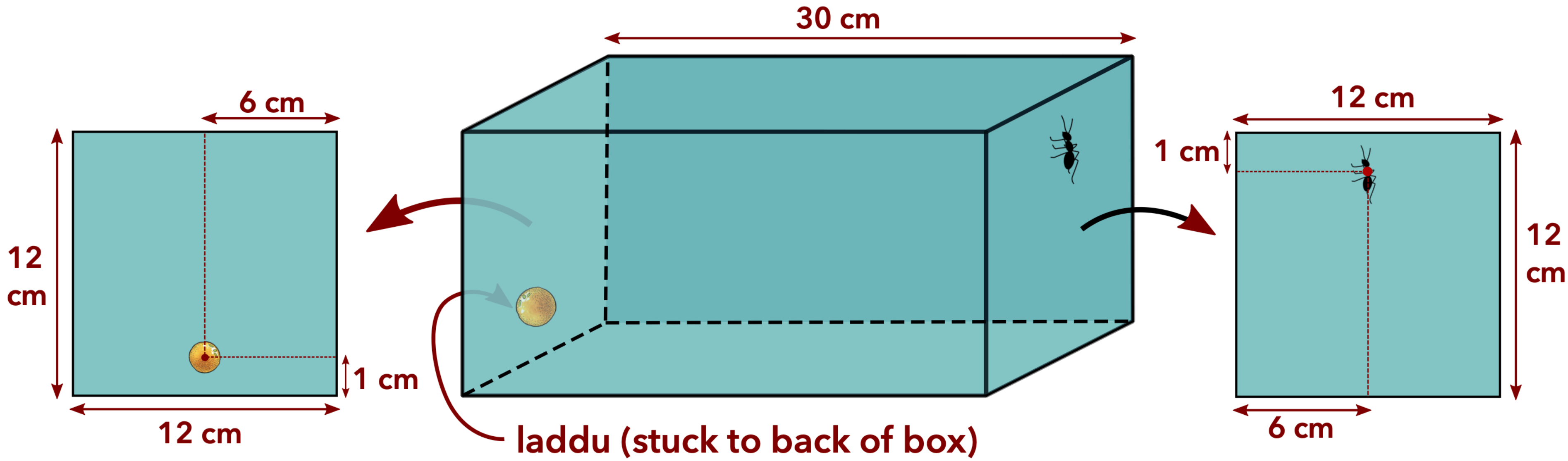
Hint: Here are a few possible *unfoldings* of the box:



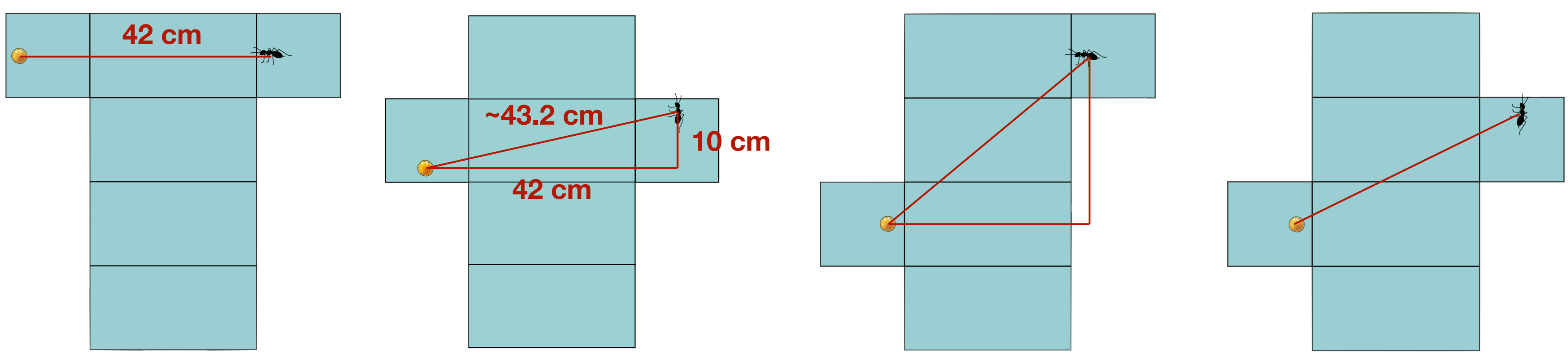


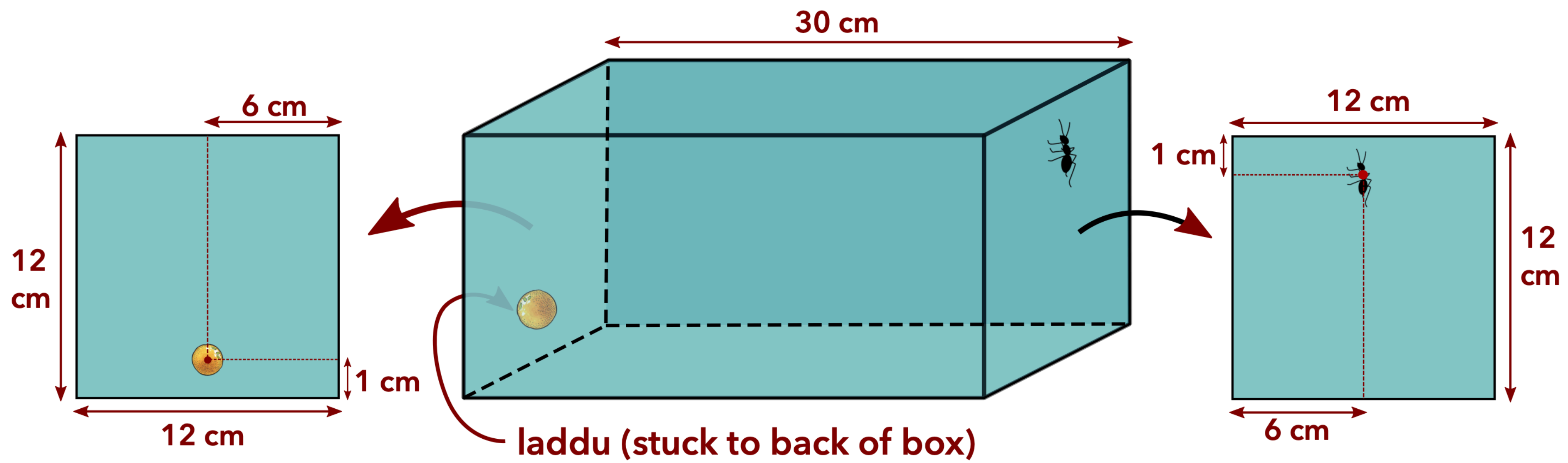
Hint: Here are a few possible *unfoldings* of the box:



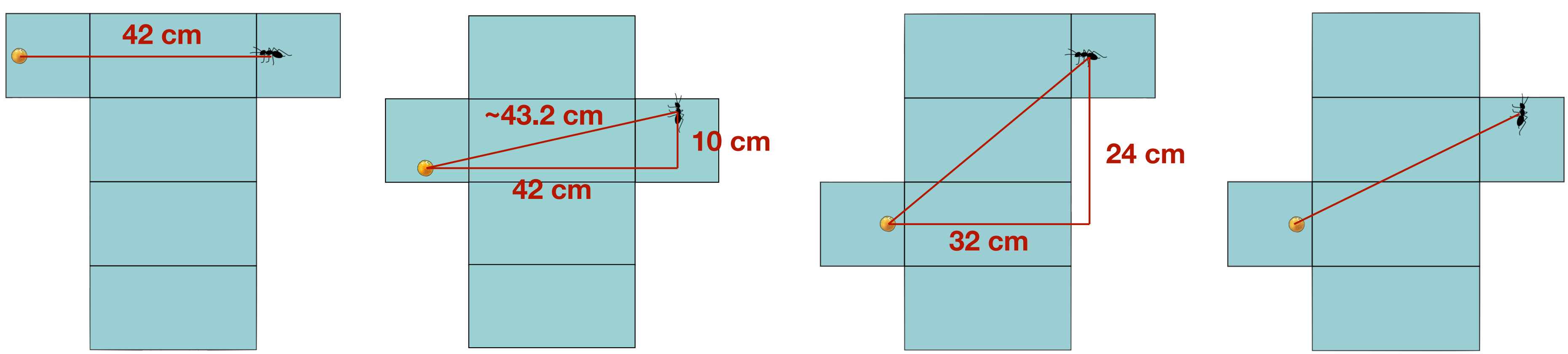


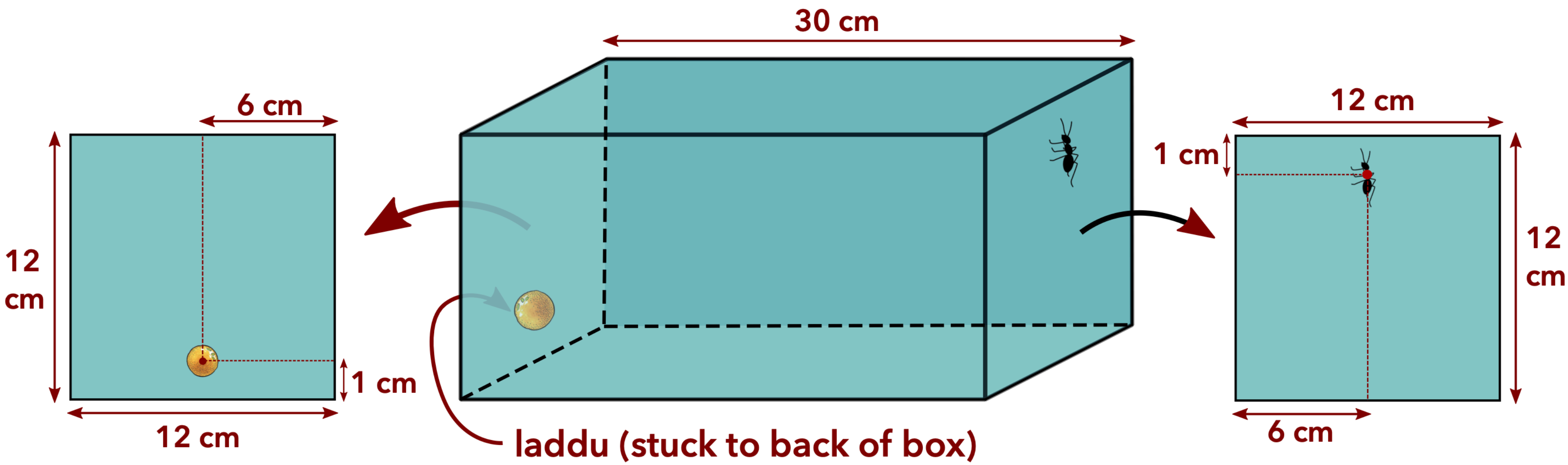
Hint: Here are a few possible *unfoldings* of the box:



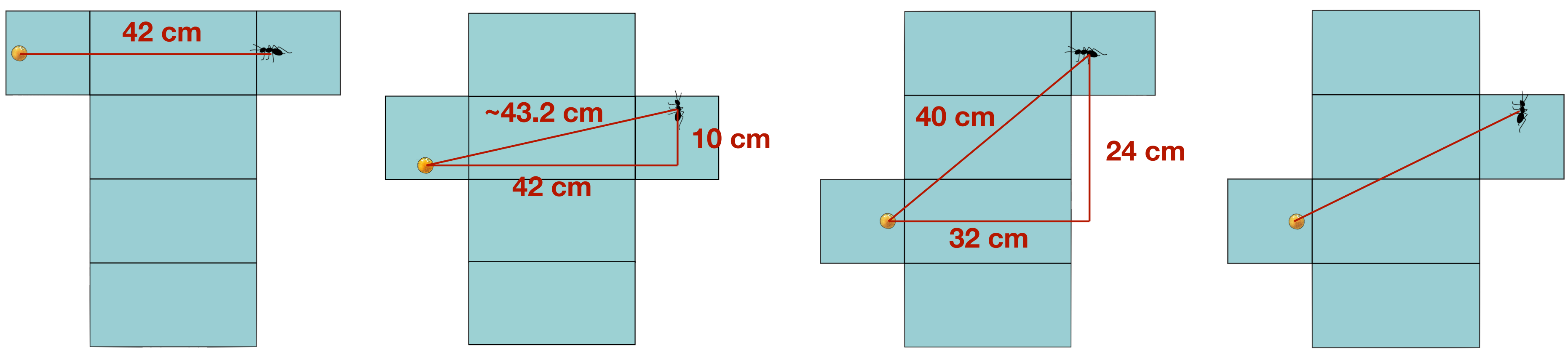


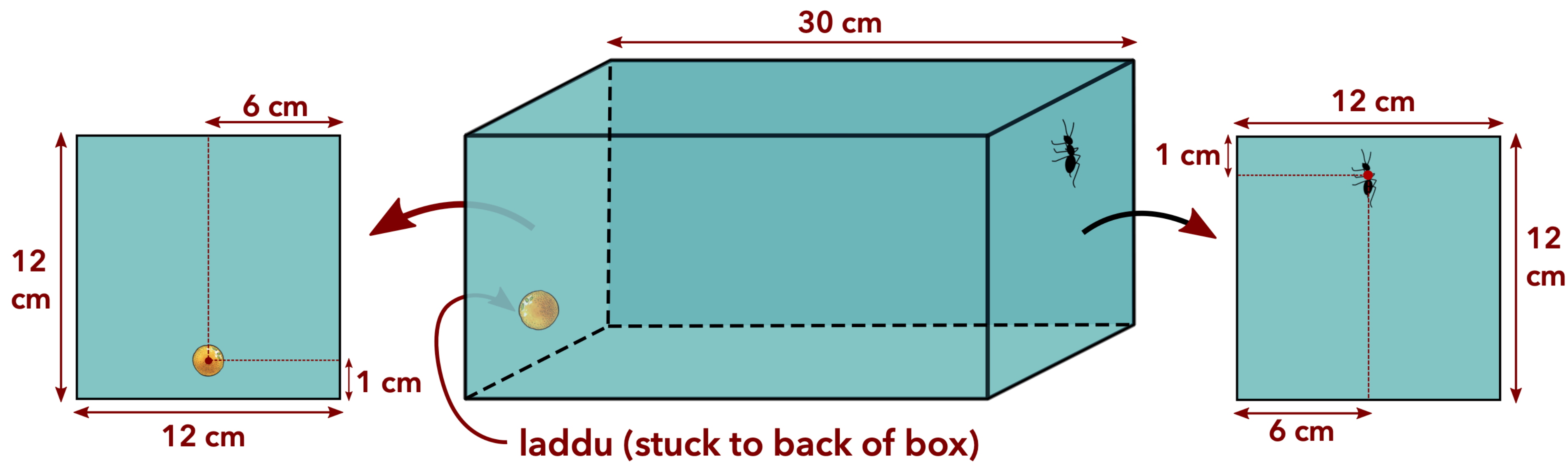
Hint: Here are a few possible *unfoldings* of the box:



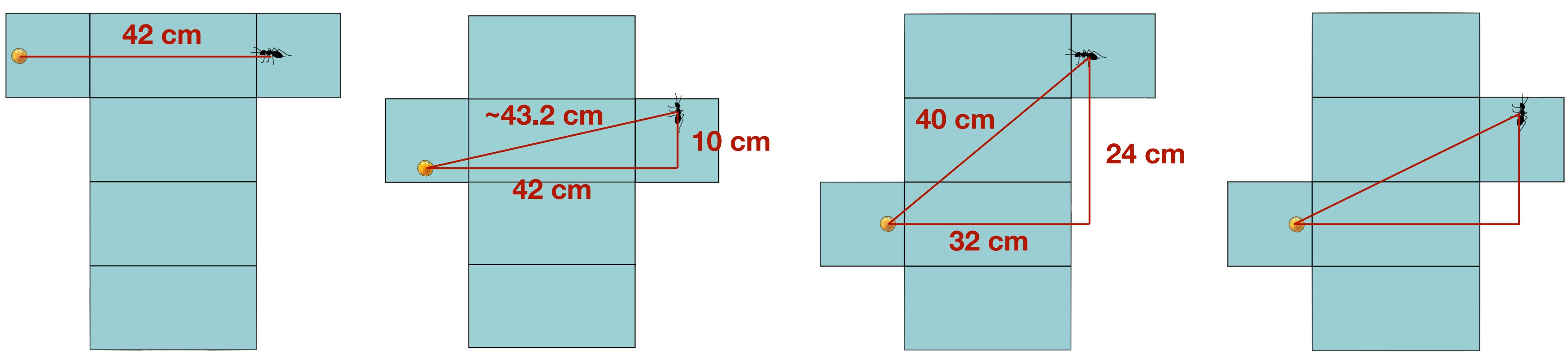


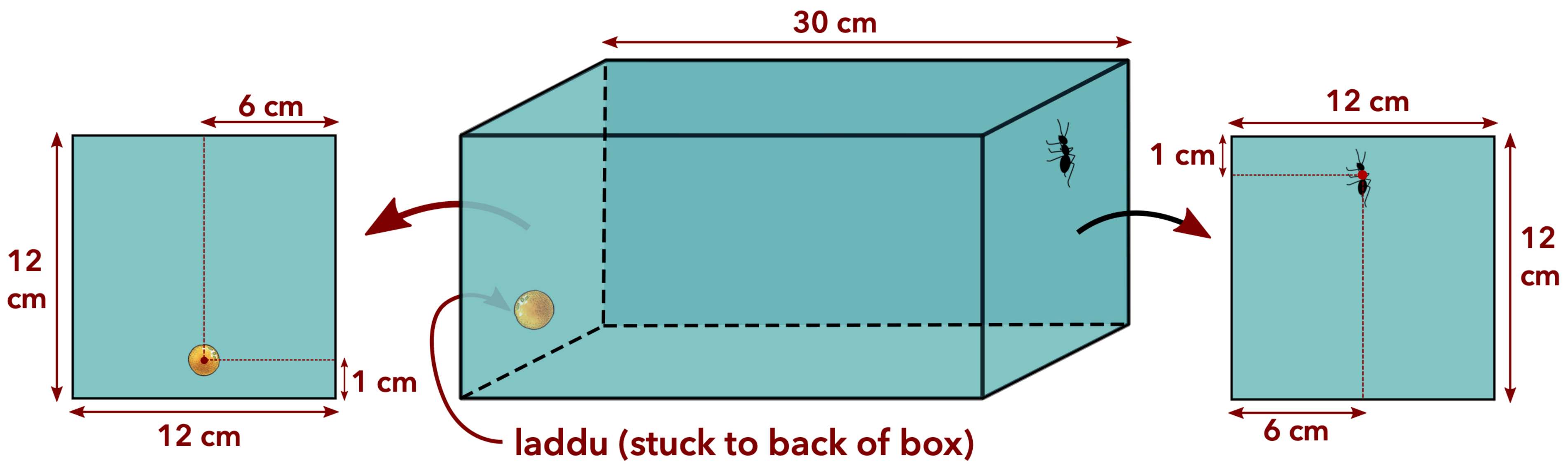
Hint: Here are a few possible *unfoldings* of the box:



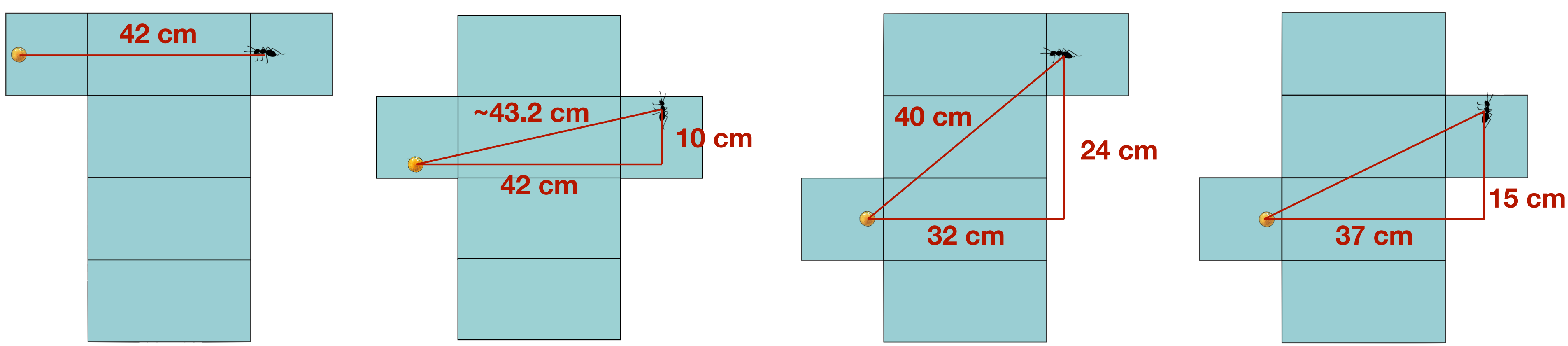


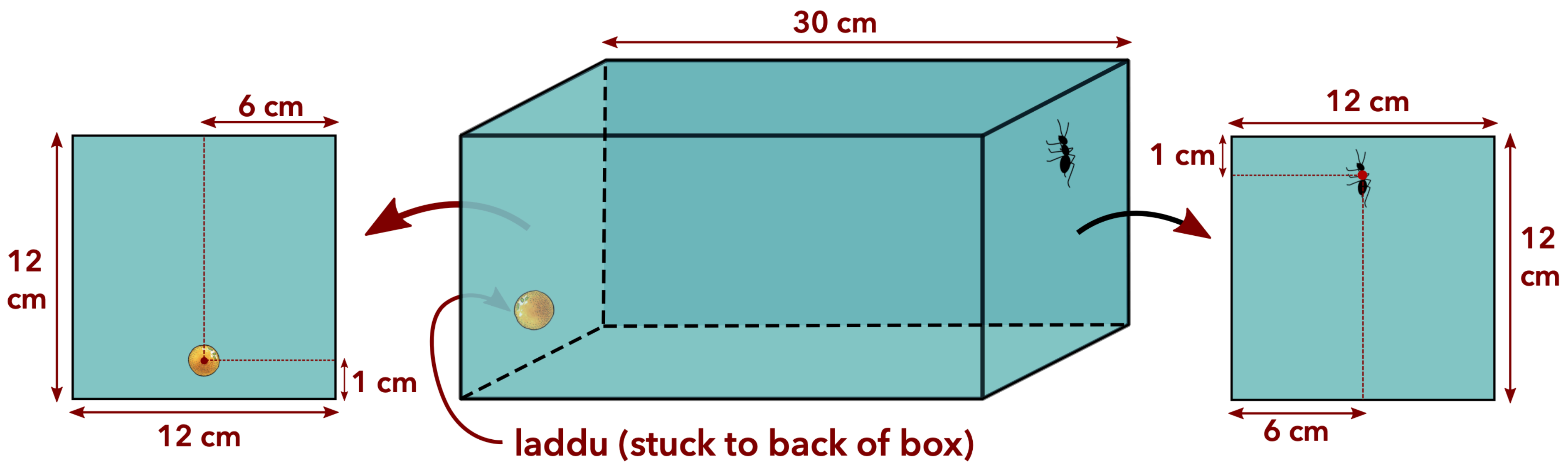
Hint: Here are a few possible *unfoldings* of the box:



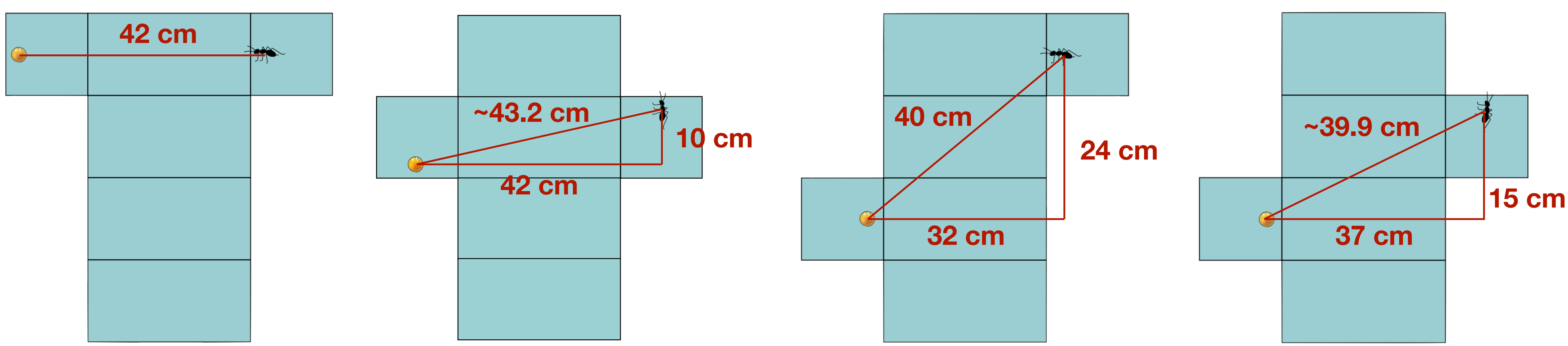


Hint: Here are a few possible *unfoldings* of the box:





Hint: Here are a few possible *unfoldings* of the box:



Definition

A *geodesic* is a path that is intrinsically straight.

Definition

A *geodesic* is a path that is intrinsically straight.

(i.e. locally distance minimizing)
(i.e. locally energy minimizing)

Definition

A *geodesic* is a path that is intrinsically straight.

Example

On the Euclidean plane, geodesics are simply straight lines.

Definition

A *geodesic* is a path that is intrinsically straight.

Example

On the Euclidean plane, geodesics are simply straight lines.

Question

Is a geodesic between two points always the shortest path between them?

Definition

A *geodesic* is a path that is intrinsically straight.

Example

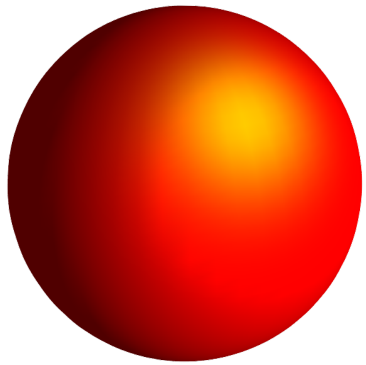
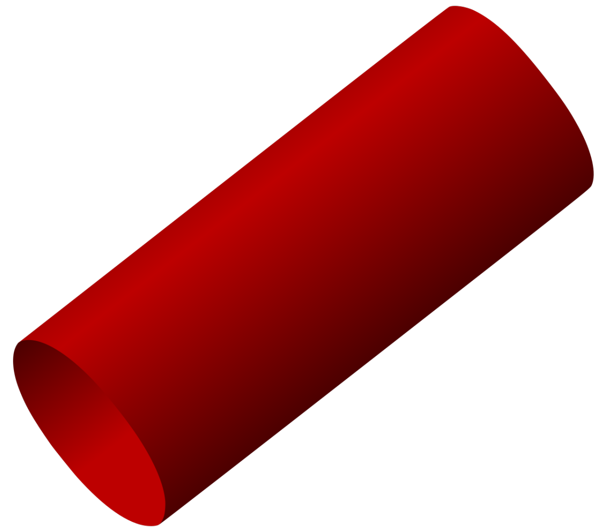
On the Euclidean plane, geodesics are simply straight lines.

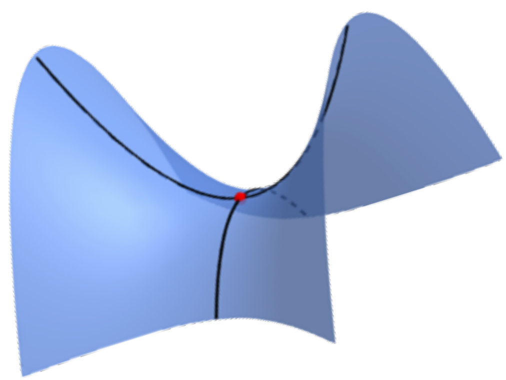
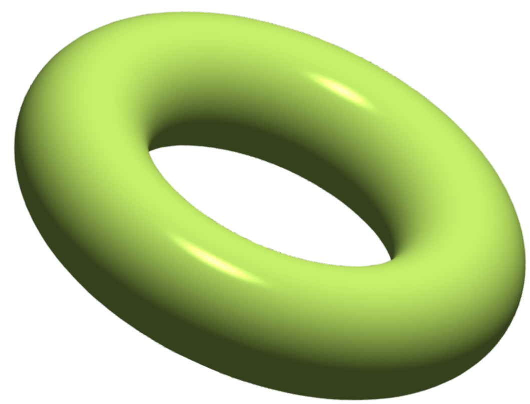
Question

Is a geodesic between two points always the shortest path between them?

Is the shortest path between two points always a geodesic?

Geodesics on other surfaces

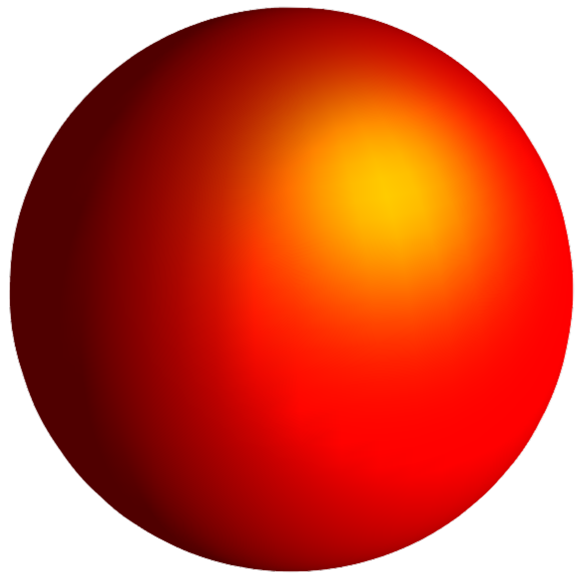




Geodesics on other surfaces

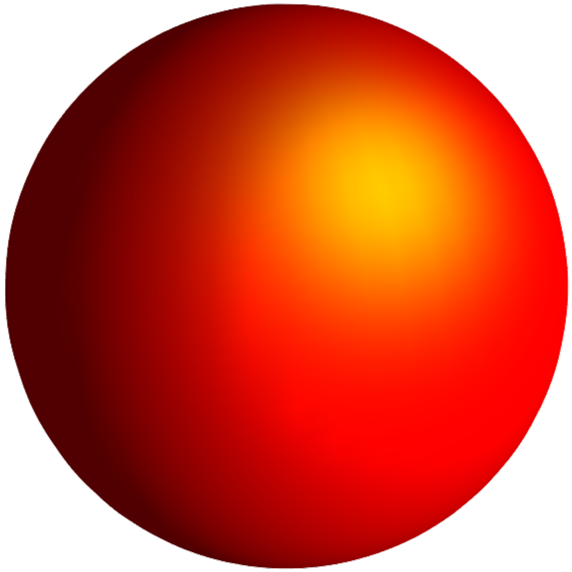
Can you describe the geodesics on the sphere?

Remember, a geodesic is a *locally straight* path!



Geodesics on other surfaces

Can you describe the geodesics on the sphere?



Remember, a geodesic is a *locally straight* path!

A paper ribbon is one useful way to create a geodesic!

Geodesics on other surfaces

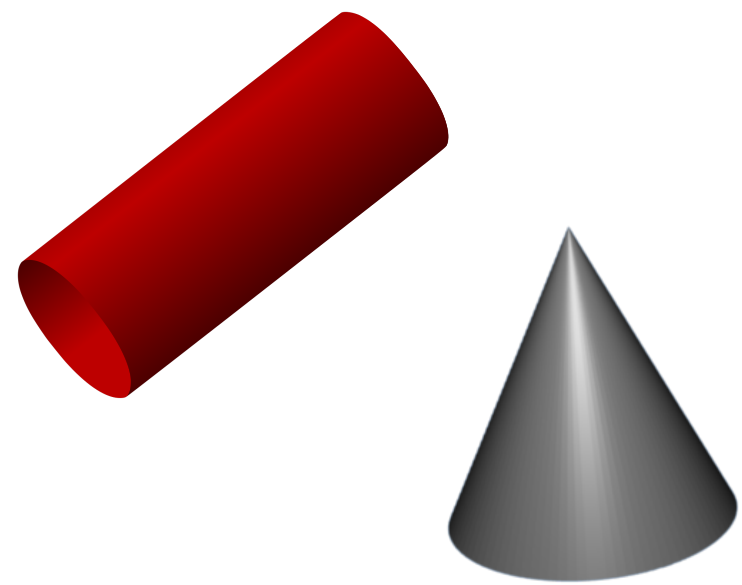
Can you *launch* geodesics on this gourd, using the ribbon method?

Properties of Geodesics

Let’s turn our attention to some surfaces
we can easy build: cylinders and cones!

Properties of Geodesics

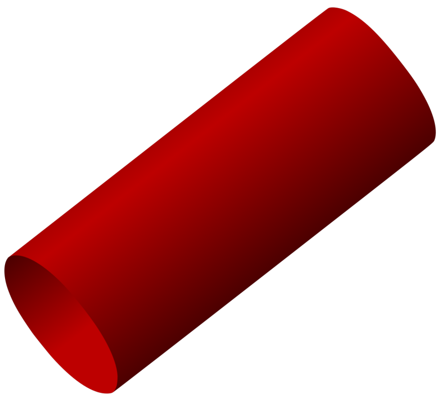
Let’s turn our attention to some surfaces we can easy build: cylinders and cones!

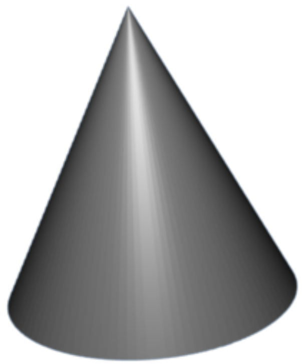


Properties of Geodesics

Let’s turn our attention to some surfaces we can easy build: cylinders and cones!

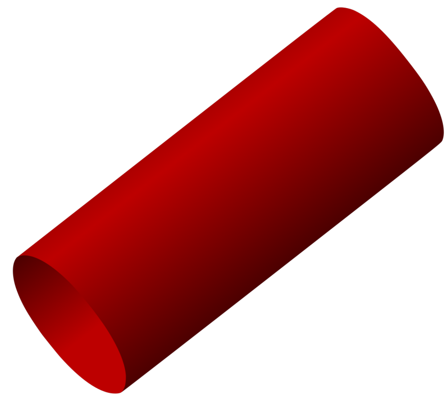
We don’t want to consider surfaces with boundary. So from now on we will imagine all cylinders and cones are infinitely long!

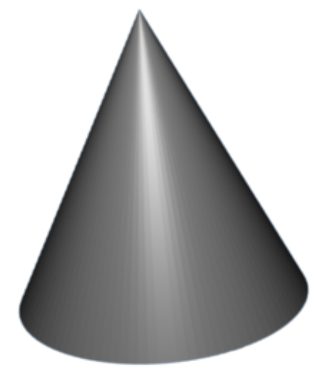




Properties of Geodesics

Let's turn our attention to some surfaces we can easy build: cylinders and cones!





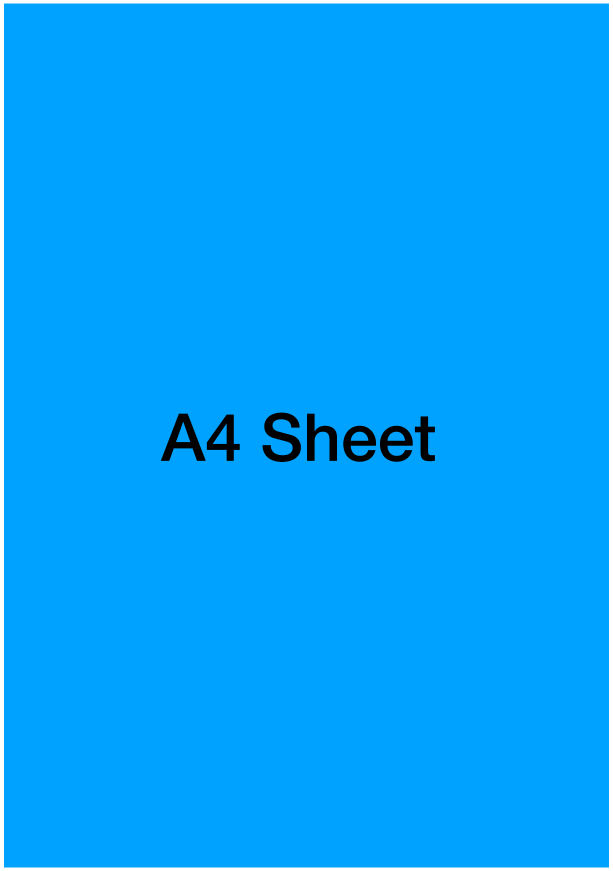
We don't want to consider surfaces with boundary. So from now on we will imagine all cylinders and cones are infinitely long!

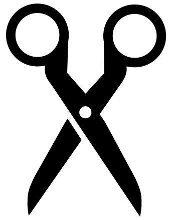
How do geodesics on cylinders and cones compare to geodesics on the plane?

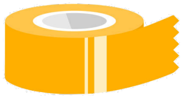
Construct a cylinder, a cone, and some geodesics

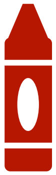
Make sure you have:

1) A4 Sheet
2) Scissors
3) Tape
4) Crayons or Sketchpens

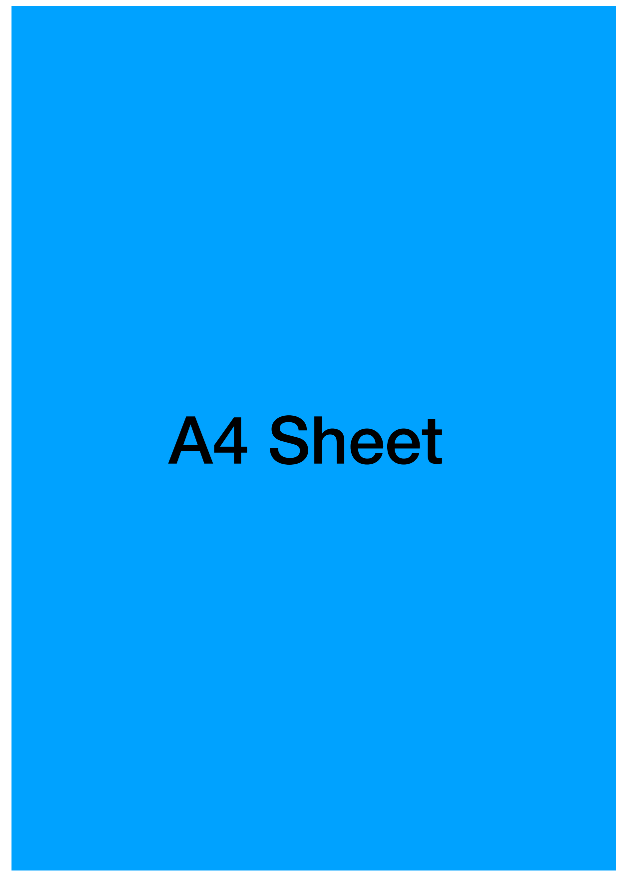






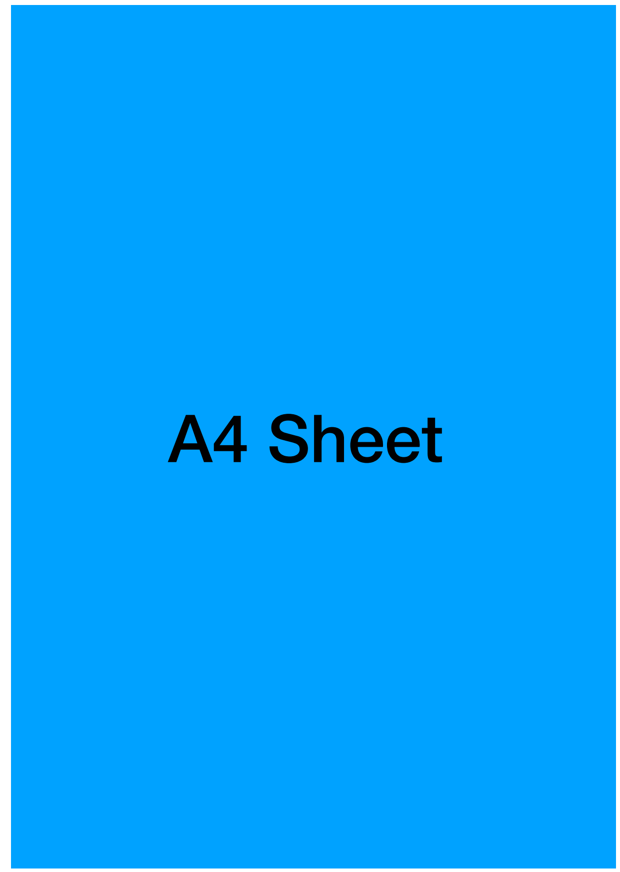


Construct a cylinder, a cone, and some geodesics



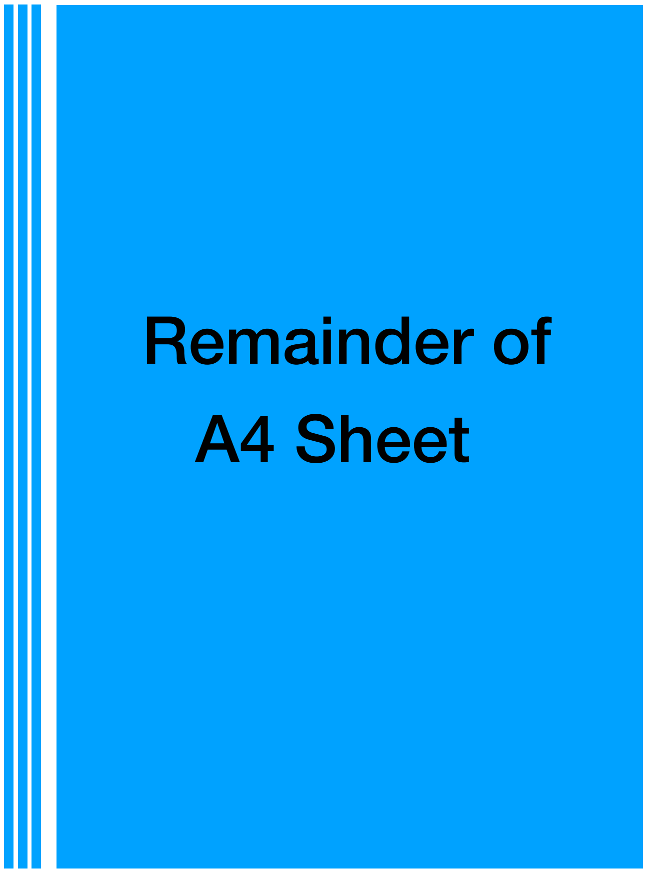
Construct a cylinder, a cone, and some geodesics

1) Cut three thin ribbons from the long side of your A4 sheet.



Construct a cylinder, a cone, and some geodesics

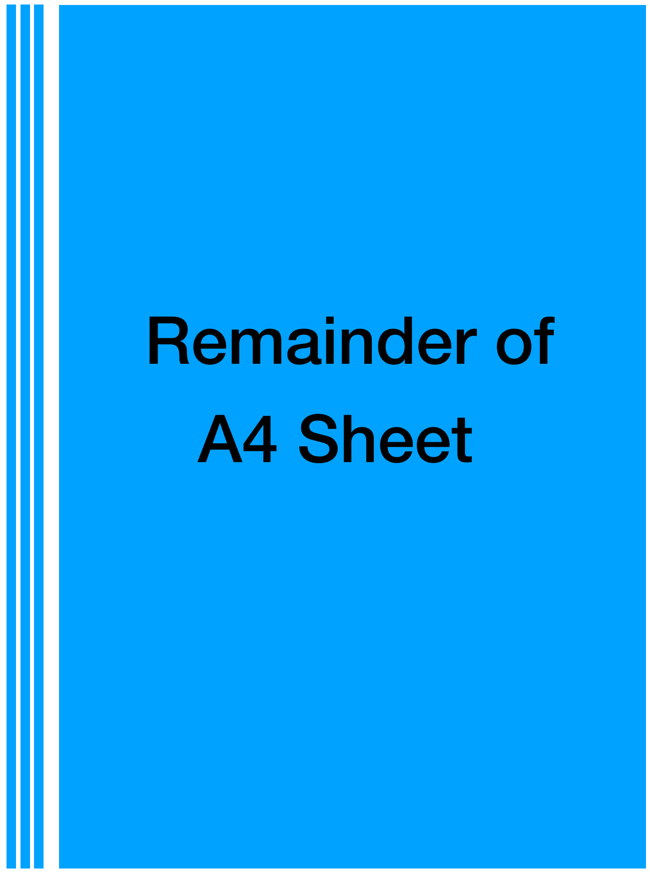
1) Cut three thin ribbons from the long side of your A4 sheet.



Construct a cylinder, a cone, and some geodesics

1) Cut three thin ribbons from the long side of your A4 sheet.

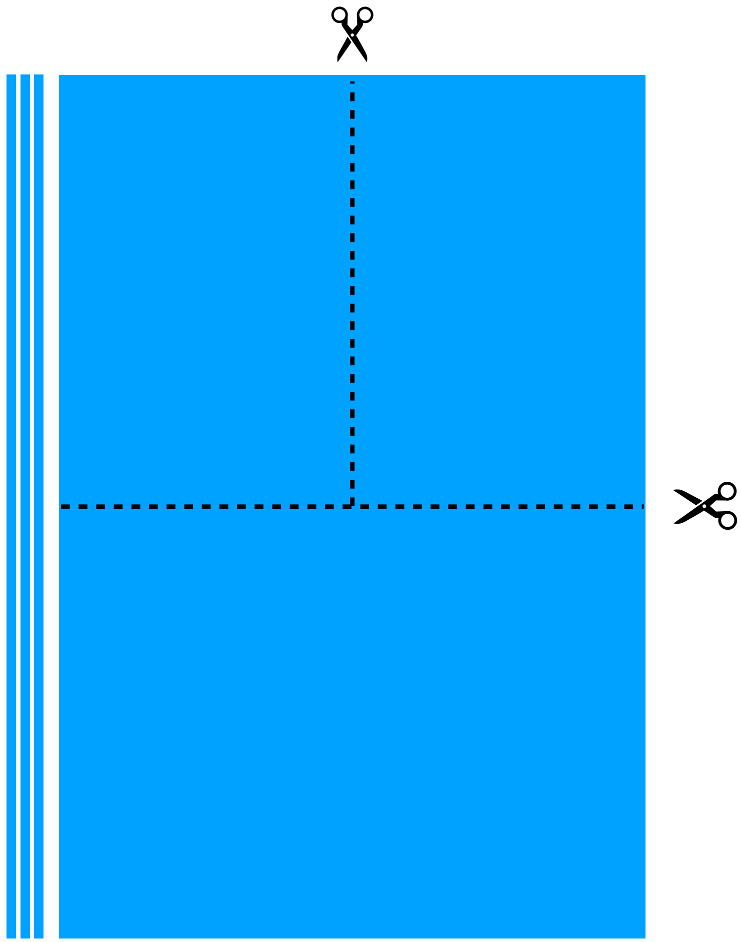
2) Fold and cut the remainder of your A4 sheet into three parts



Construct a cylinder, a cone, and some geodesics

1) Cut three thin ribbons from the long side of your A4 sheet.

2) Fold and cut the remainder of your A4 sheet into three parts

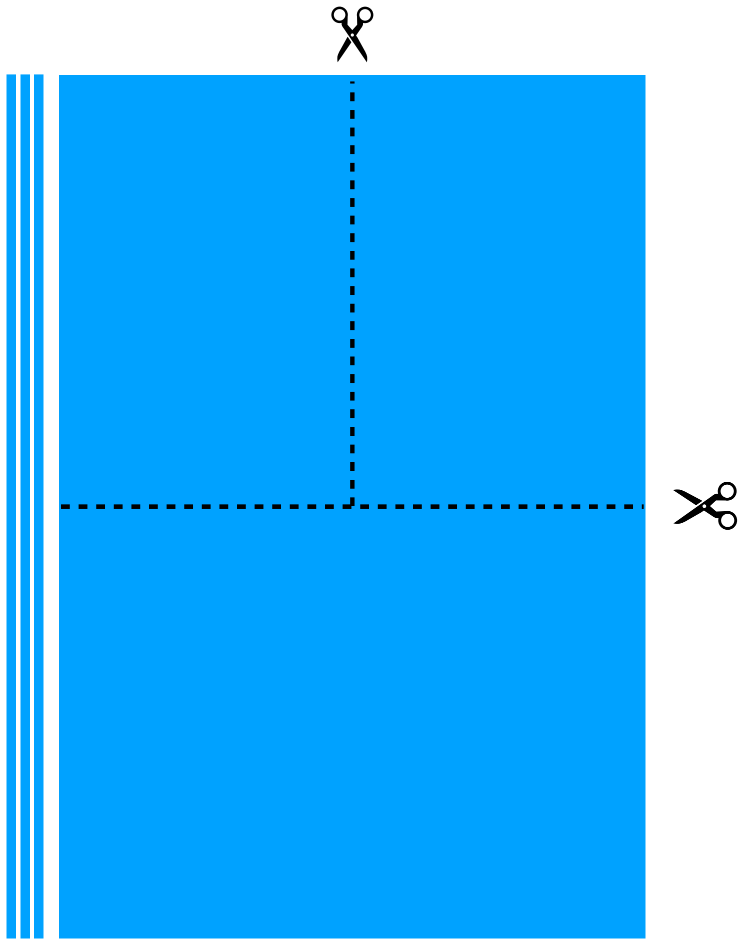


Construct a cylinder, a cone, and some geodesics

1) Cut three thin ribbons from the long side of your A4 sheet.

2) Fold and cut the remainder of your A4 sheet into three parts

3) Make one of the small rectangles into a cylinder, and make the large rectangle into a cone.

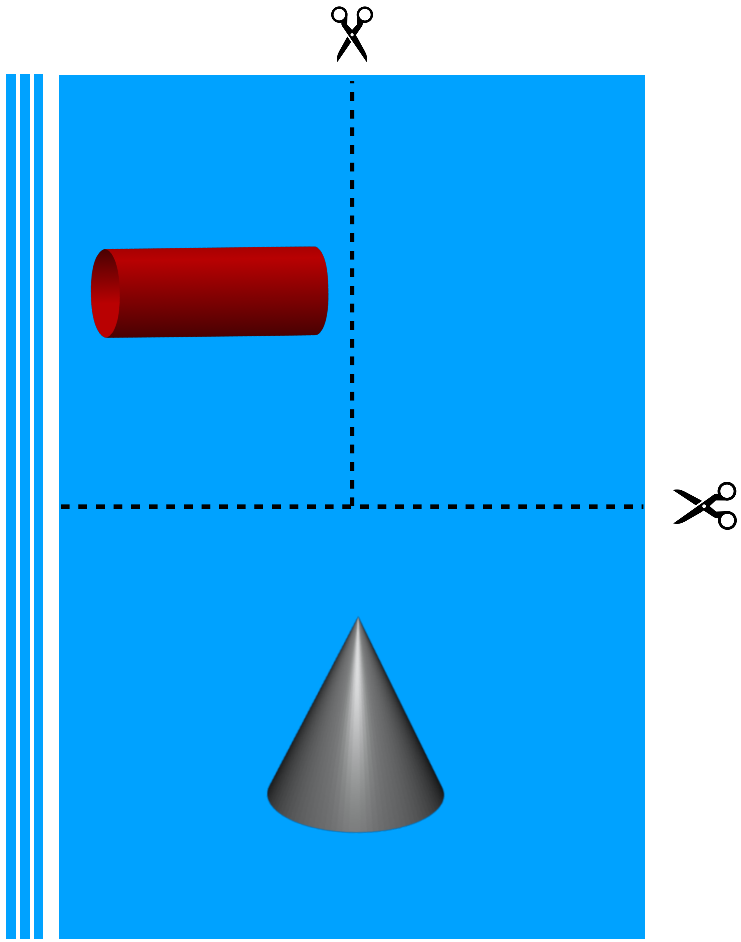


Construct a cylinder, a cone, and some geodesics

1) Cut three thin ribbons from the long side of your A4 sheet.

2) Fold and cut the remainder of your A4 sheet into three parts

3) Make one of the small rectangles into a cylinder, and make the large rectangle into a cone.



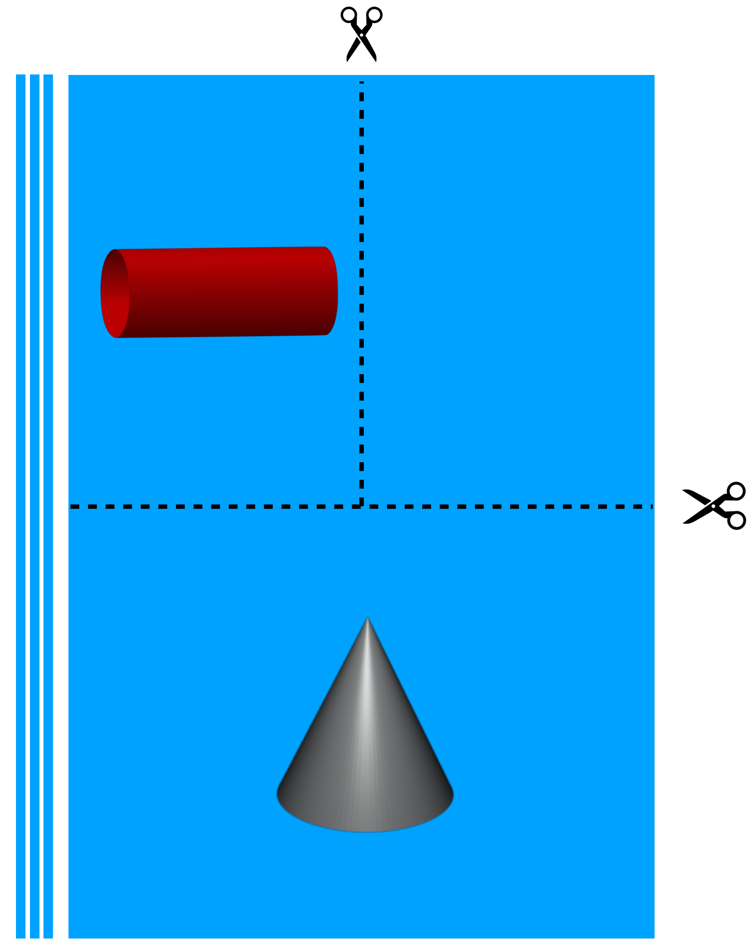
Construct a cylinder, a cone, and some geodesics

1) Cut three thin ribbons from the long side of your A4 sheet.

2) Fold and cut the remainder of your A4 sheet into three parts

3) Make one of the small rectangles into a cylinder, and make the large rectangle into a cone.

4) Color the ribbons different colors using crayons or sketch pens.



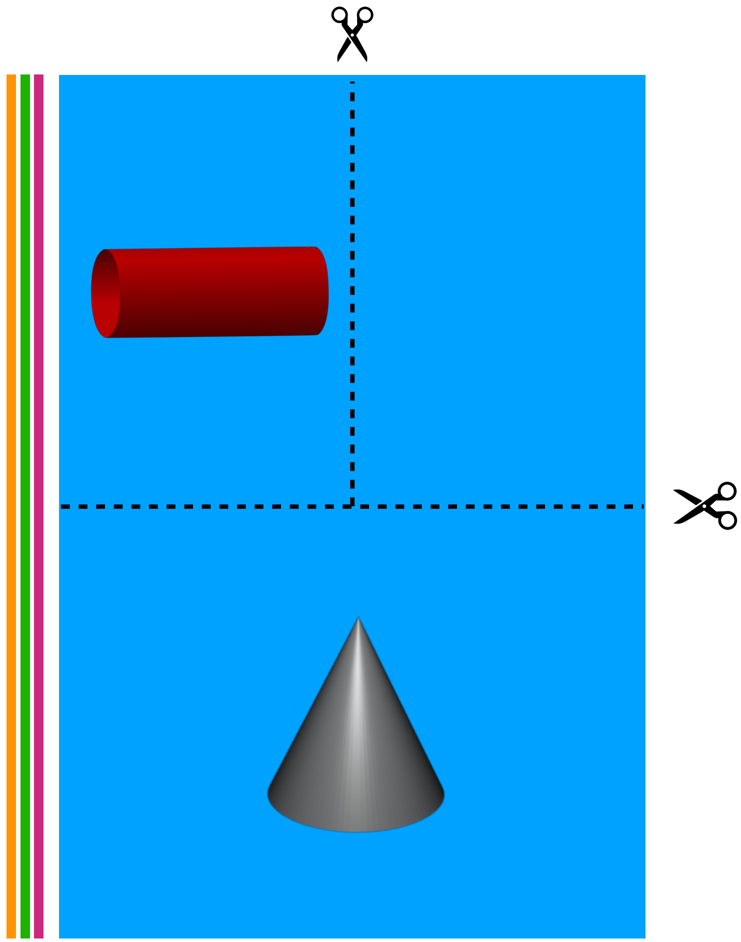
Construct a cylinder, a cone, and some geodesics

1) Cut three thin ribbons from the long side of your A4 sheet.

2) Fold and cut the remainder of your A4 sheet into three parts

3) Make one of the small rectangles into a cylinder, and make the large rectangle into a cone.

4) Color the ribbons different colors using crayons or sketch pens.



Definition of Geodesic

Let’s refine our definition slightly:

From now on, a ***geodesic*** is a <u>maximal</u> path that is intrinsically straight.

A ***geodesic segment*** is a finite portion of a geodesic.

What do geodesics look like on the cylinder?

Use your geodesic ribbons and see what types of geodesics you can create!

What do geodesics look like on the cylinder?

Use your geodesic ribbons and see what types of geodesics you can create!

What do geodesics look like on the cone?

Can anyone share a geodesic they've launched?

Properties of Geodesics

Here are some familiar properties of geodesics in the plane (a.k.a. *lines*).

Properties of Geodesics

Here are some familiar properties of geodesics in the plane (a.k.a. *lines*).

(i) There is at least one geodesic through any two points.

Properties of Geodesics

Here are some familiar properties of geodesics in the plane (a.k.a. *lines*).

(i) There is at least one geodesic through any two points.

(ii) There is a unique geodesic through any two points.

Properties of Geodesics

Here are some familiar properties of geodesics in the plane (a.k.a. *lines*).

(i) There is at least one geodesic through any two points.

(ii) There is a unique geodesic through any two points.

(iii) Two geodesics meet in at most one point.

Properties of Geodesics

Here are some familiar properties of geodesics in the plane (a.k.a. *lines*).

(i) There is at least one geodesic through any two points.

(ii) There is a unique geodesic through any two points.

(iii) Two geodesics meet in at most one point.

(iv) There exist geodesics which do not meet.

Properties of Geodesics

Here are some familiar properties of geodesics in the plane (a.k.a. *lines*).

(i) There is at least one geodesic through any two points.

(ii) There is a unique geodesic through any two points.

(iii) Two geodesics meet in at most one point.

(iv) There exist geodesics which do not meet.

(v) All geodesics have infinite length.

Properties of Geodesics

Here are some familiar properties of geodesics in the plane (a.k.a. *lines*).

(i) There is at least one geodesic through any two points.

(ii) There is a unique geodesic through any two points.

(iii) Two geodesics meet in at most one point.

(iv) There exist geodesics which do not meet.

(v) All geodesics have infinite length.

(vi) A geodesic segment always gives the shortest distance between two points.

Properties of Geodesics

Here are some familiar properties of geodesics in the plane (a.k.a. *lines*).

(i) There is at least one geodesic through any two points.

(ii) There is a unique geodesic through any two points.

(iii) Two geodesics meet in at most one point.

(iv) There exist geodesics which do not meet.

(v) All geodesics have infinite length.

(vi) A geodesic segment always gives the shortest distance between two points.

(vii) The shortest distance between two points is always a geodesic segment.

Properties of Geodesics

Here are some familiar properties of geodesics in the plane (a.k.a. *lines*).

(i) There is at least one geodesic through any two points.

(ii) There is a unique geodesic through any two points.

(iii) Two geodesics meet in at most one point.

(iv) There exist geodesics which do not meet.

(v) All geodesics have infinite length.

(vi) A geodesic segment always gives the shortest distance between two points.

(vii) The shortest distance between two points is always a geodesic segment.

(viii) A geodesic never crosses itself.

Properties of Geodesics

Do these properties hold for cylinders and cones?

(i) There is at least one geodesic through any two points.

(ii) There is a unique geodesic through any two points.

(iii) Two geodesics meet in at most one point.

(iv) There exist geodesics which do not meet.

(v) All geodesics have infinite length.

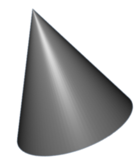
(vi) A geodesic segment always gives the shortest distance between two points.

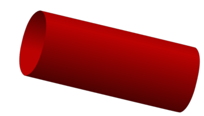
(vii) The shortest distance between two points is always a geodesic segment.

(viii) A geodesic never crosses itself.

Properties of Geodesics

Do these properties hold for cylinders and cones?





(i) There is at least one geodesic through any two points.

(ii) There is a unique geodesic through any two points.

(iii) Two geodesics meet in at most one point.

(iv) There exist geodesics which do not meet.

(v) All geodesics have infinite length.

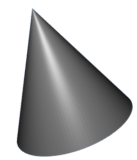
(vi) A geodesic segment always gives the shortest distance between two points.

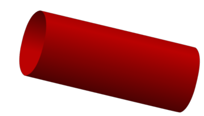
(vii) The shortest distance between two points is always a geodesic segment.

(viii) A geodesic never crosses itself.

Properties of Geodesics

Do these properties hold for cylinders and cones?





(i) There is at least one geodesic through any two points.

(ii) There is a unique geodesic through any two points.

(iii) Two geodesics meet in at most one point.

(iv) There exist geodesics which do not meet.

(v) All geodesics have infinite length.

(vi) A geodesic segment always gives the shortest distance between two points.

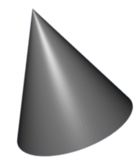
(vii) The shortest distance between two points is always a geodesic segment.

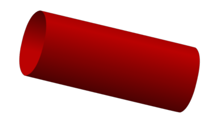
(viii) A geodesic never crosses itself.

Which properties no longer hold?
Any guesses?

Properties of Geodesics

Let's focus on these three properties.





(i) There is at least one geodesic through any two points.

(ii) There is a unique geodesic through any two points.

(iii) Two geodesics meet in at most one point.

(iv) There exist geodesics which do not meet.

(v) All geodesics have infinite length.

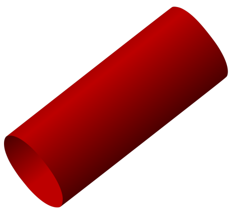
(vi) A geodesic segment always gives the shortest distance between two points.

(vii) The shortest distance between two points is always a geodesic segment.

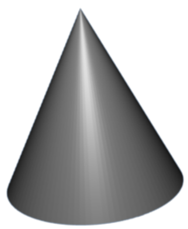
(viii) A geodesic never crosses itself.

Can you find <u>counterexamples?</u>

Activity: Constructing Counterexamples



Infinite Cylinder



Infinite Cone

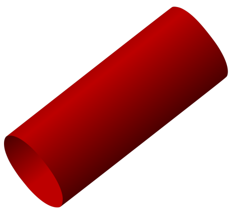
On *each* surface, can you construct:

— two points with multiple geodesics between them?

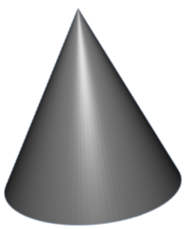
— a finite length geodesic?

— a geodesic with self-crossings?

Activity: Constructing Counterexamples



Infinite Cylinder



Infinite Cone

On *each* surface, can you construct:

— two points with multiple geodesics between them?

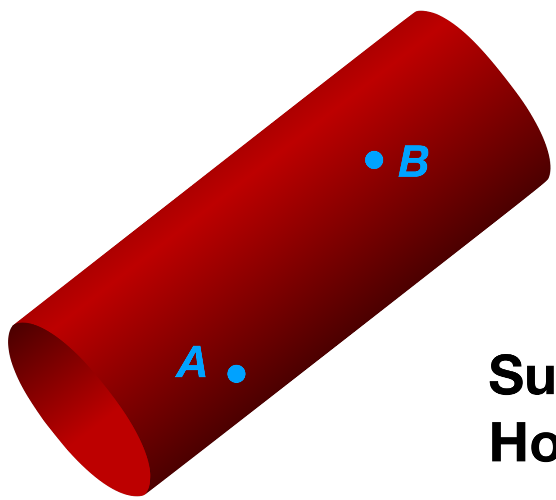
— a finite length geodesic?

— a geodesic with self-crossings?

When you have constructed an example, come to the stage and share!
If you don’t believe an example exists, explain why.

A question

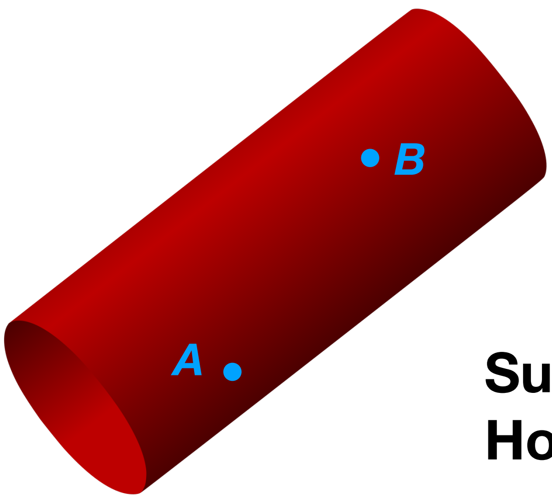
We've seen that on some surfaces, there can be more than one geodesic between two points…



Suppose *A* and *B* are two points on a cylinder. How many geodesics are there between *A* and *B*?

A question

We've seen that on some surfaces, there can be more than one geodesic between two points…

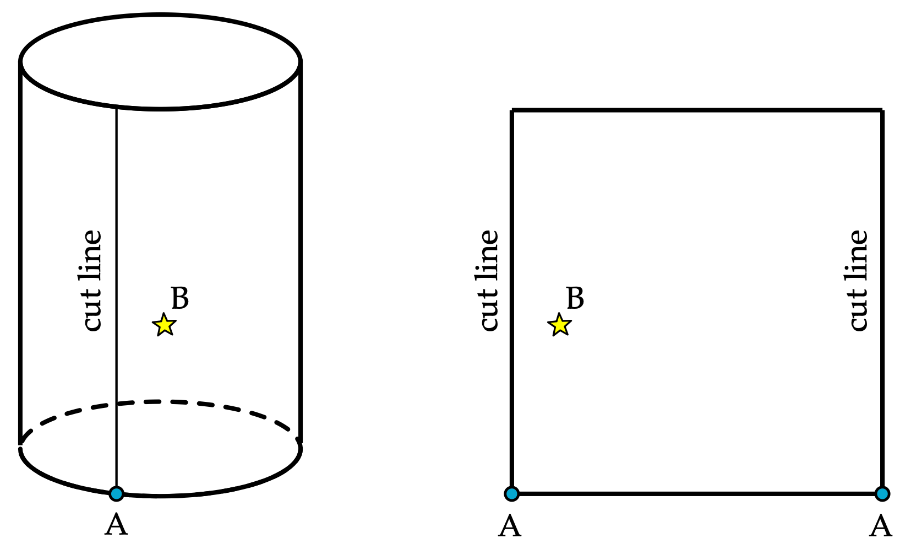


Suppose *A* and *B* are two points on a cylinder.
How many geodesics are there between *A* and *B*?

Let's try *drawing* all the geodesics between *A* and *B*.

An *n-sheeted covering* of a cylinder

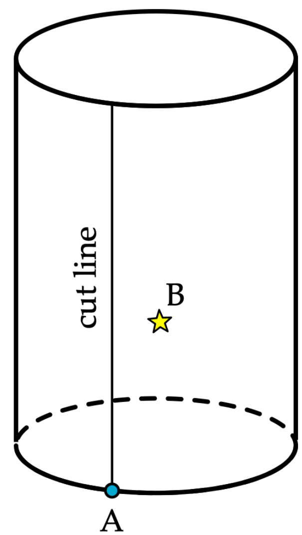
The simplest covering is a 1-sheeted covering, which we get by cutting the cylinder along any vertical *cut-line*, and then unrolling it.

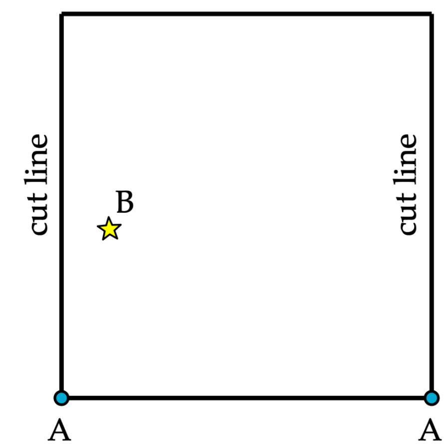


An *n-sheeted covering* of a cylinder

The simplest covering is a 1-sheeted covering, which we get by cutting the cylinder along any vertical *cut-line*, and then unrolling it.

Construct a cylinder as shown, with point A lying on the cut line and point B lying off of the cut line.



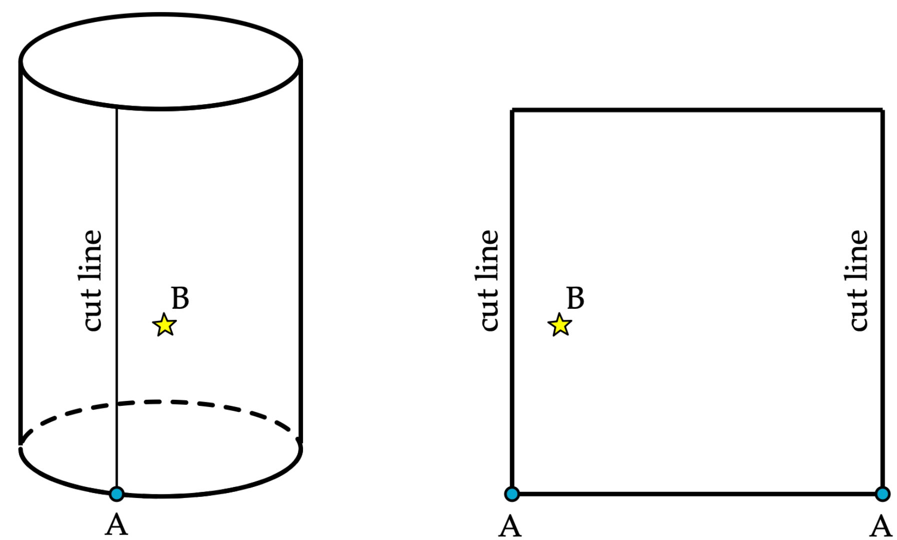


An *n-sheeted covering* of a cylinder

The simplest covering is a 1-sheeted covering, which we get by cutting the cylinder along any vertical *cut-line*, and then unrolling it.

Construct a cylinder as shown, with point A lying on the cut line and point B lying off of the cut line.

Draw two distinct geodesics through A and B on the 1-sheeted covering.

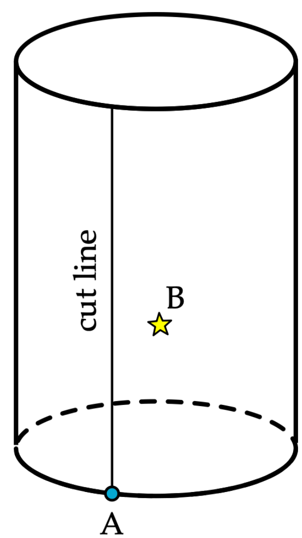


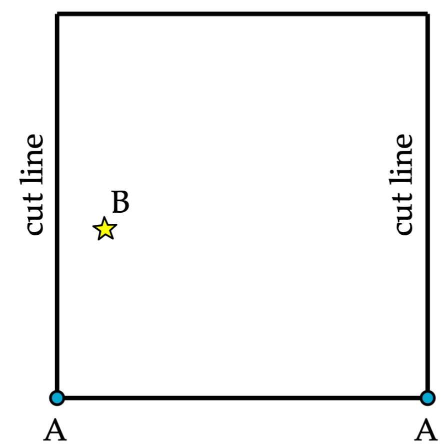
An *n-sheeted covering* of a cylinder

The simplest covering is a 1-sheeted covering, which we get by cutting the cylinder along any vertical *cut-line*, and then unrolling it.

Construct a cylinder as shown, with point A lying on the cut line and point B lying off of the cut line.

Draw two distinct geodesics through A and B on the 1-sheeted covering.

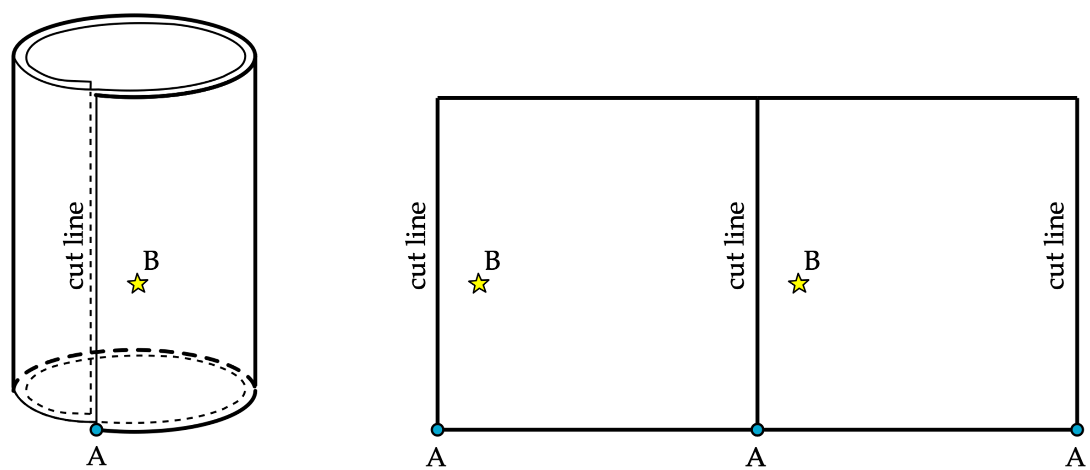




Use the ribbon test to verify these are truly geodesics on the rolled-up cylinder.

An *n-sheeted covering* of a cylinder

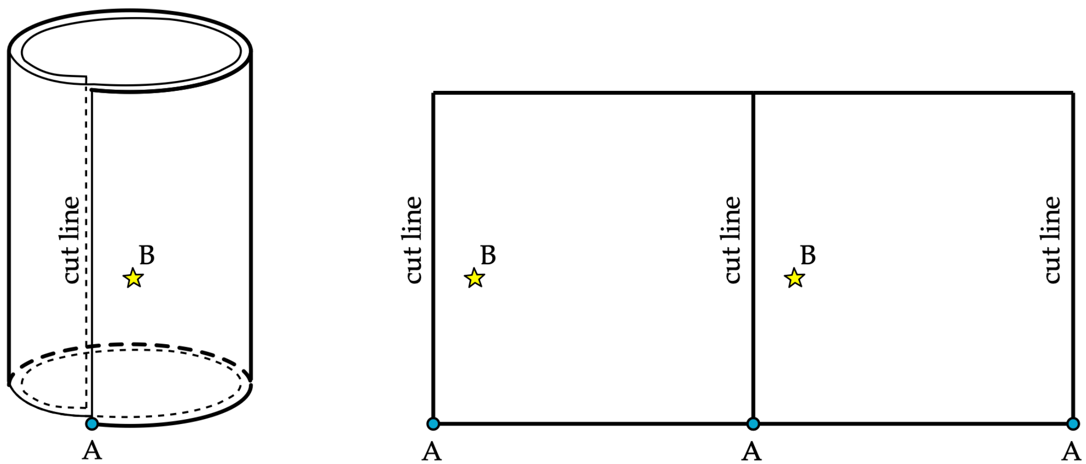
Construct a 2-sheeted covering of the same cylinder, as shown. Mark points A and B as precisely as possible on both sheets.



How many geodesics does the 2-sheeted cover allow you to draw? Draw them, and verify they are indeed geodesics on the rolled up cylinder.

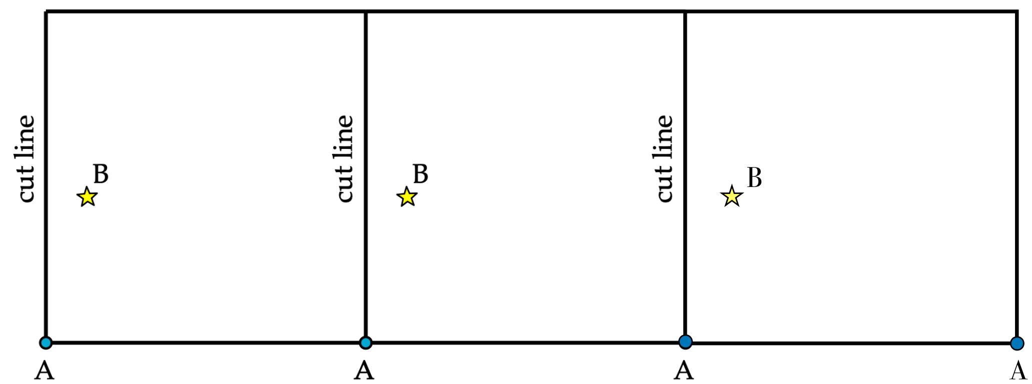
An *n-sheeted covering* of a cylinder

Construct a 2-sheeted covering of the same cylinder, as shown. Mark points A and B as precisely as possible on both sheets.



How many geodesics does the 2-sheeted cover allow you to draw? Draw them, and verify they are indeed geodesics on the rolled up cylinder. **A special prize for the first group that does this correctly!**

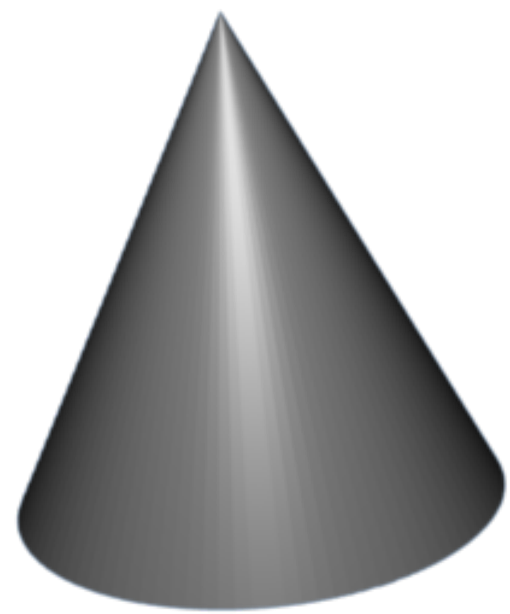
An *n-sheeted covering* of a cylinder



How many geodesics would a 3-sheeted cover allow you to draw?

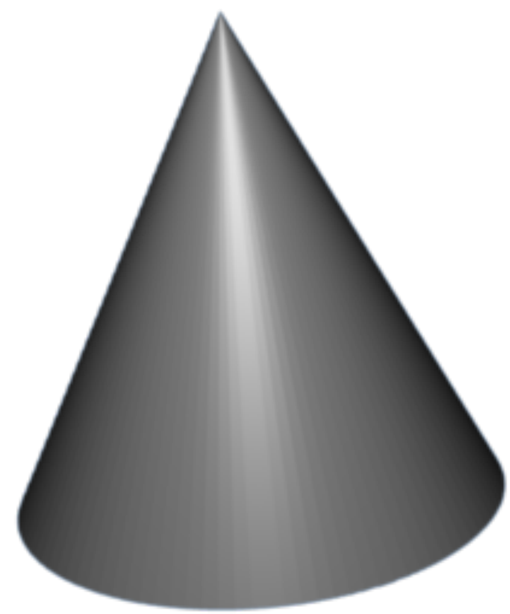
What is the total number of geodesics between A and B?

Let’s return to the infinite cone



Let's return to the infinite cone

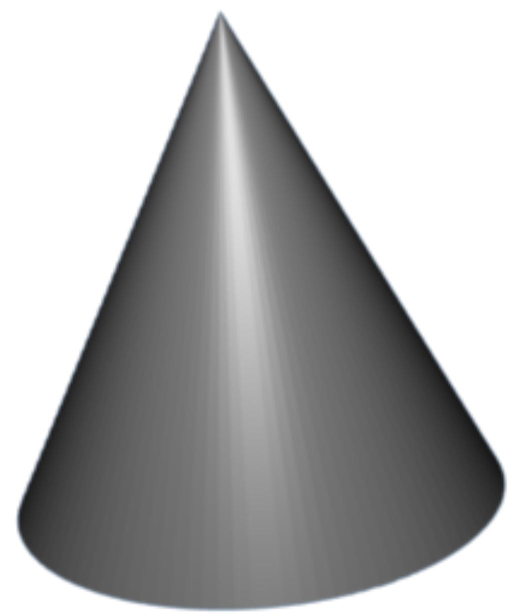
What geodesics have we seen on the cone so far?



Let’s return to the infinite cone

What geodesics have we seen on the cone so far?

Remember that a geodesic is a *maximal* intrinsically straight path.

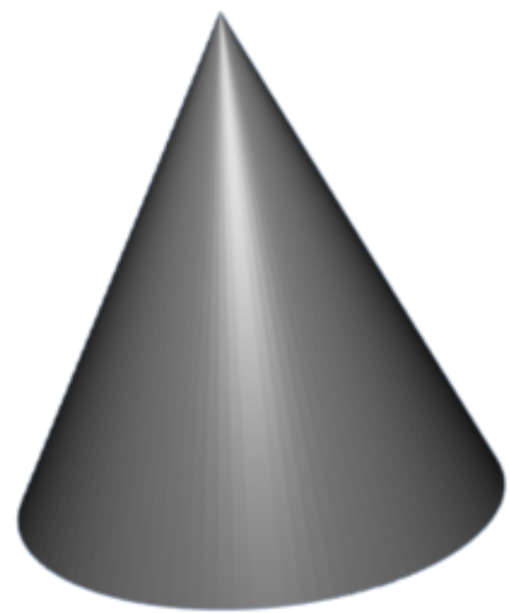


Let’s return to the infinite cone

What geodesics have we seen on the cone so far?

Remember that a geodesic is a *maximal* intrinsically straight path.

What happens to a geodesic when it hits the vertex?

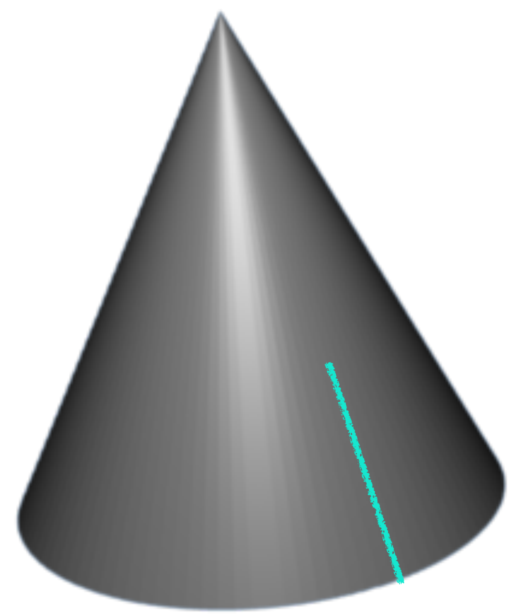


Let’s return to the infinite cone

What geodesics have we seen on the cone so far?

Remember that a geodesic is a *maximal* intrinsically straight path.

What happens to a geodesic when it hits the vertex?

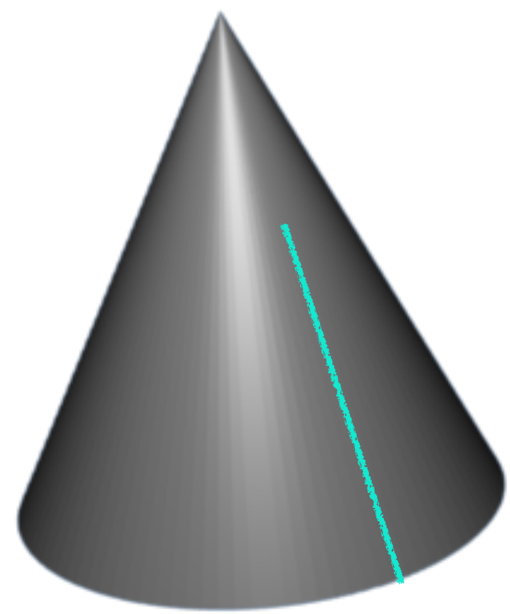


Let’s return to the infinite cone

What geodesics have we seen on the cone so far?

Remember that a geodesic is a *maximal* intrinsically straight path.

What happens to a geodesic when it hits the vertex?

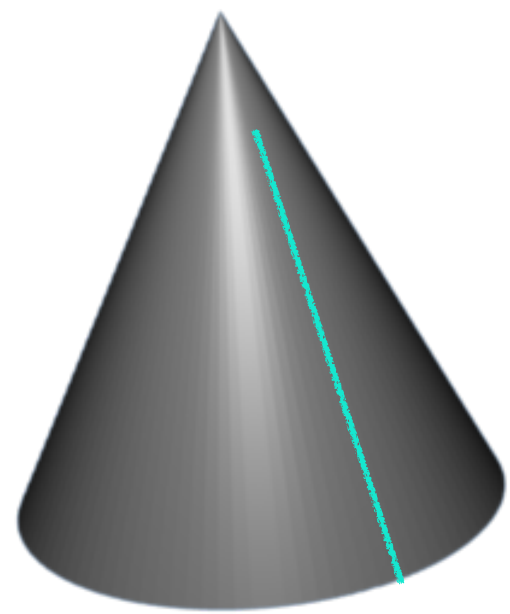


Let’s return to the infinite cone

What geodesics have we seen on the cone so far?

Remember that a geodesic is a *maximal* intrinsically straight path.

What happens to a geodesic when it hits the vertex?

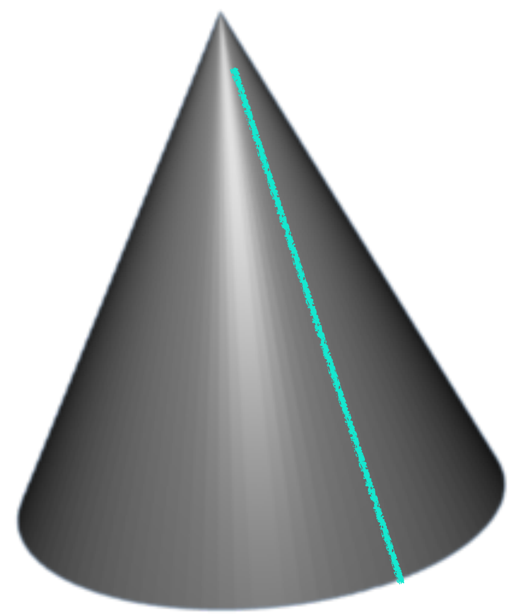


Let’s return to the infinite cone

What geodesics have we seen on the cone so far?

Remember that a geodesic is a *maximal* intrinsically straight path.

What happens to a geodesic when it hits the vertex?

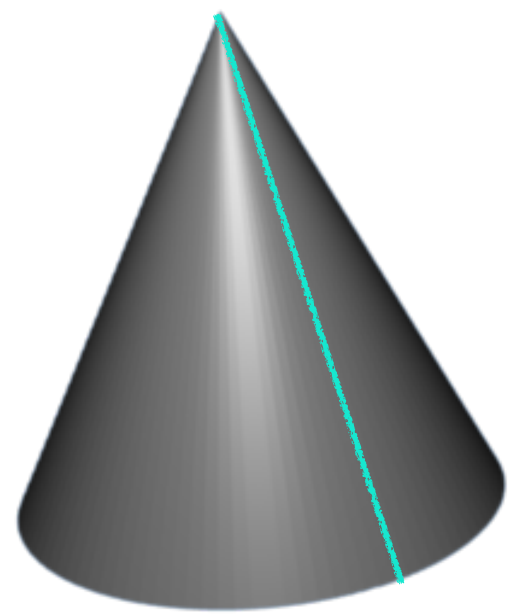


Let's return to the infinite cone

What geodesics have we seen on the cone so far?

Remember that a geodesic is a *maximal* intrinsically straight path.

What happens to a geodesic when it hits the vertex?



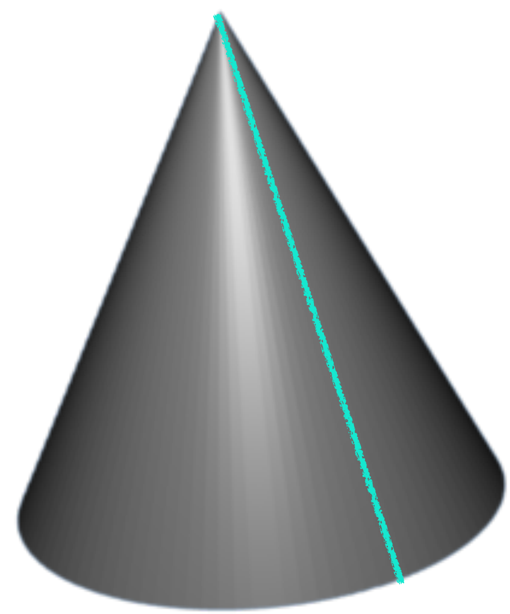
Let’s return to the infinite cone

What geodesics have we seen on the cone so far?

Remember that a geodesic is a *maximal* intrinsically straight path.

What happens to a geodesic when it hits the vertex?

Can a geodesic continue past the vertex?

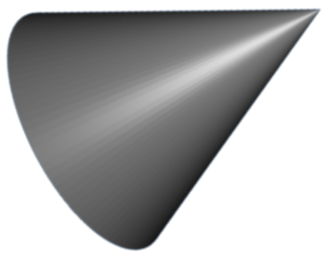


One last question before we break:

One last question before we break:

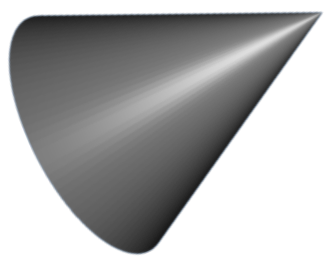
How many geodesics are there between two points on a cone?

How many geodesics are there between two points on a cone?



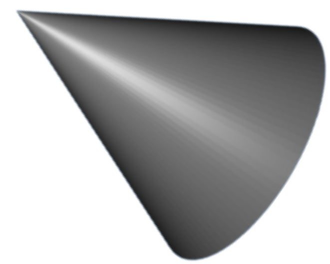
Are there infinitely many?

How many geodesics are there between two points on a cone?

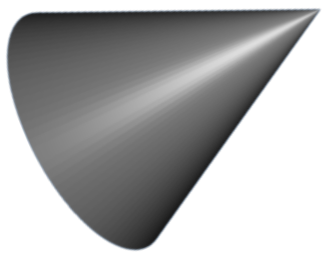


Are there infinitely many?

Are there finitely many?

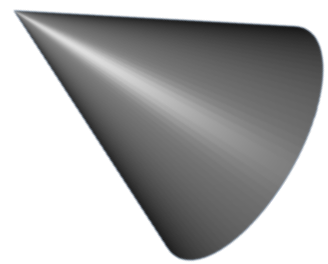


How many geodesics are there between two points on a cone?



Are there infinitely many?

Are there finitely many?



We'll explore this question tomorrow!

Geodesics on Surfaces

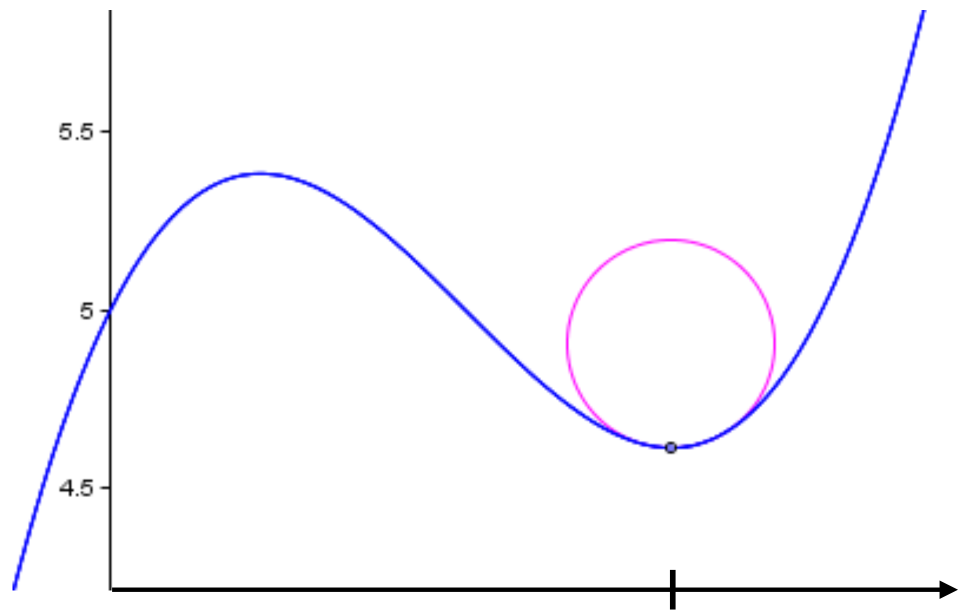
Day Two

May 22, 2025
FACETS @ IMSc Chennai

Vijay Ravikumar
Azim Premji University, Bengaluru

Prologue

the curvature of a curve



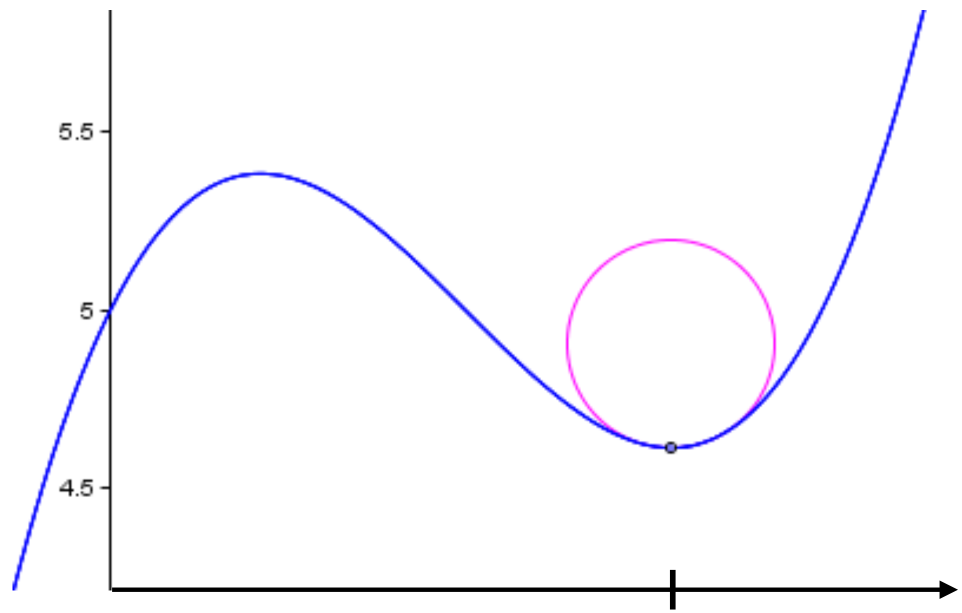
We'll start today with a glimpse of a seemingly unrelated topic: *curvature*

Prologue

the curvature of a curve

Some of you may have seen curvature in *multivariable calculus* or *differential geometry*.

The curvature of a 1-D curve is defined as follows:

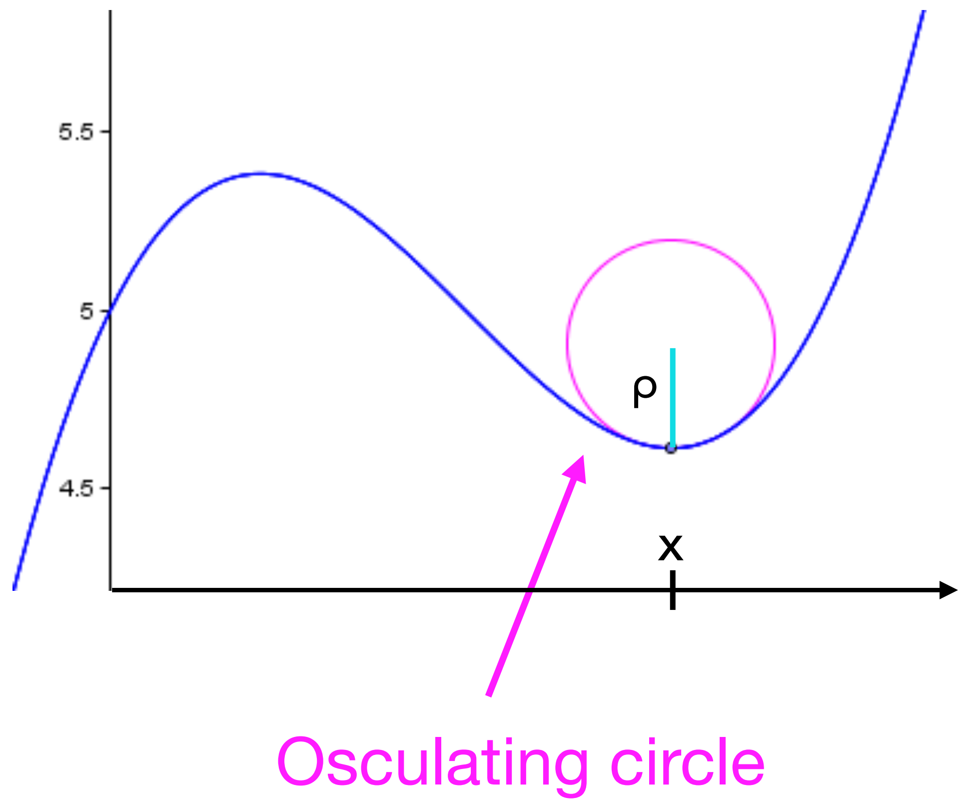


Prologue

the curvature of a curve

Some of you may have seen curvature in *multivariable calculus* or *differential geometry*.

The curvature of a 1-D curve is defined as follows:

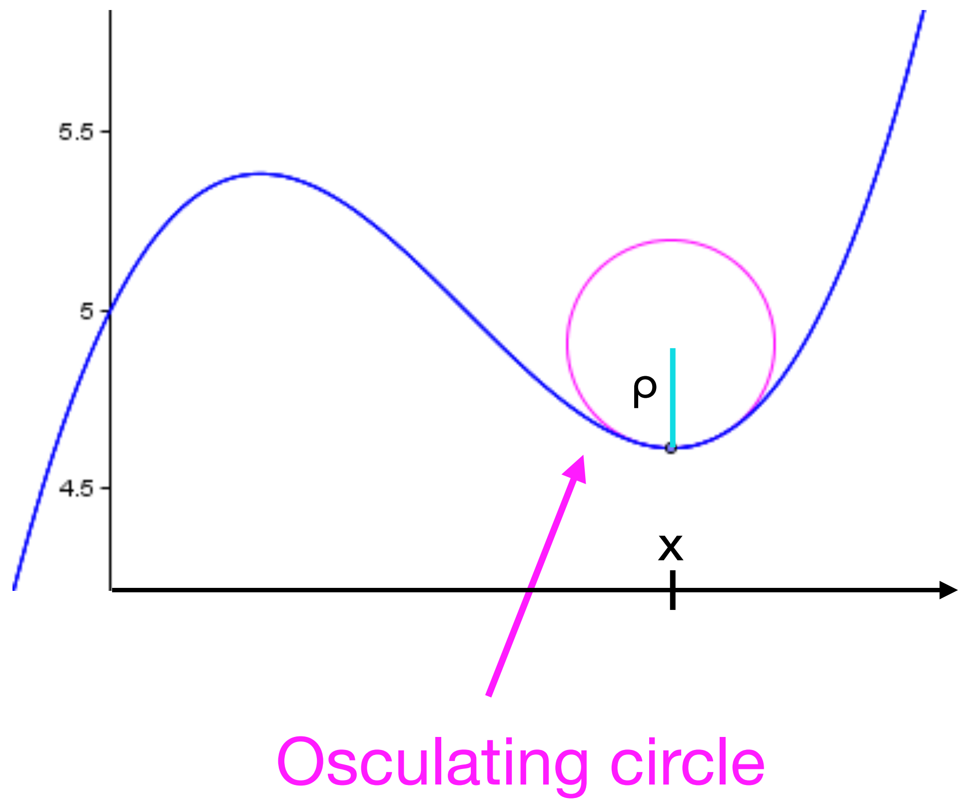


Prologue

the curvature of a curve

Some of you may have seen curvature in *multivariable calculus* or *differential geometry*.

The curvature of a 1-D curve is defined as follows:



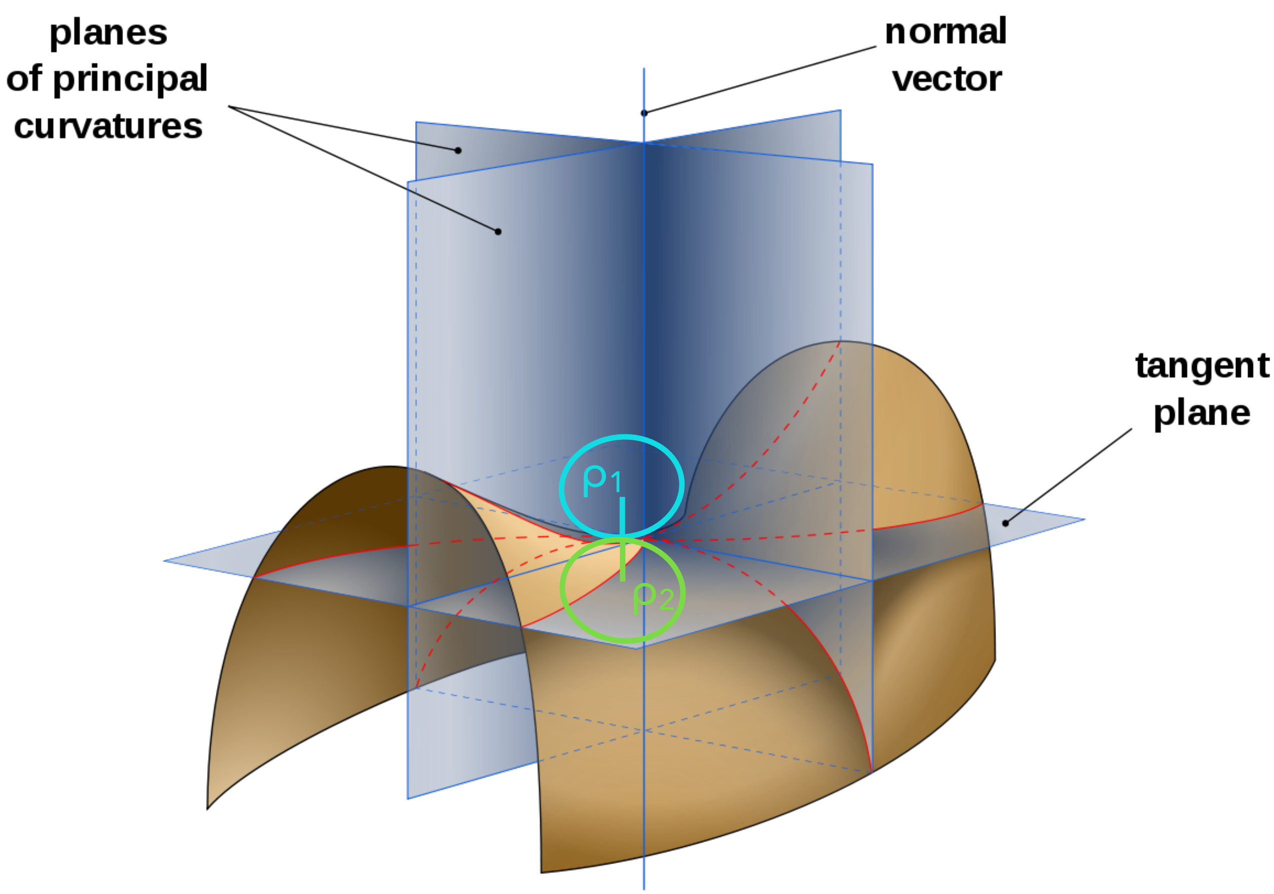
Curvature at x = $\kappa(x) = 1/\rho$

Prologue

the curvature of a surface

The curvature of a 2-D surface is defined via the (signed) curvature of the most extreme cross sectional curves:

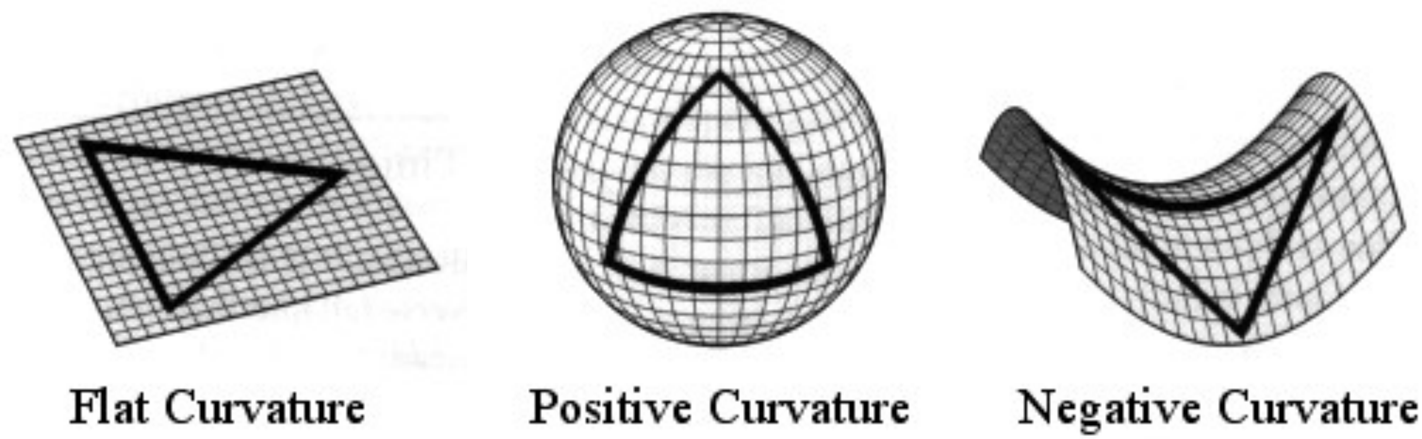
$$\kappa = \frac{1}{\rho_1 \rho_2}.$$



Prologue

the curvature of a surface

You may have seen other characterizations of *constant* curvature surfaces:



angle sum = 180° | **angle sum > 180°** | **angle sum < 180°**

Prologue

the curvature of a surface

How do geodesics and curvature interact?

Prologue

the curvature of a surface

How do geodesics and curvature interact?

Recall

A *geodesic* is a maximal path that is intrinsically straight.

Prologue

the curvature of a surface

How do geodesics and curvature interact?

Recall

A *geodesic* is a maximal path that is intrinsically straight.

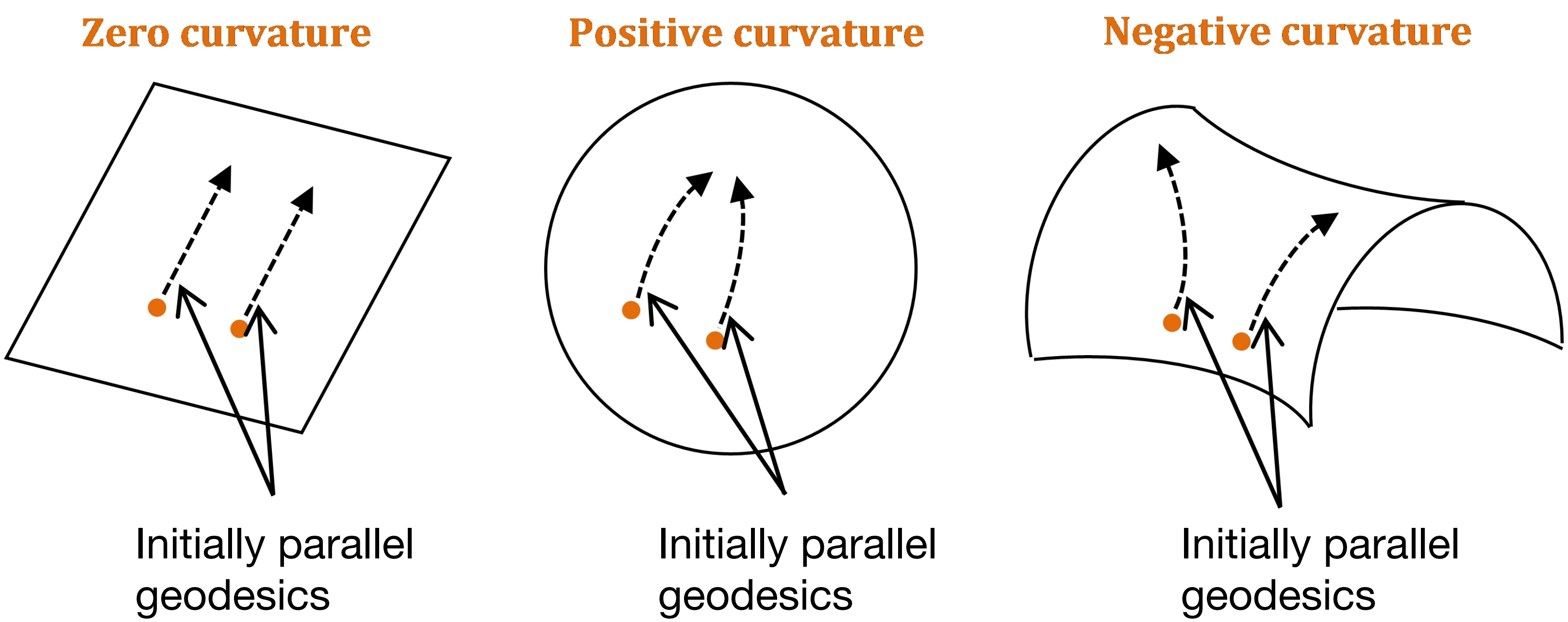
Aside: we can define this precisely using the idea of the *osculating plane,* also called a *plane of curvature*.

If a path is intrinsically straight, its osculating plane will always be perpendicular to its tangent plane. (Equivalently, its unit normal vector will always be normal to the surface.)

Prologue

the curvature of a surface

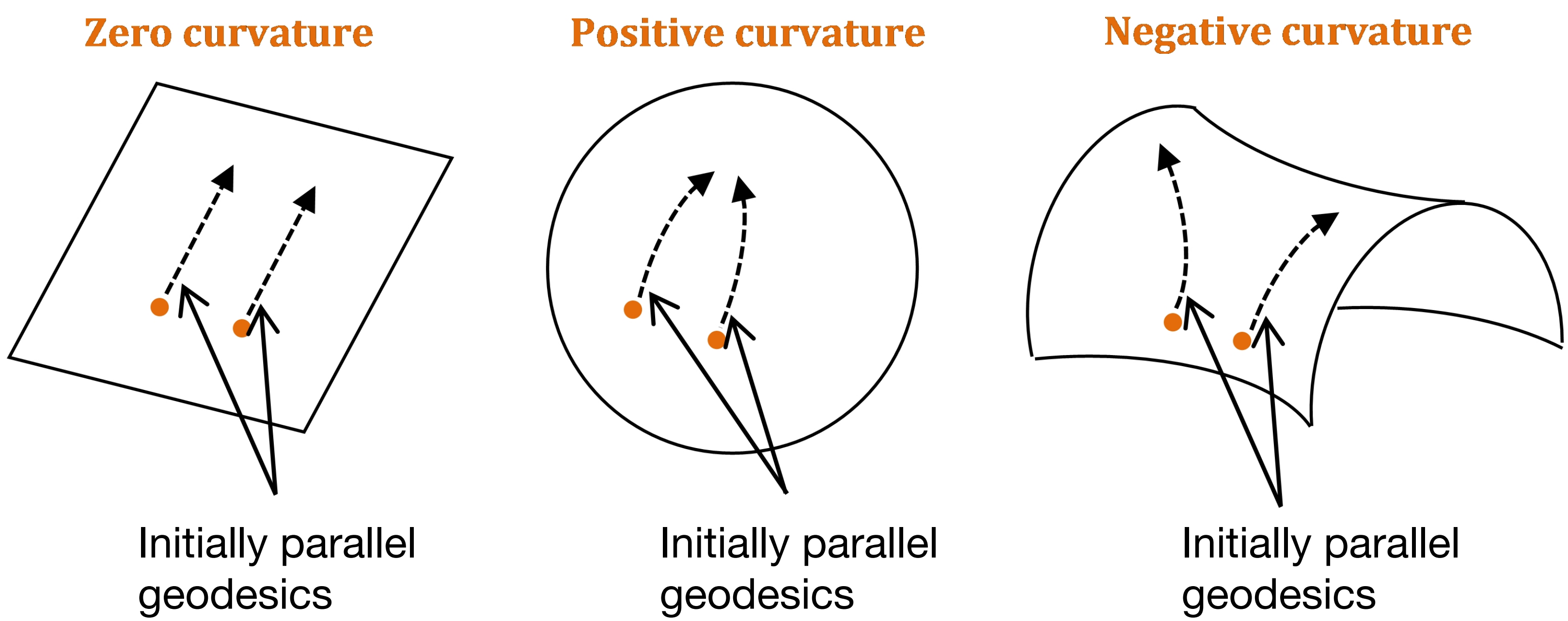
You may have seen this characterization of *constant* curvature surfaces as well:



Prologue

the curvature of a surface

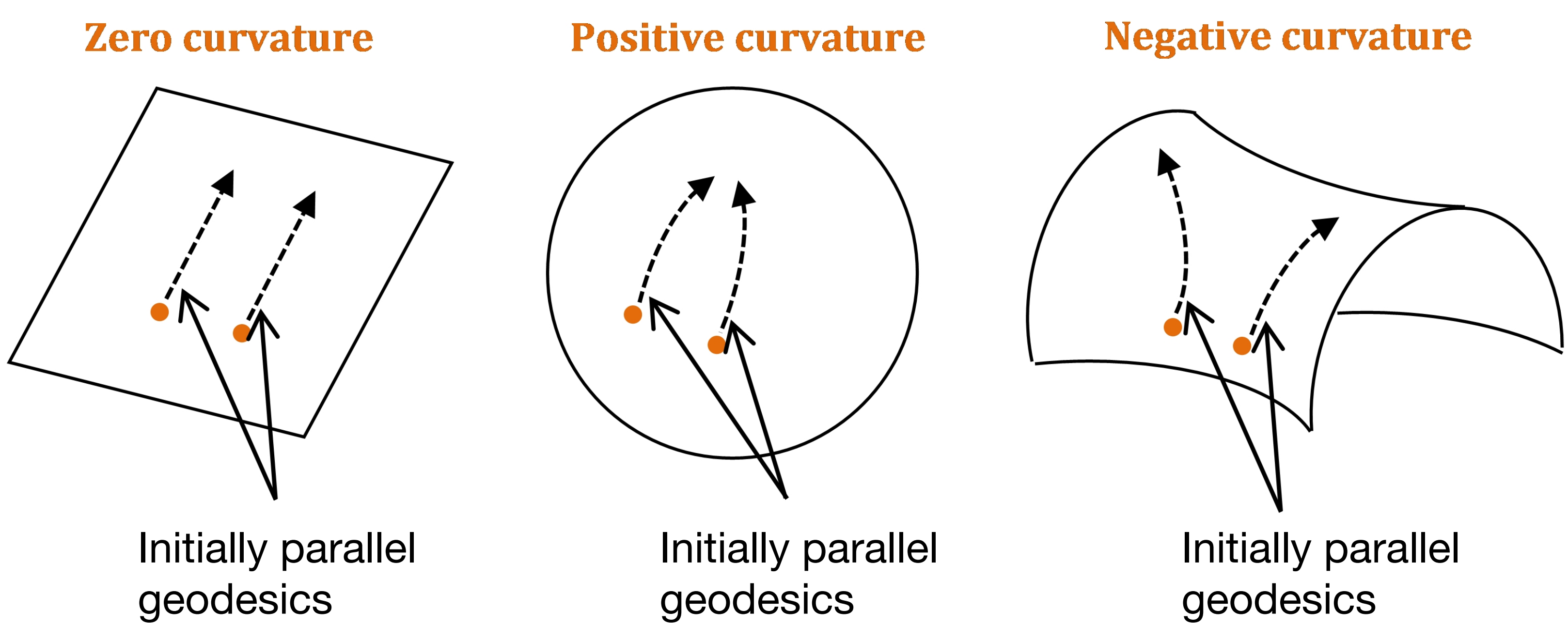
Under nonzero curvature, initially parallel geodesics do not remain parallel



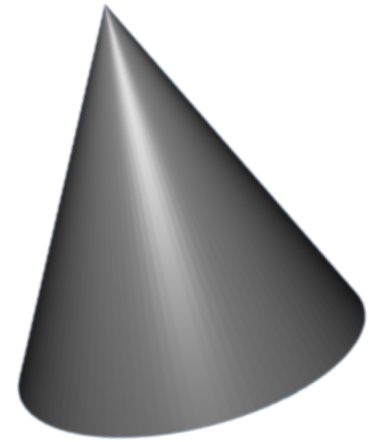
Prologue

the curvature of a surface

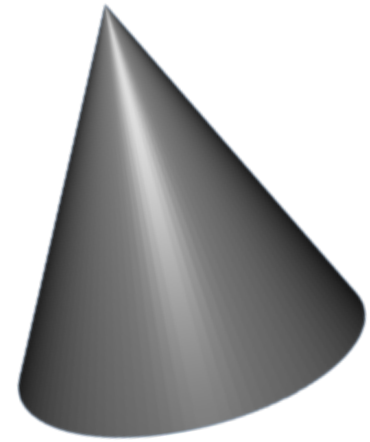
In this way, geodesics can help us explore curvature.



How flat is the infinite cone?

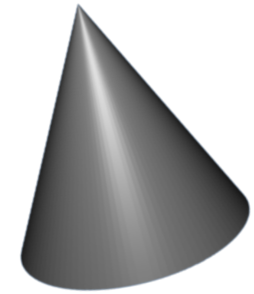


How flat is the infinite cone?

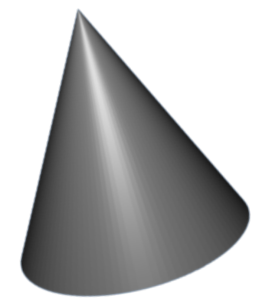


Do initially parallel geodesics remain parallel on the infinite cone?

Activity: Do initially parallel geodesics remain parallel on the infinite cone?



Activity: Do initially parallel geodesics remain parallel on the infinite cone?

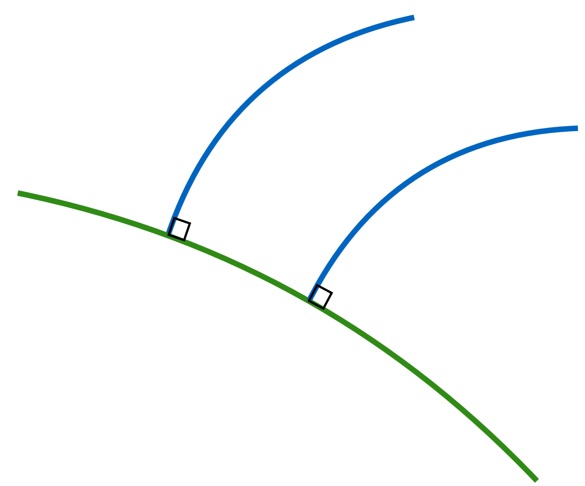


How do we make sure the two geodesics start out parallel?

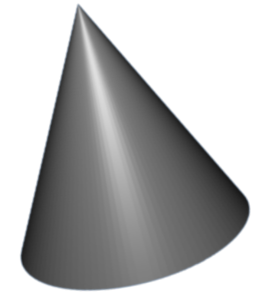
We'll need an additional geodesic to act as a *base*.

We can then launch the two geodesics *at right angles* from this base.

So you'll need *three* geodesic ribbons in total.



Activity: Do initially parallel geodesics remain parallel on the infinite cone?

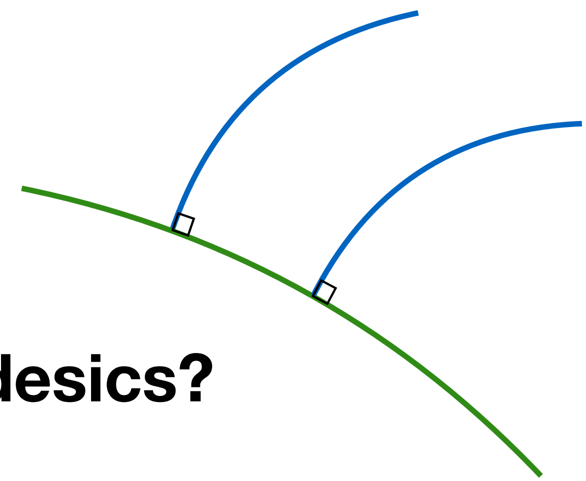


How do we make sure the two geodesics start out parallel?

We'll need an additional geodesic to act as a *base*.

We can then launch the two geodesics *at right angles* from this base.

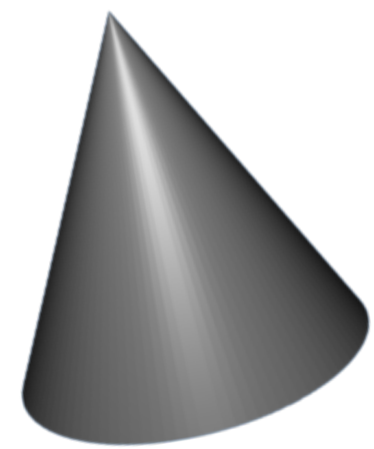
So you'll need *three* geodesic ribbons in total.



What happens to the initially parallel geodesics?

Yesterday we asked:

How many geodesics are there between two points on a cone?

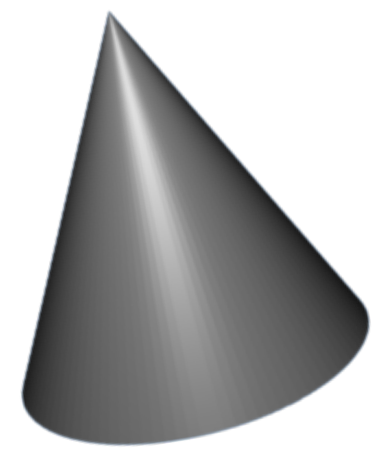


Yesterday we asked:

How many geodesics are there between two points on a cone?

Are there infinitely many?

Are there finitely many?



Yesterday we asked:

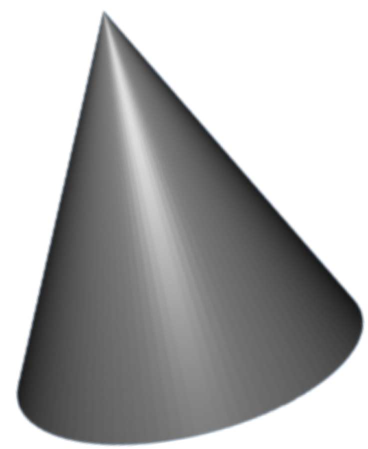
How many geodesics are there between two points on a cone?

Are there infinitely many?

Are there finitely many?

We also considered the question:

How many self-intersections will a geodesic have?



Yesterday we asked:

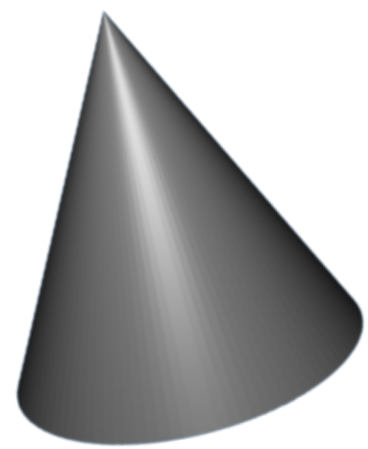
How many geodesics are there between two points on a cone?

Are there infinitely many?

Are there finitely many?

We also considered the question:

How many self-intersections will a geodesic have?



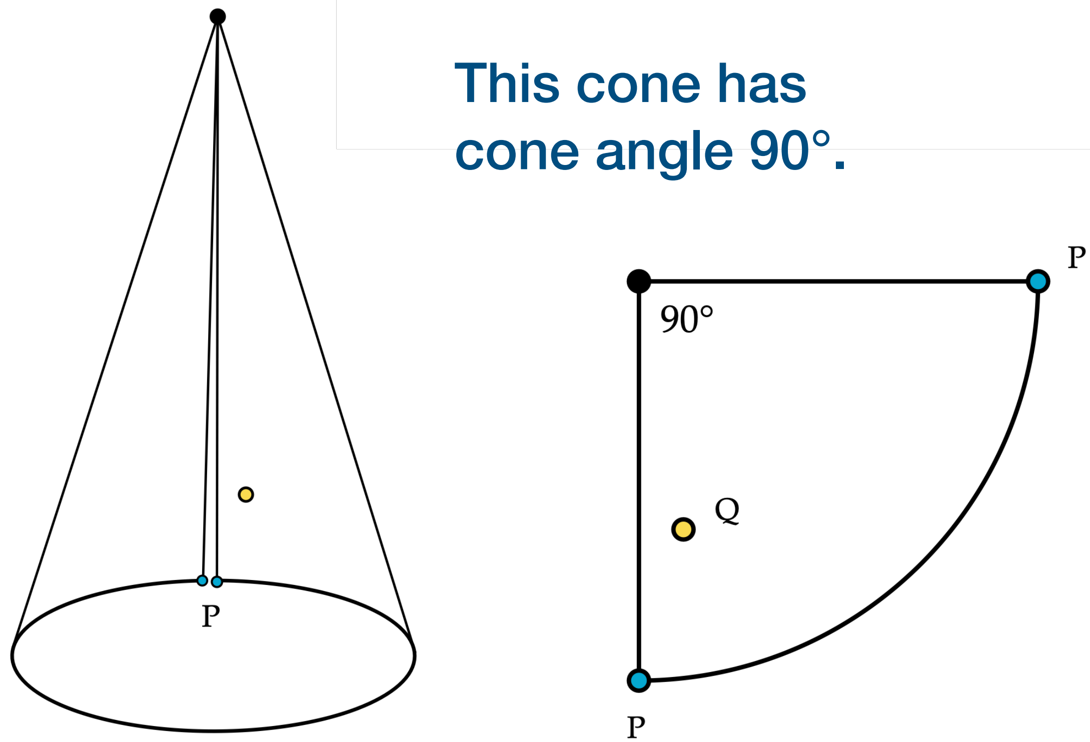
An n-sheeted covering will help us answer these questions!

An *n-sheeted covering* of a cone

We get a 1-sheeted covering by cutting the cone along any vertical *cut-line*, and then unrolling it.

Unlike a cylinder, the properties of a cone depend on its *cone angle!*

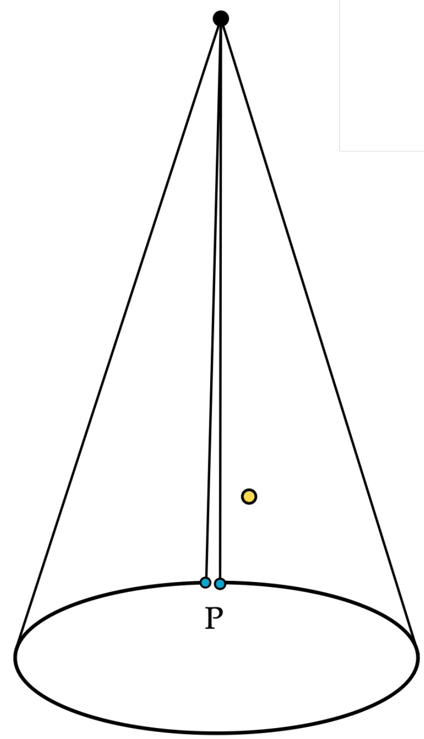
This cone has cone angle 90°.



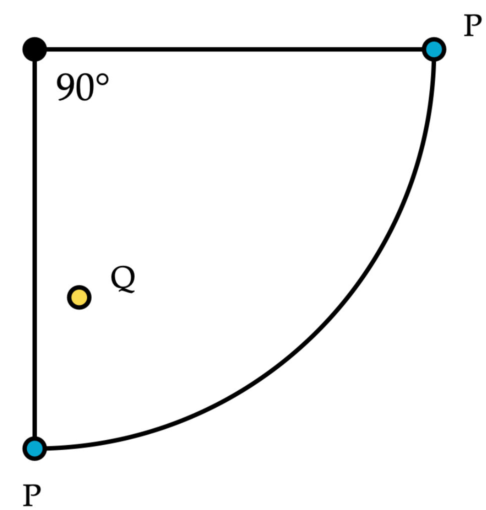
An *n-sheeted covering* of a cone

Exercise 1: Consider the points P and Q shown on this 90° cone.

Use a one-sheeted covering to find two distinct geodesics through P and Q.



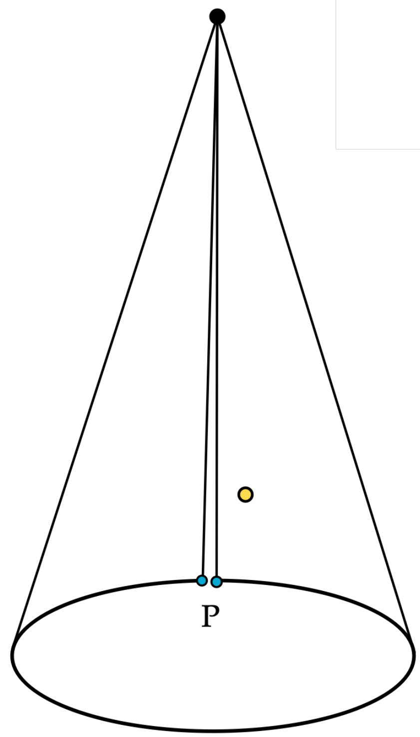
This cone has cone angle 90°.



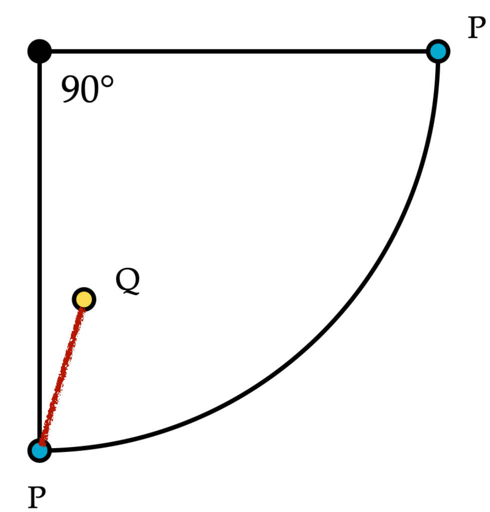
An *n-sheeted covering* of a cone

Exercise 1: Consider the points P and Q shown on this 90° cone.

Use a one-sheeted covering to find two distinct geodesics through P and Q.



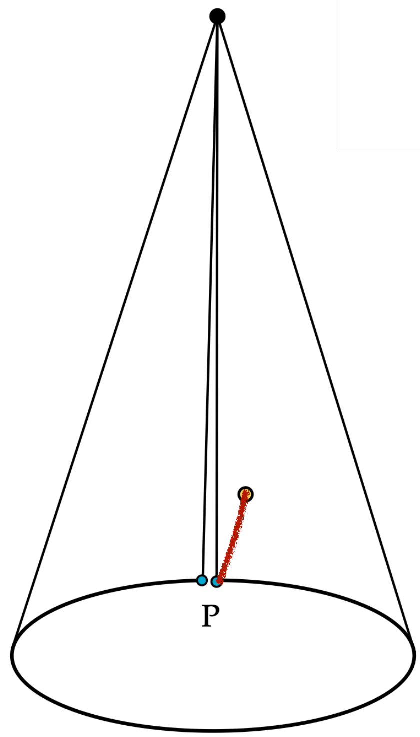
This cone has cone angle 90°.



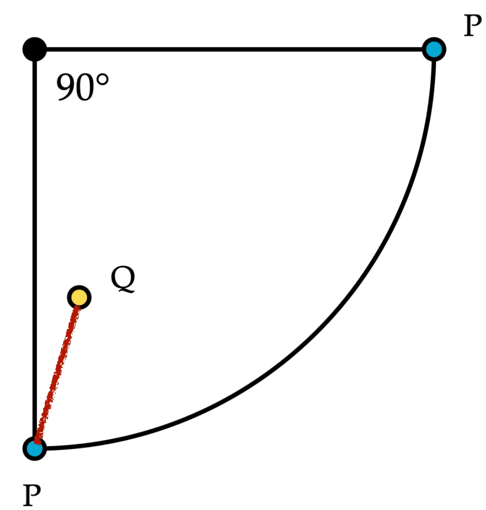
An *n-sheeted covering* of a cone

Exercise 1: Consider the points P and Q shown on this 90° cone.

Use a one-sheeted covering to find two distinct geodesics through P and Q.



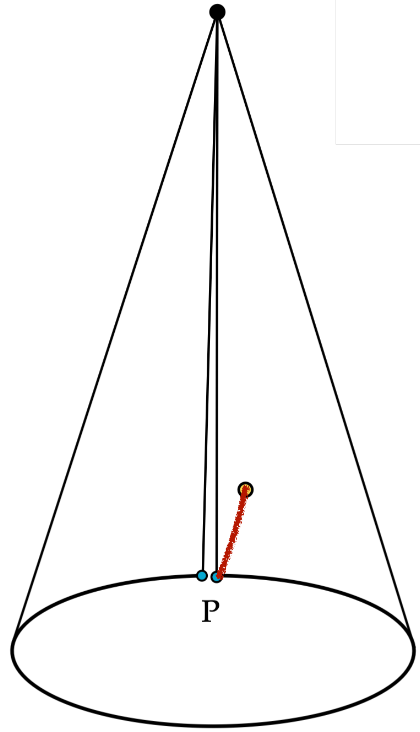
This cone has cone angle 90°.



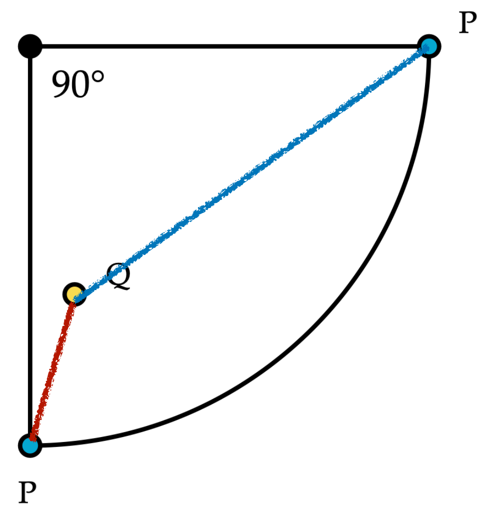
An *n-sheeted covering* of a cone

Exercise 1: Consider the points P and Q shown on this 90° cone.

Use a one-sheeted covering to find two distinct geodesics through P and Q.



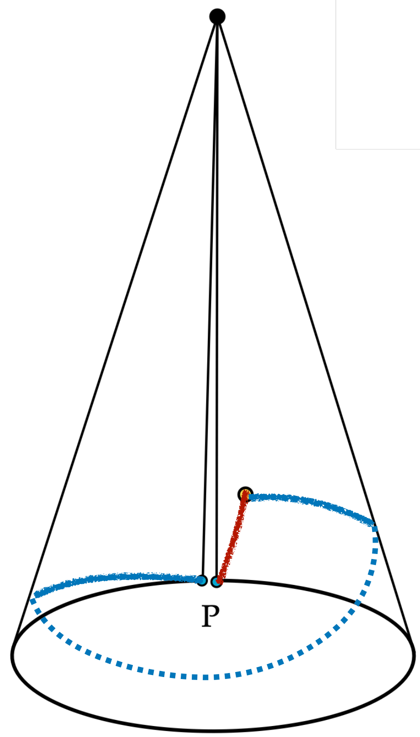
This cone has cone angle 90°.



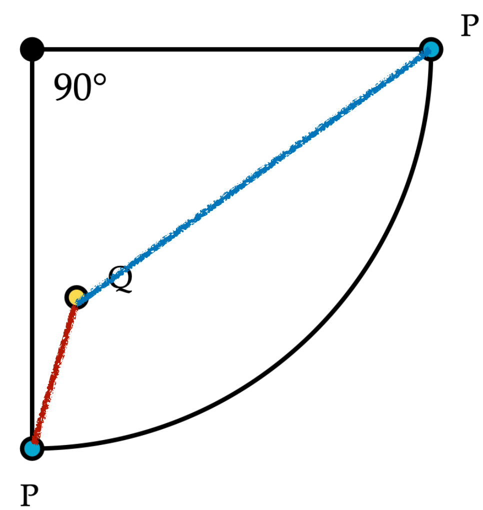
An *n-sheeted covering* of a cone

Exercise 1: Consider the points P and Q shown on this 90° cone.

Use a one-sheeted covering to find two distinct geodesics through P and Q.

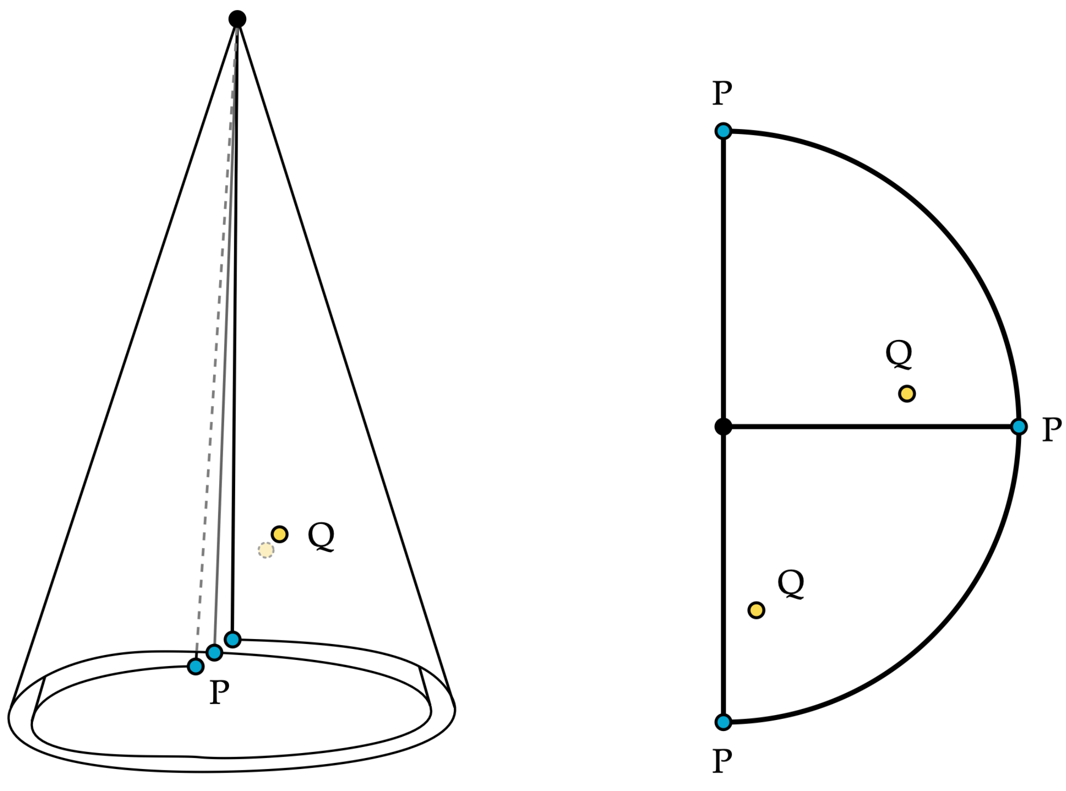


This cone has cone angle 90°.



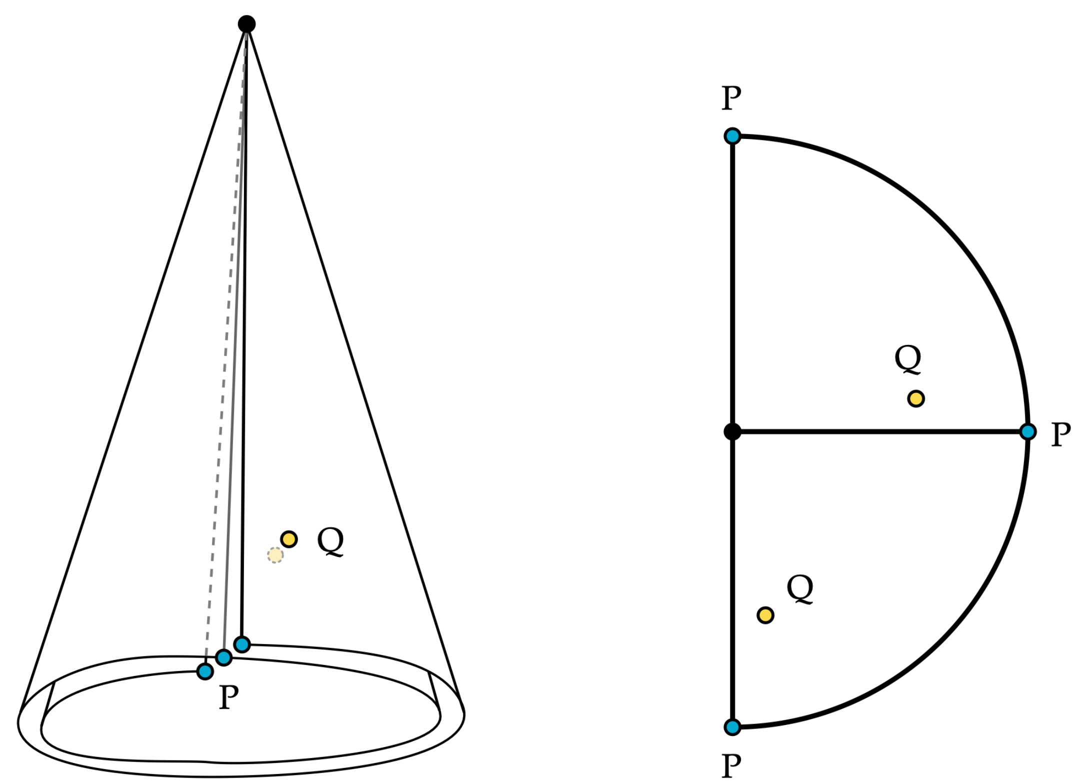
An *n-sheeted covering* of a cone

But, we can add more sheets! Here is a 2-sheeted covering of the same cone:



An *n-sheeted covering* of a cone

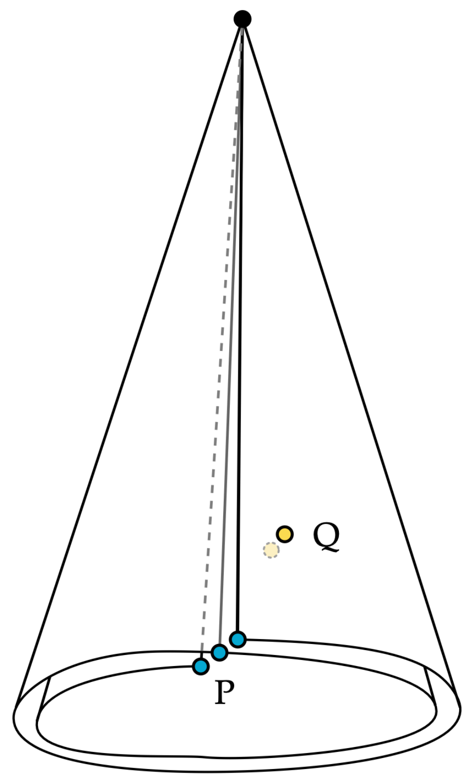
Exercise 2: Using a 2-sheeted covering, can you find any additional geodesics. Draw these precisely on your paper model!

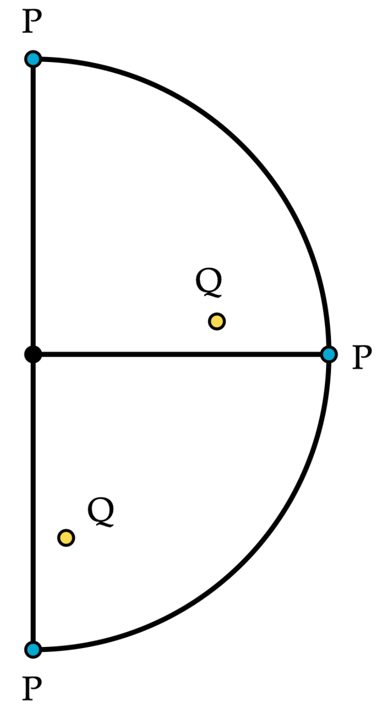


An *n-sheeted covering* of a cone

Exercise 2: Using a 2-sheeted covering, can you find any additional geodesics. Draw these precisely on your paper model!

What is the *total* number of geodesics between P and Q? Feel free to add a third or fourth sheet!



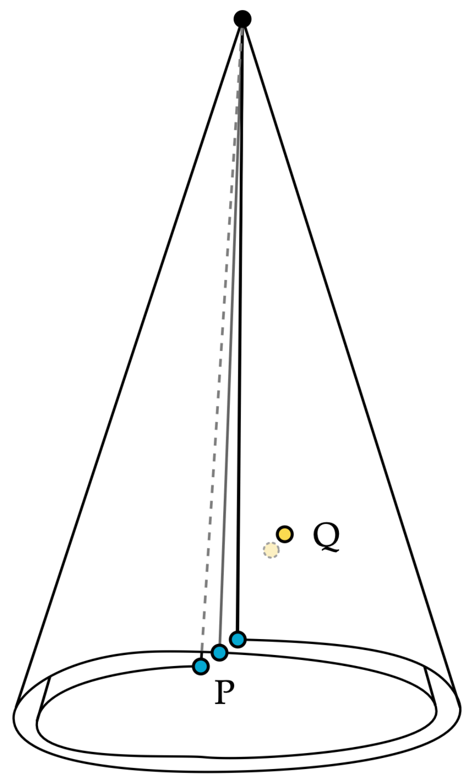


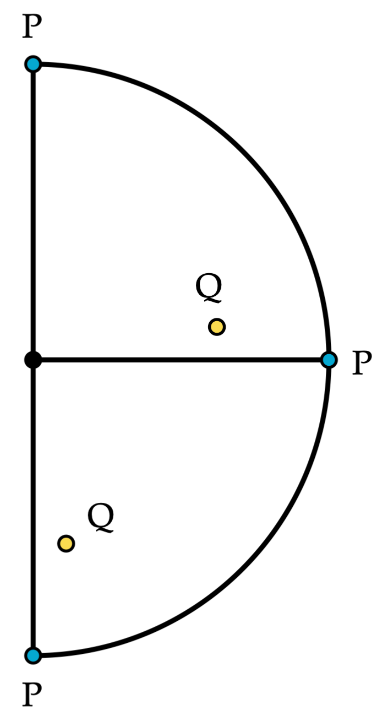
An *n-sheeted covering* of a cone

Exercise 2: Using a 2-sheeted covering, can you find any additional geodesics. Draw these precisely on your paper model!

What is the *total* number of geodesics between P and Q? Feel free to add a third or fourth sheet!

Does this hold for *any* two points P and Q on the cone?





An *n-sheeted covering* of a cone

Exercise 2: Using a 2-sheeted covering, can you find any additional geodesics. Draw these precisely on your paper model!

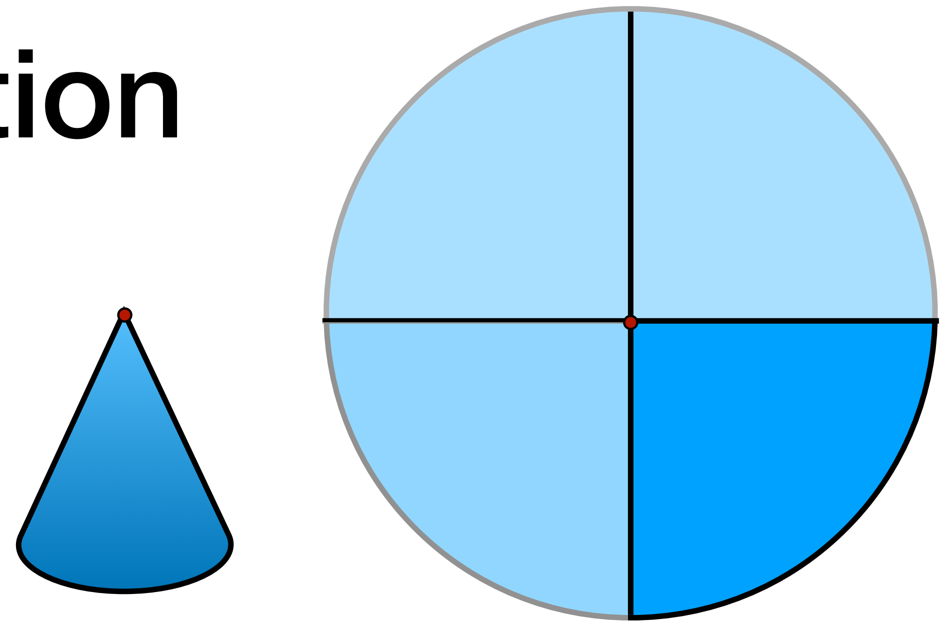
What is the *total* number of geodesics between P and Q? Feel free to add a third or fourth sheet!

Does this hold for *any* two points P and Q on the cone?

What is the total number of geodesics if the cone angle is 60°?

Bonus Question

Consider a cone with cone-angle 90°.

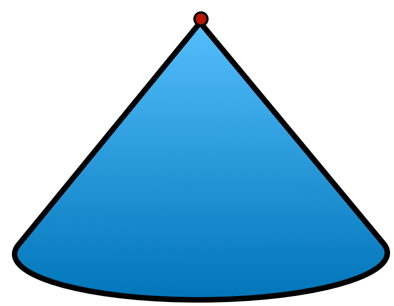


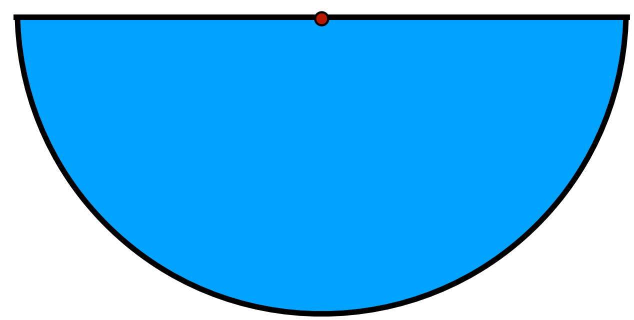
How many self-intersections can a geodesic on this cone have? Is there a limit?

What about on cones with other cone-angles?

Extra Bonus

Construct a cone with cone angle 180°.

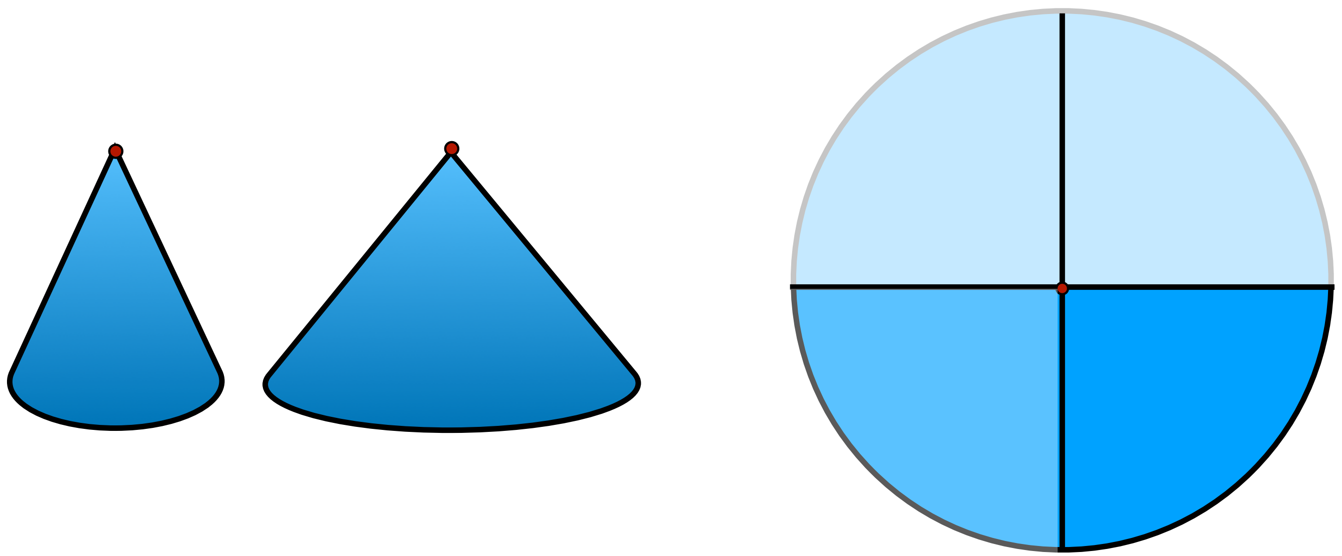




Prove **that geodesics on this cone will** ***never*** **have any self-intersections.**

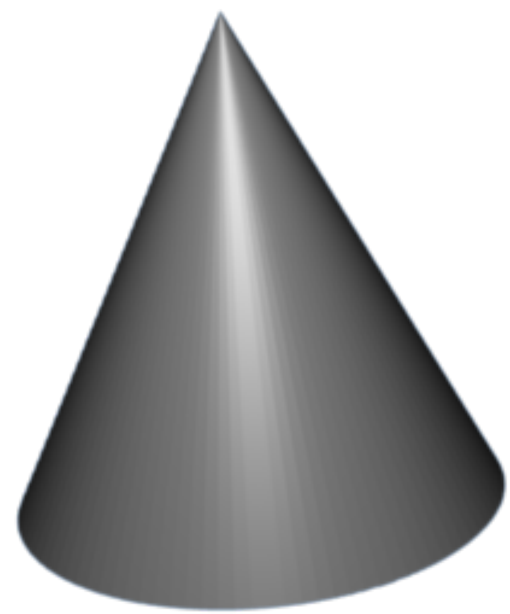
Important Bonus

Consider a cone with cone angle strictly 90° or strictly 180°.

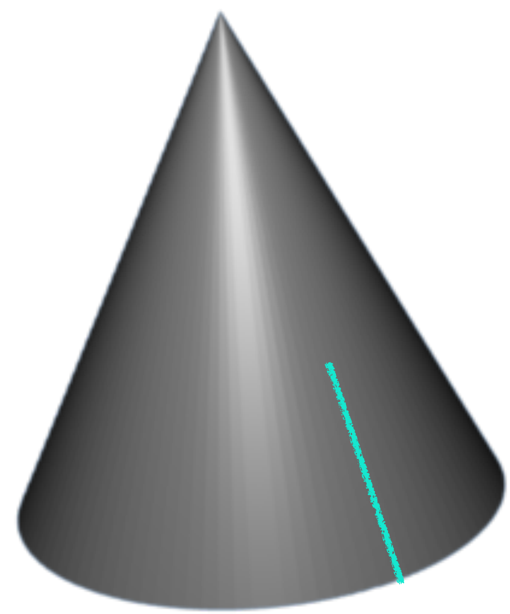


Do initially parallel geodesics remain parallel?

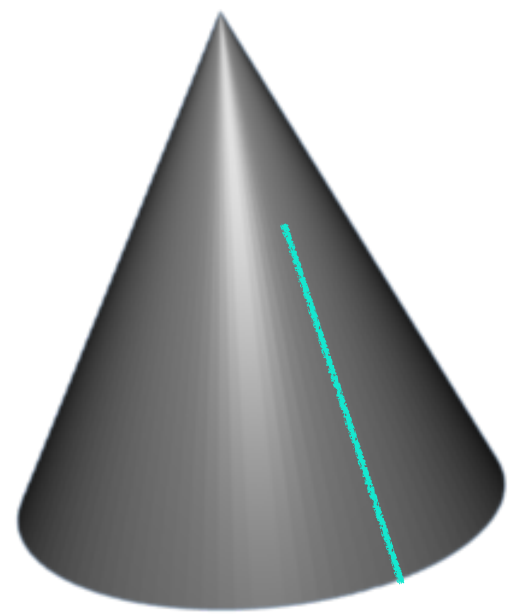
What happens if one geodesic *hits* the vertex?



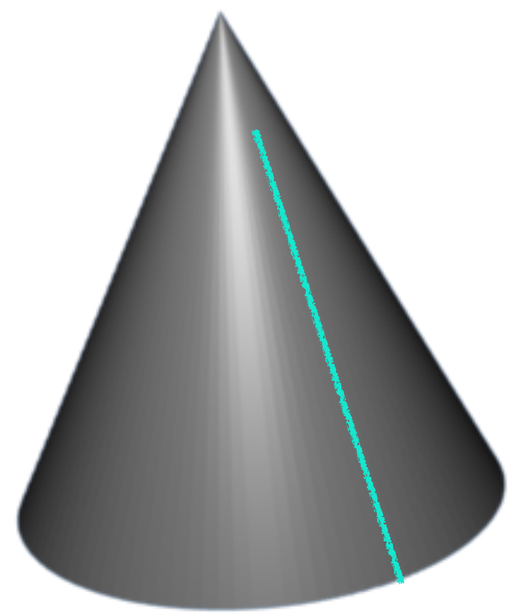
What happens if one geodesic *hits* the vertex?



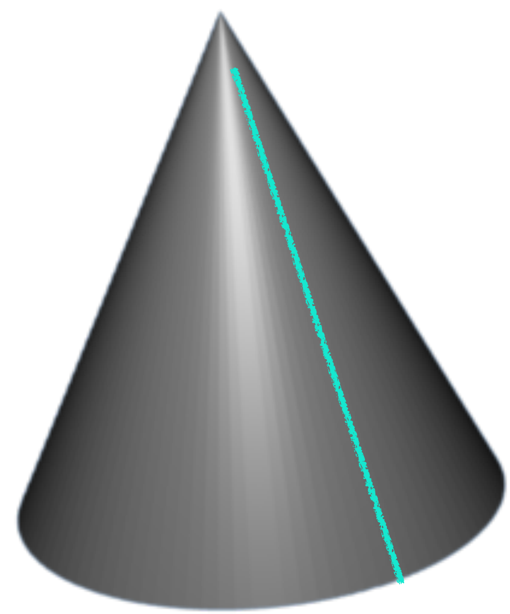
What happens if one geodesic *hits* the vertex?



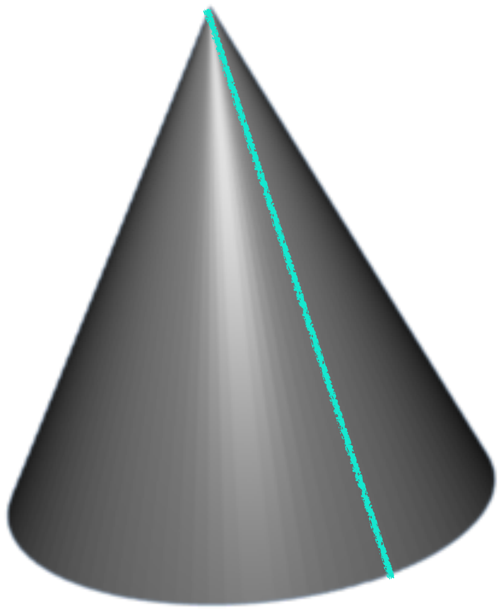
What happens if one geodesic *hits* the vertex?



What happens if one geodesic *hits* the vertex?

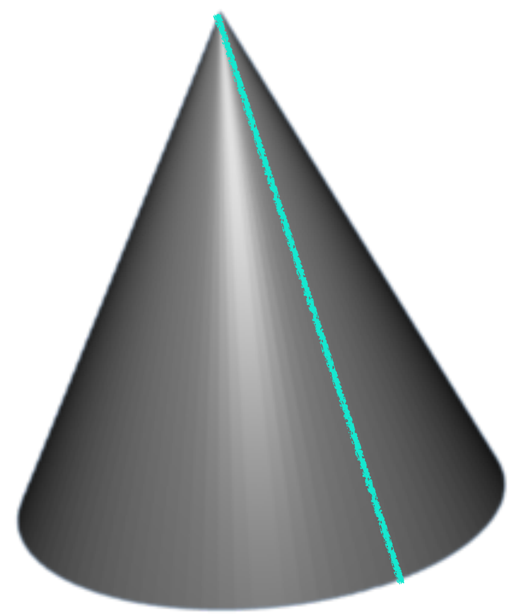


What happens if one geodesic *hits* the vertex?



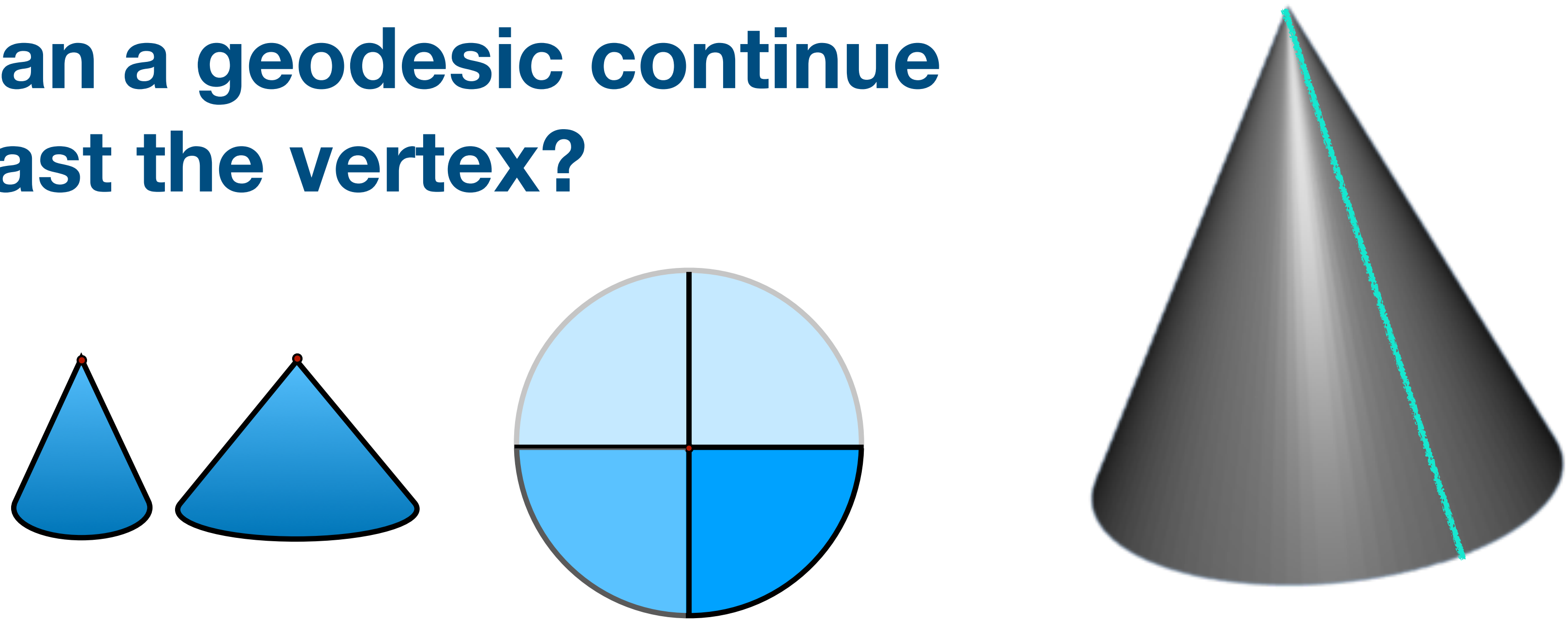
What happens if one geodesic *hits* the vertex?

Can a geodesic continue past the vertex?



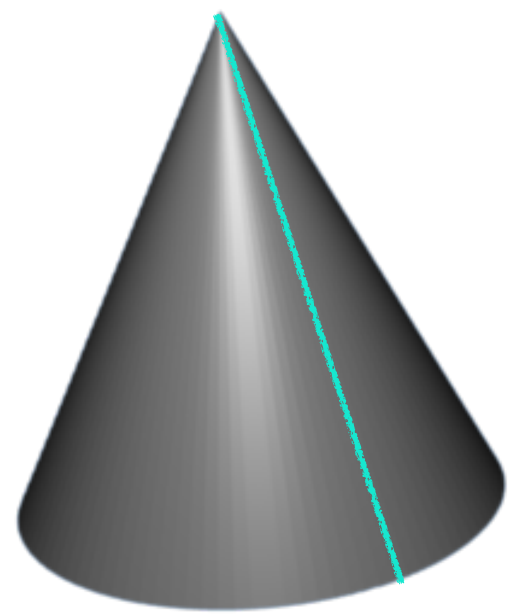
What happens if one geodesic *hits* the vertex?

Can a geodesic continue past the vertex?

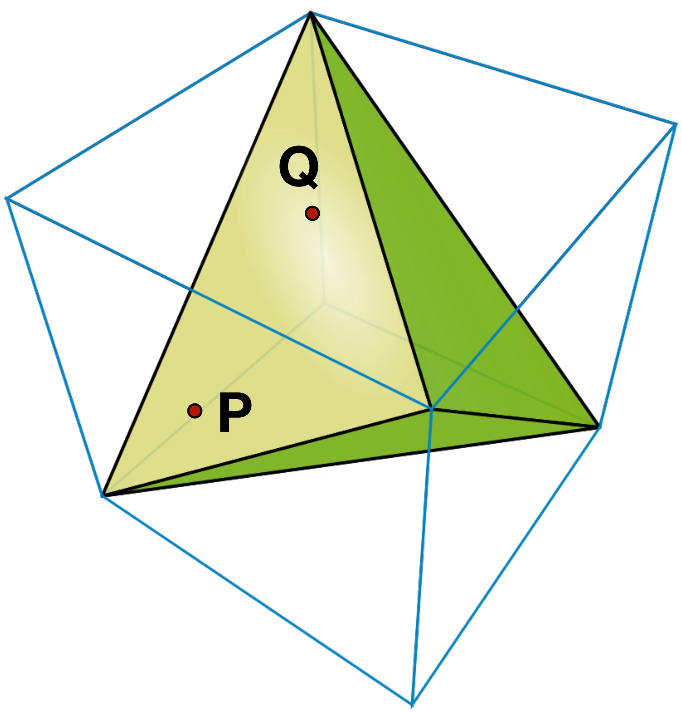


If the cone angle is 2π/n for some natural number n, then something special occurs!

In this case we get an *orbifold*. Although it has singularities, they are surprisingly well behaved, and geodesics can be continued through them.



Let’s turn our attention to one final surface: the tetrahedron!



How many geodesics are there that connect two given points?

Let’s construct a tetrahedron and see…

Lizard, 1942,
by MC Escher

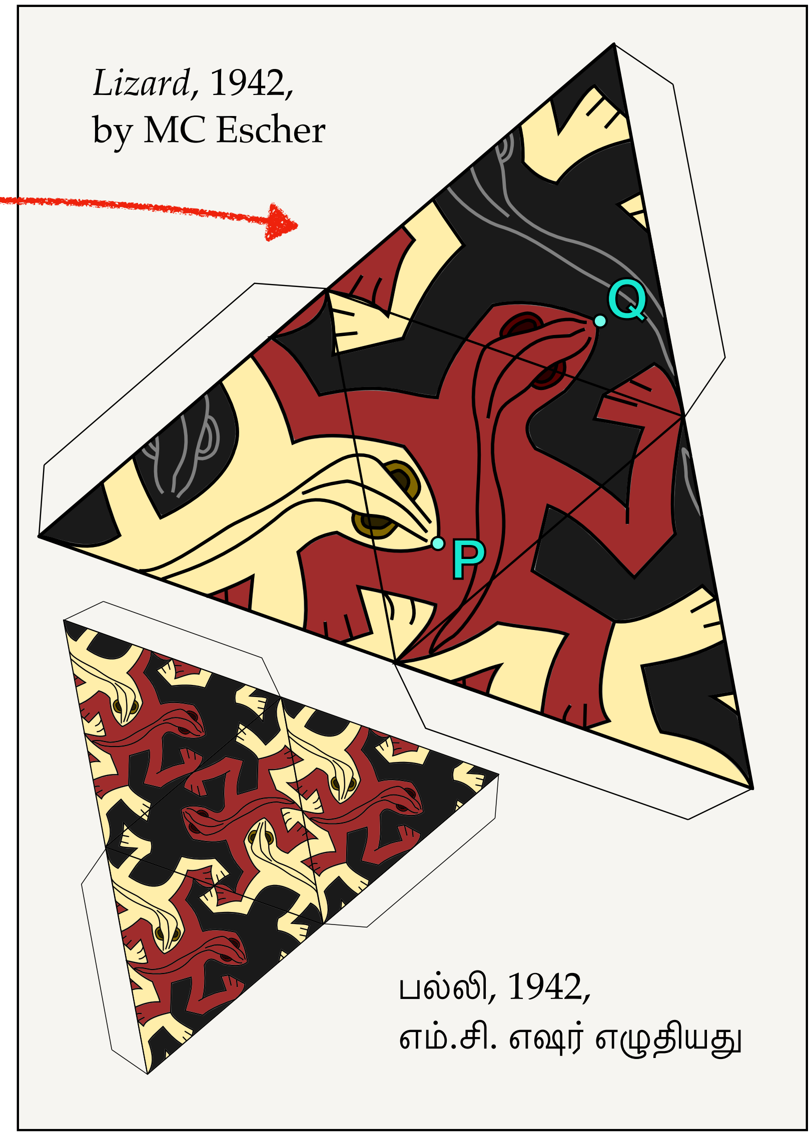
பல்லி, 1942,
எம்.சி. எஷர் எழுதியது

Instructions

1) **Cut out the larger tetrahedral net.**

2) **Fold it to create a tetrahedron, but do NOT tape or paste it shut just yet!**

3) **How many geodesics can you find containing both the points P and Q?**

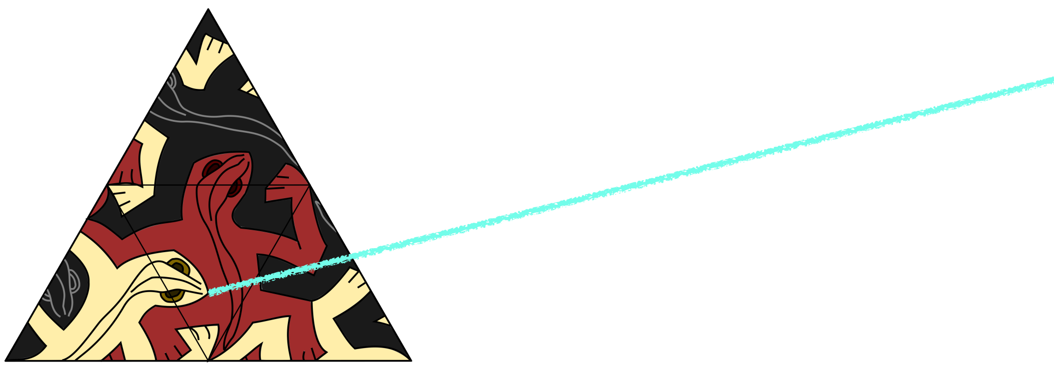


A n-sheeted covering of the tetrahedron?

We'll start with a 1-sheeted covering.

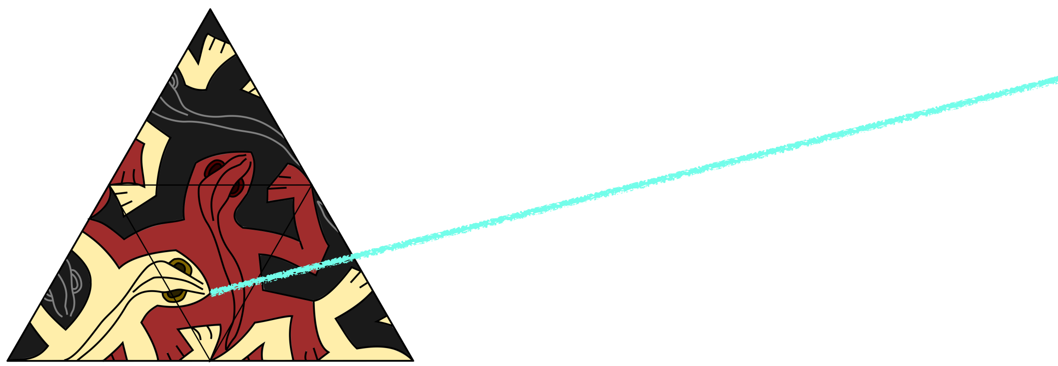
A n-sheeted covering of the tetrahedron?

We'll start with a 1-sheeted covering.



A n-sheeted covering of the tetrahedron?

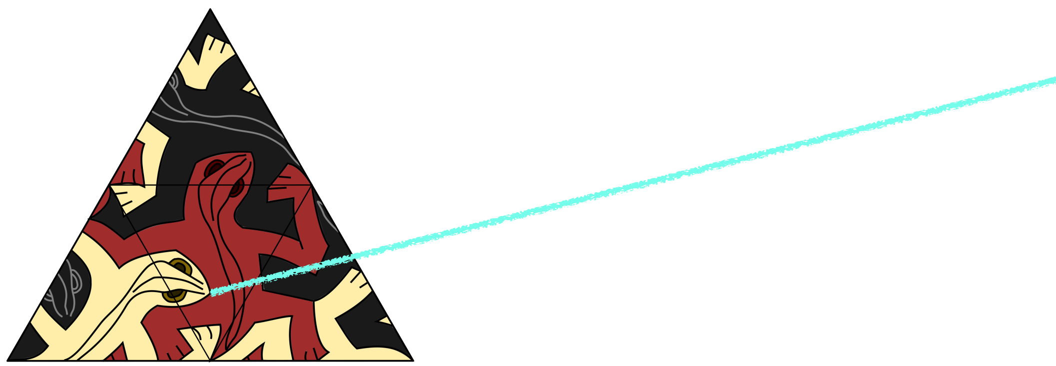
We’ll start with a 1-sheeted covering.



Where does the geodesic go next?

A n-sheeted covering of the tetrahedron?

We'll start with a 1-sheeted covering.

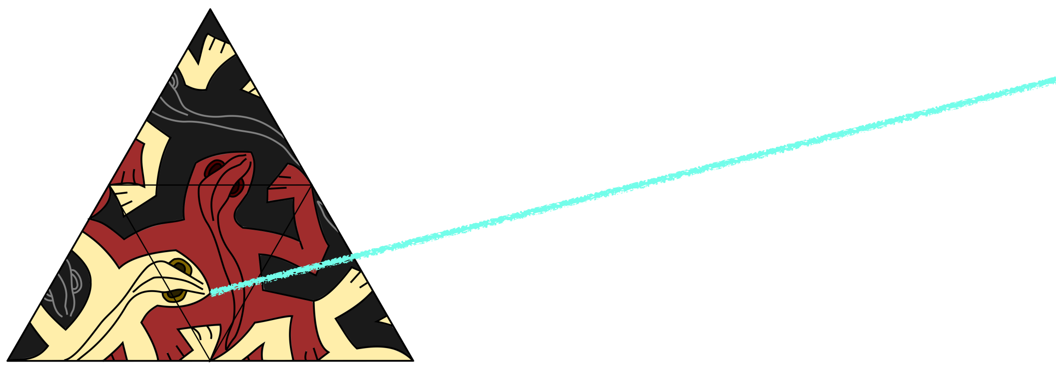


Where does the geodesic go next?

How should we arrange the second sheet?

A n-sheeted covering of the tetrahedron?

We'll start with a 1-sheeted covering.

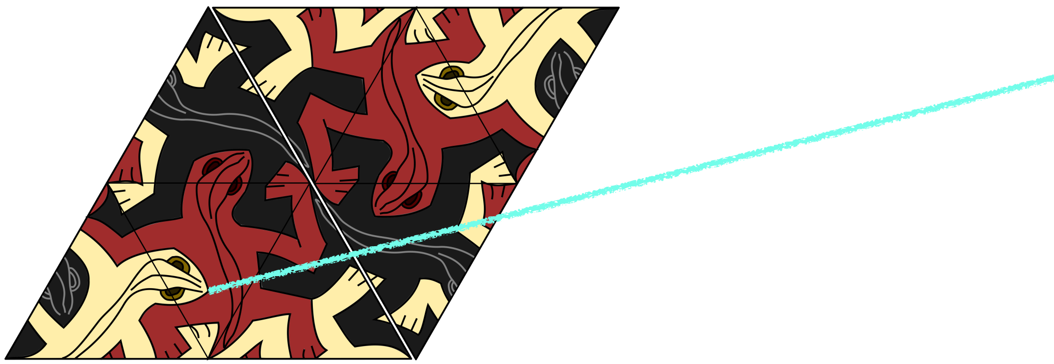


Where does the geodesic go next?

How should we arrange the second sheet?

Working in pairs, try to answer this with your cut-outs!

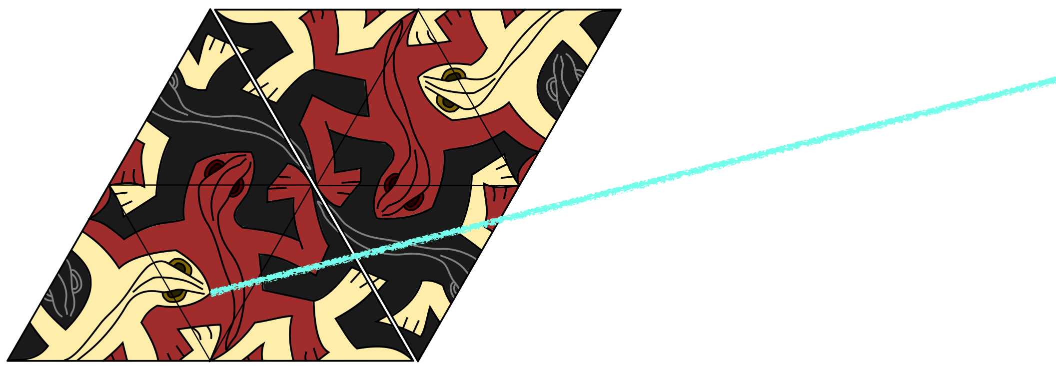
A n-sheeted covering of the tetrahedron?



Where does the geodesic go next?

How should we arrange the second sheet?

A n-sheeted covering of the tetrahedron?

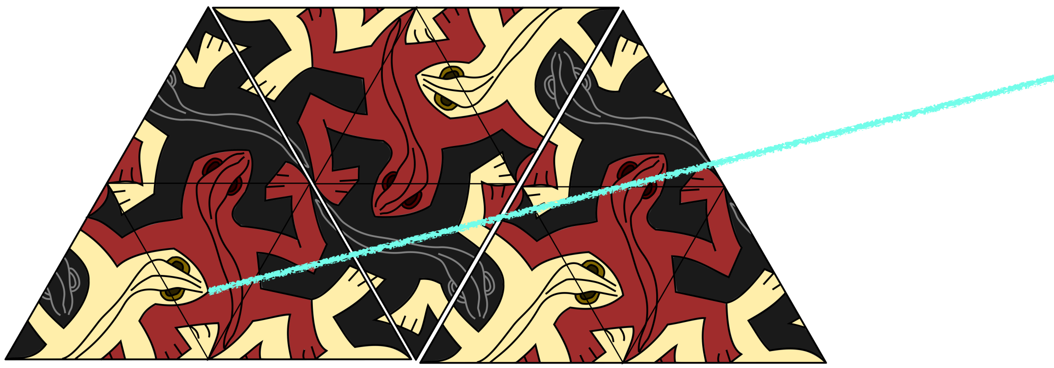


Where does the geodesic go next?

How should we arrange the second sheet?

What about the next sheet?

A n-sheeted covering of the tetrahedron?

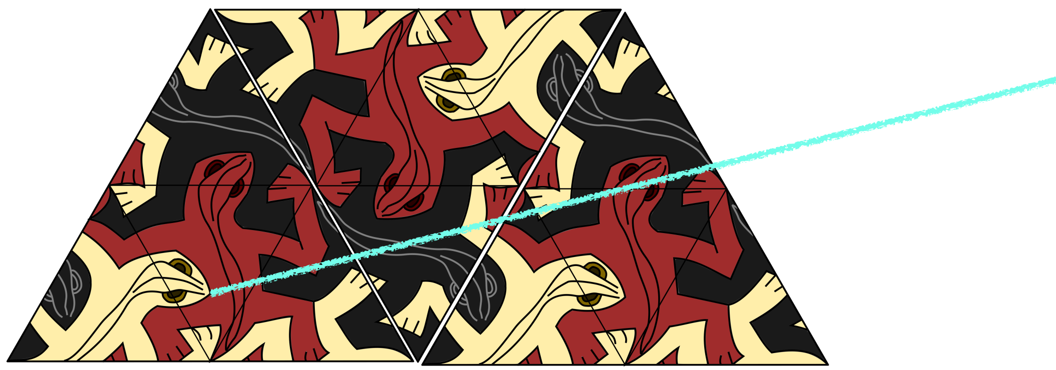


Where does the geodesic go next?

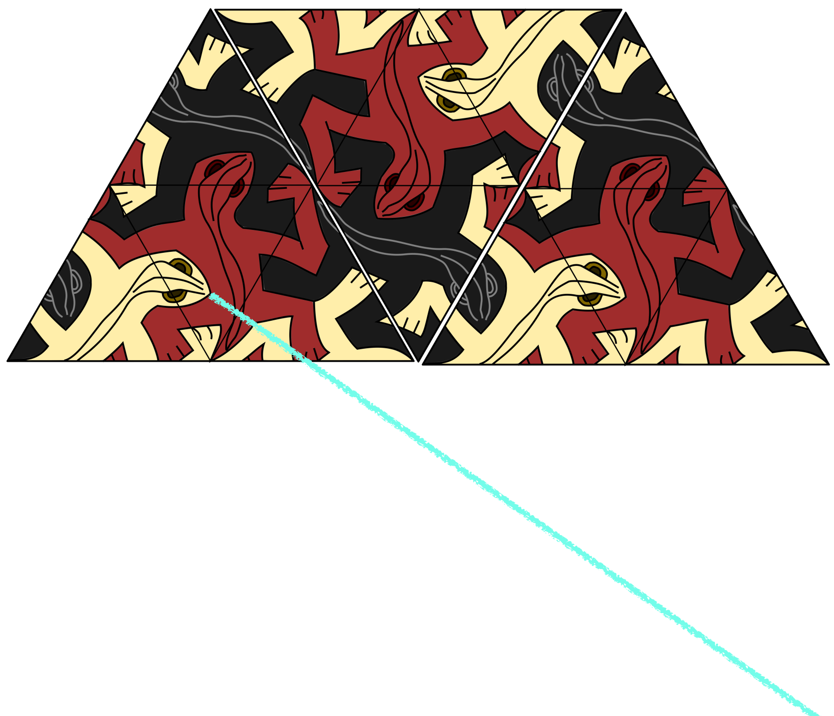
How should we arrange the second sheet?

What about the next sheet?

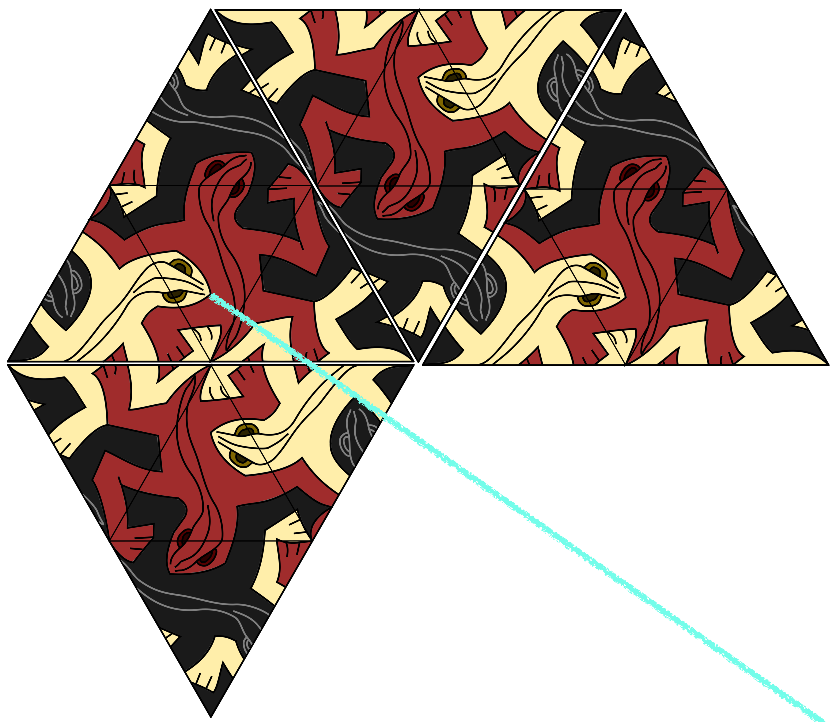
A n-sheeted covering of the tetrahedron?



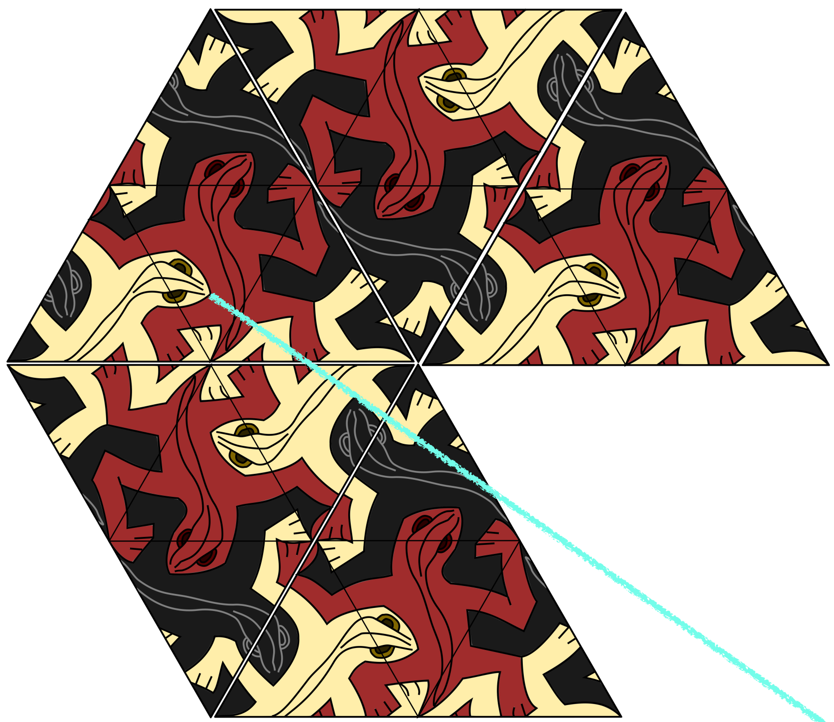
A n-sheeted covering of the tetrahedron?



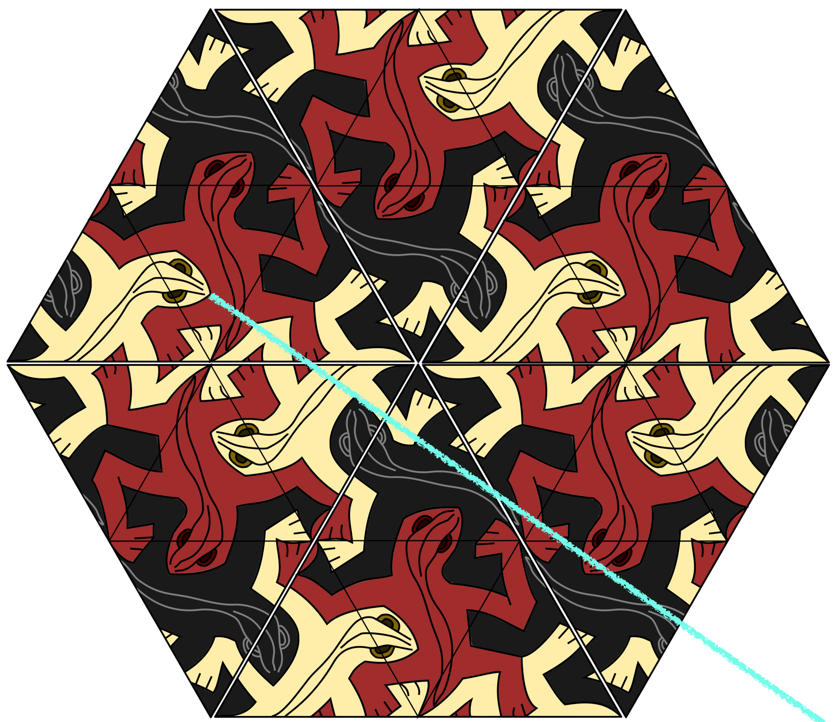
A n-sheeted covering of the tetrahedron?



A n-sheeted covering of the tetrahedron?



A n-sheeted covering of the tetrahedron?



A n-sheeted covering of the tetrahedron?

A n-sheeted covering of the tetrahedron

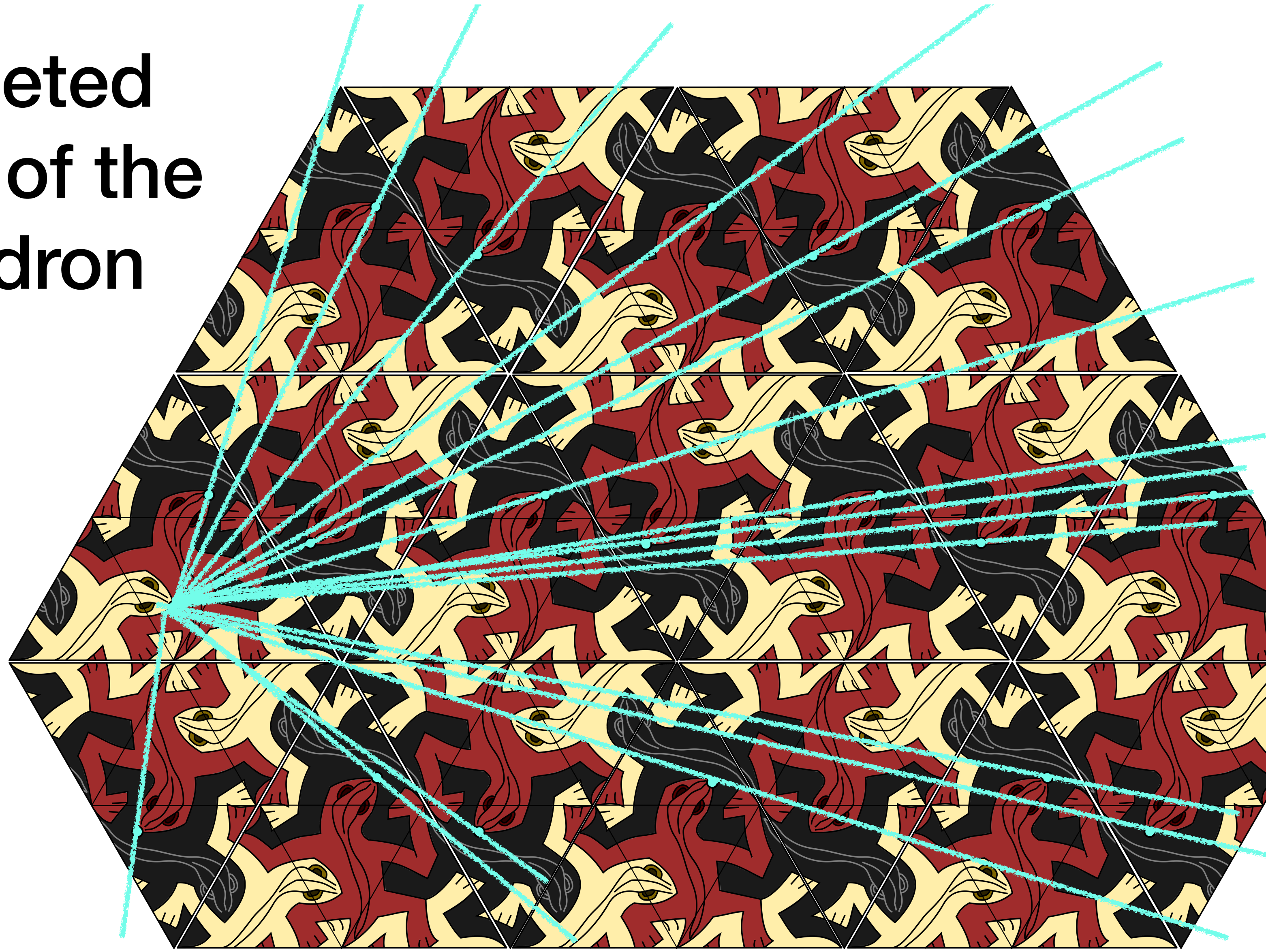
Recall

How many geodesics can you find containing both the points P and Q?

A n-sheeted covering of the tetrahedron

A n-sheeted covering of the tetrahedron

One new geodesic for every sheet!



Final Question

Can you find a *closed* (i.e. finite length) geodesic containing the point P? How many such geodesics can you find?

Can you find a *closed* (i.e. finite length) geodesic containing the point P?

How many such geodesics can you find?

Can you find a *closed* (i.e. finite length) geodesic containing the point P?

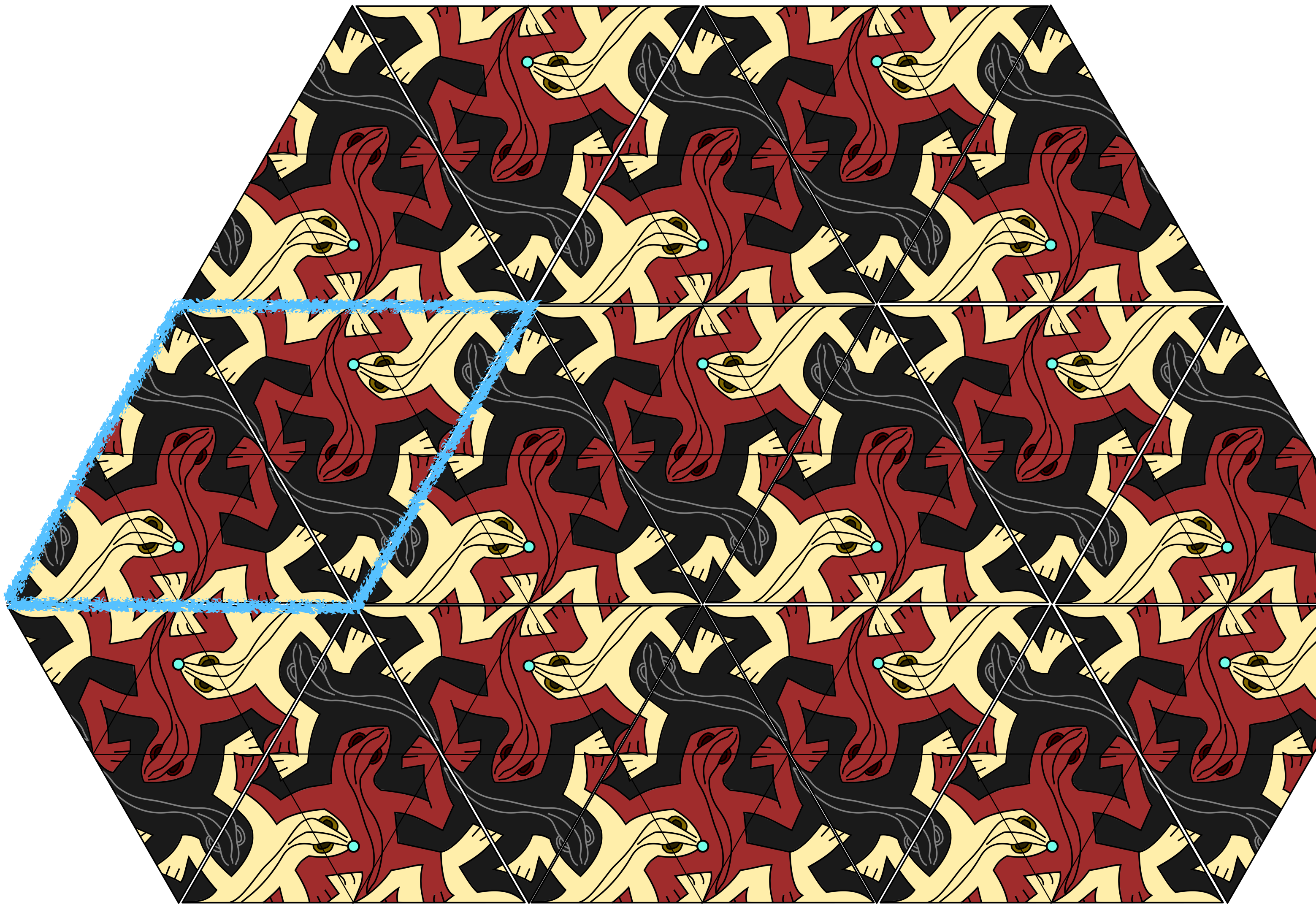
How many such geodesics can you find?

What if we only want geodesics that avoid vertices?

Can you find a *closed* (i.e. finite length) geodesic containing the point P?

How many such geodesics can you find?

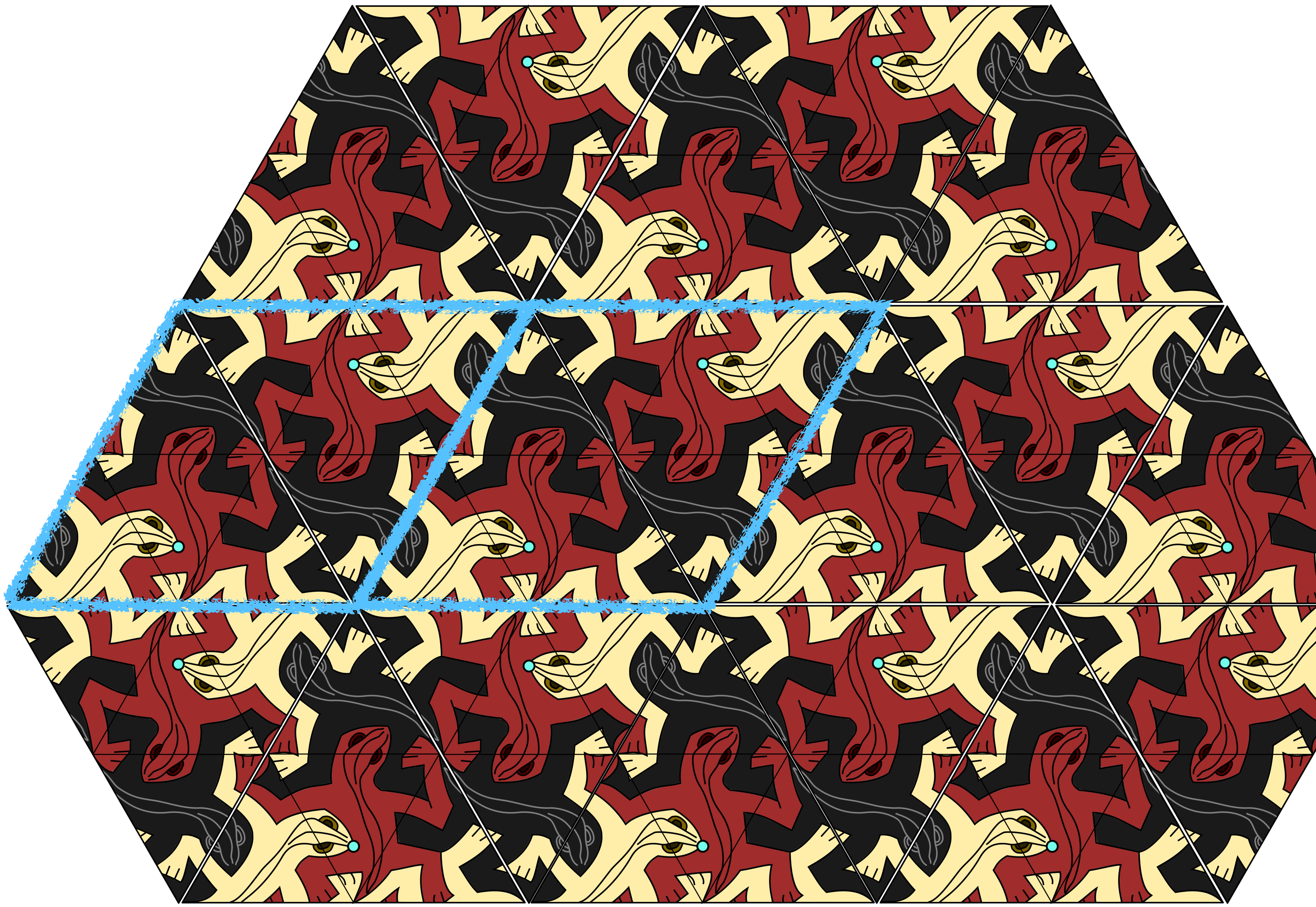
What if we only want geodesics that avoid vertices?



Can you find a *closed* (i.e. finite length) geodesic containing the point P?

How many such geodesics can you find?

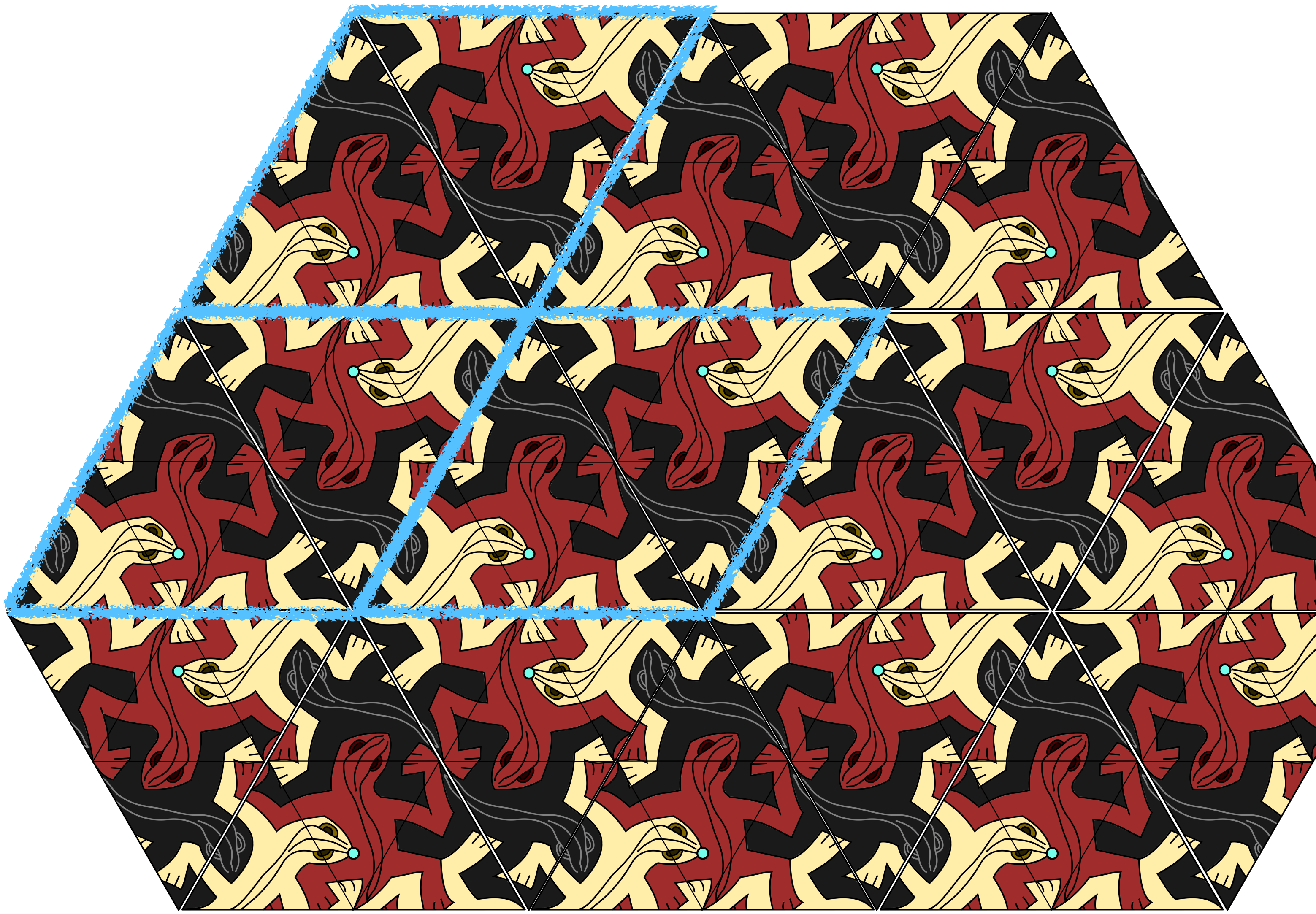
What if we only want geodesics that avoid vertices?



Can you find a *closed* (i.e. finite length) geodesic containing the point P?

How many such geodesics can you find?

What if we only want geodesics that avoid vertices?



Can you find a *closed* (i.e. finite length) geodesic containing the point P?

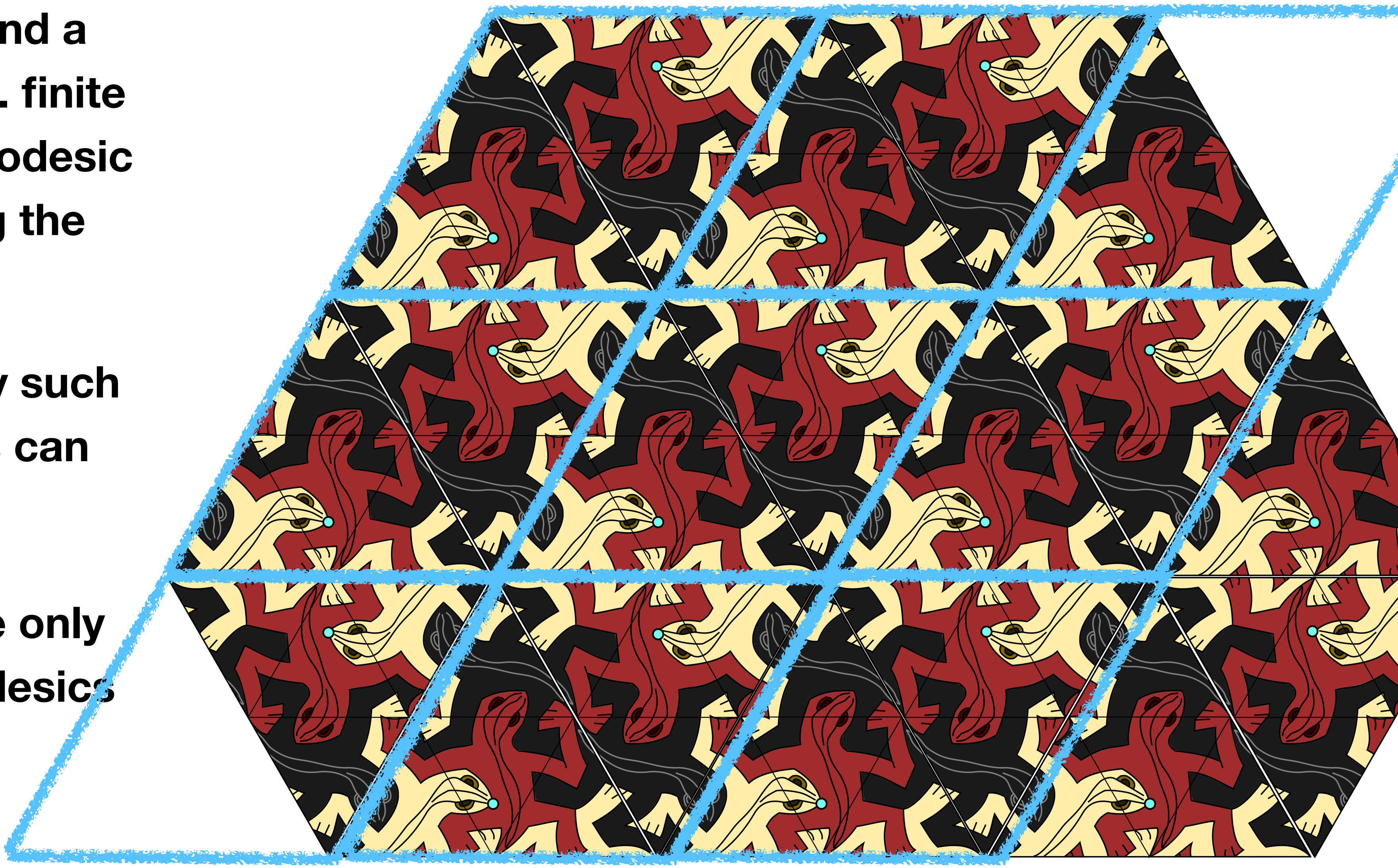
How many such geodesics can you find?

What if we only want geodesics that avoid vertices?

Can you find a *closed* (i.e. finite length) geodesic containing the point P?

How many such geodesics can you find?

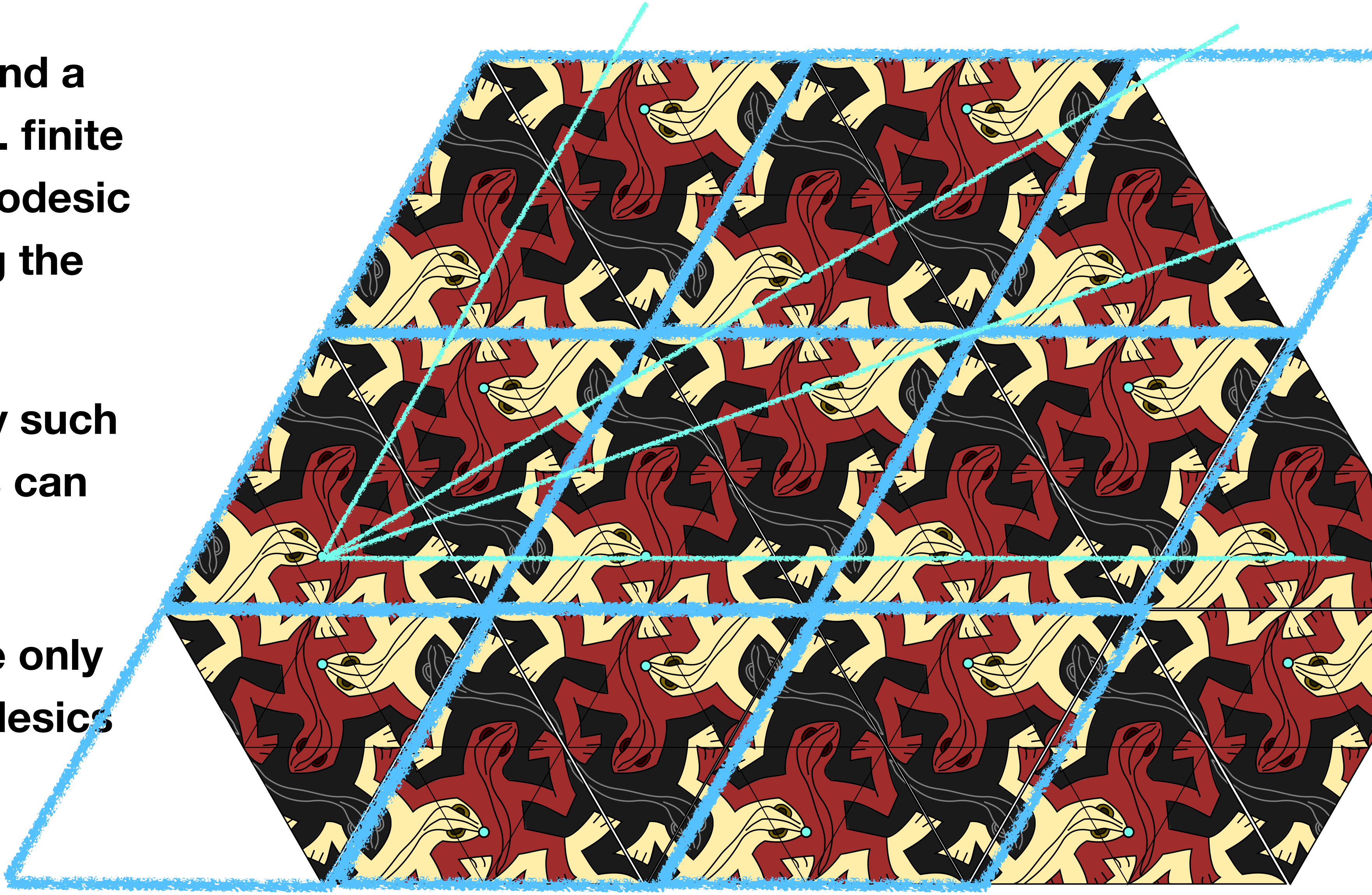
What if we only want geodesics that avoid vertices?



Can you find a *closed* (i.e. finite length) geodesic containing the point P?

How many such geodesics can you find?

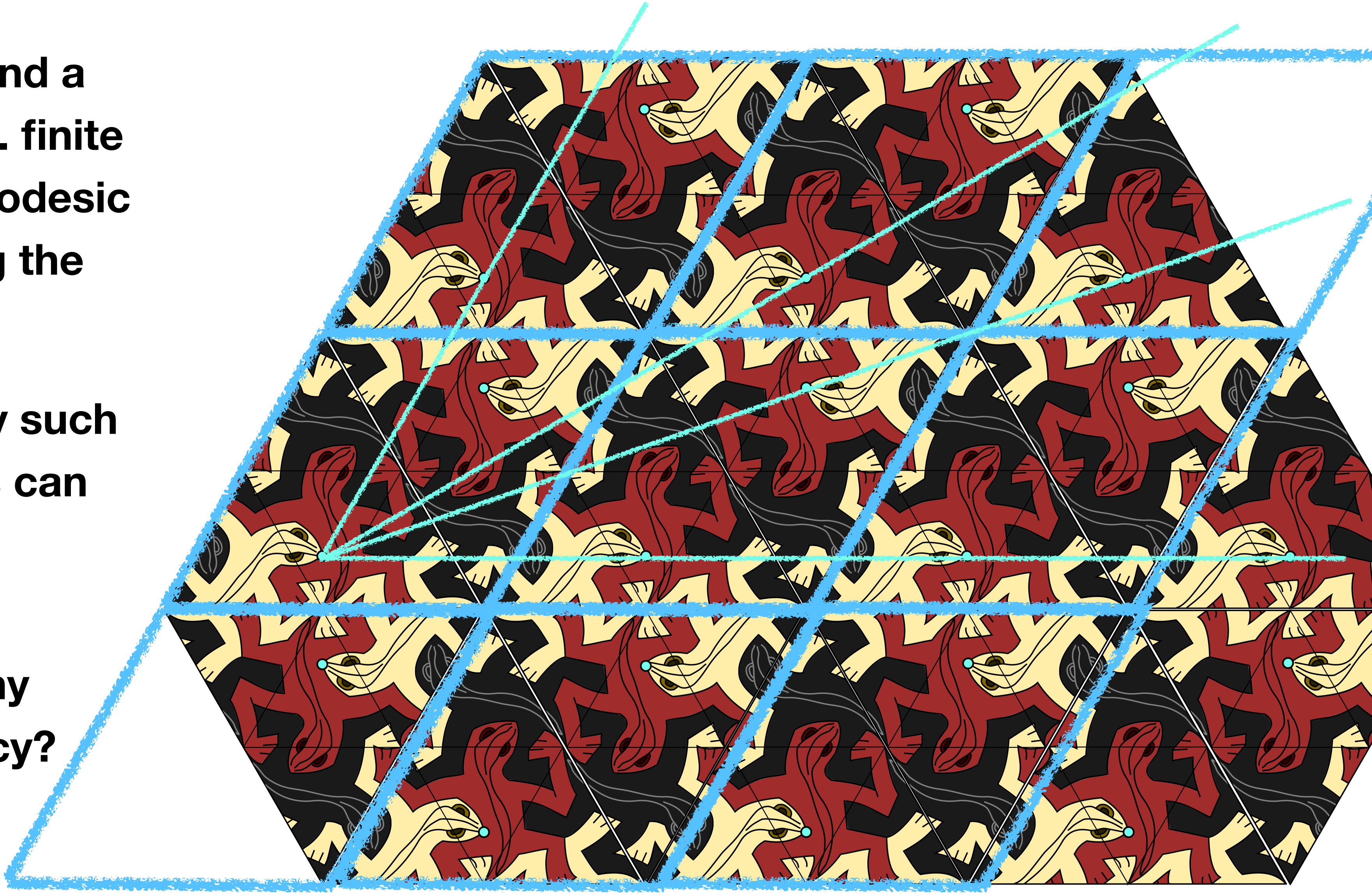
What if we only want geodesics that avoid vertices?



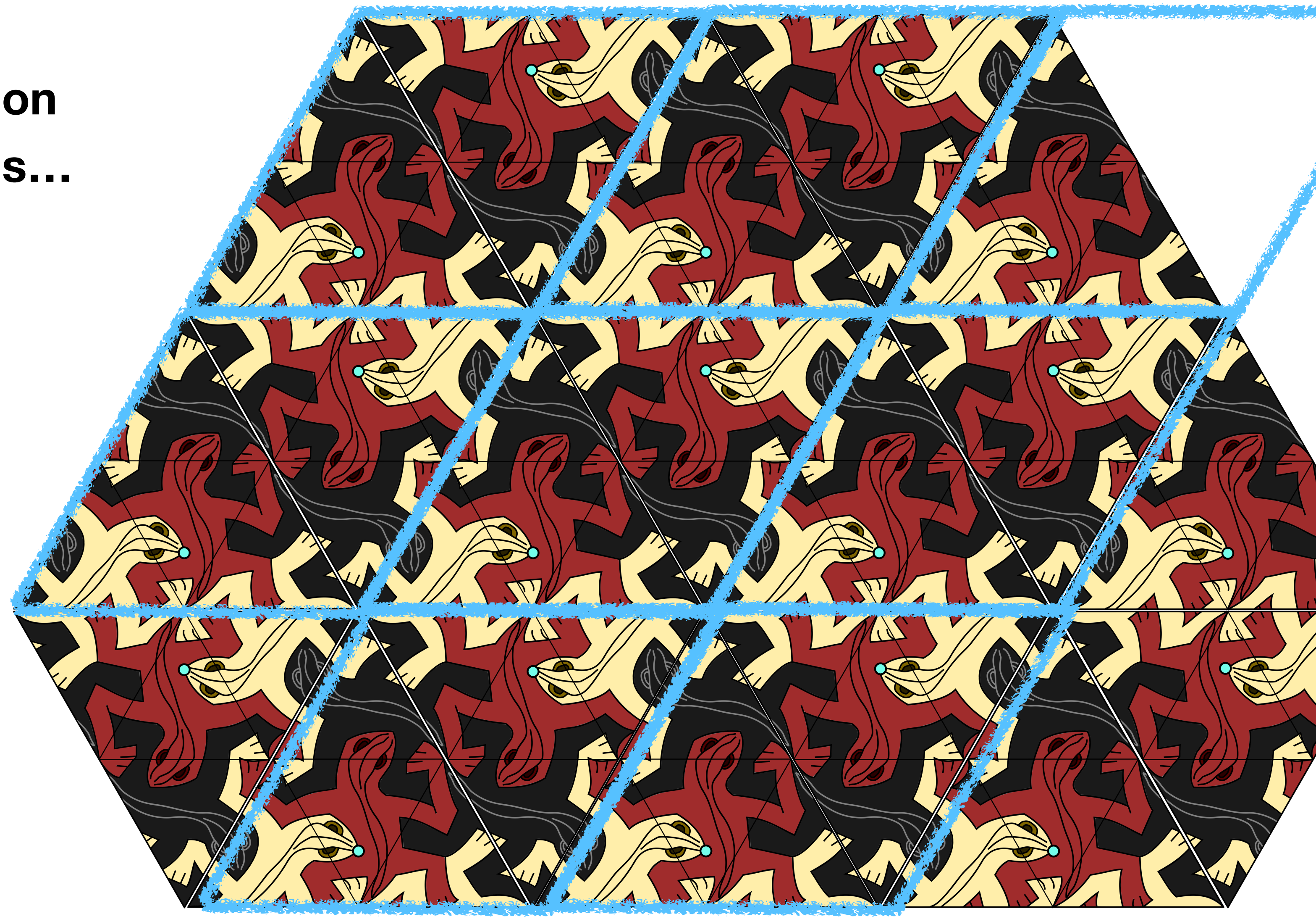
Can you find a *closed* (i.e. finite length) geodesic containing the point P?

How many such geodesics can you find?

Is there any redundancy?

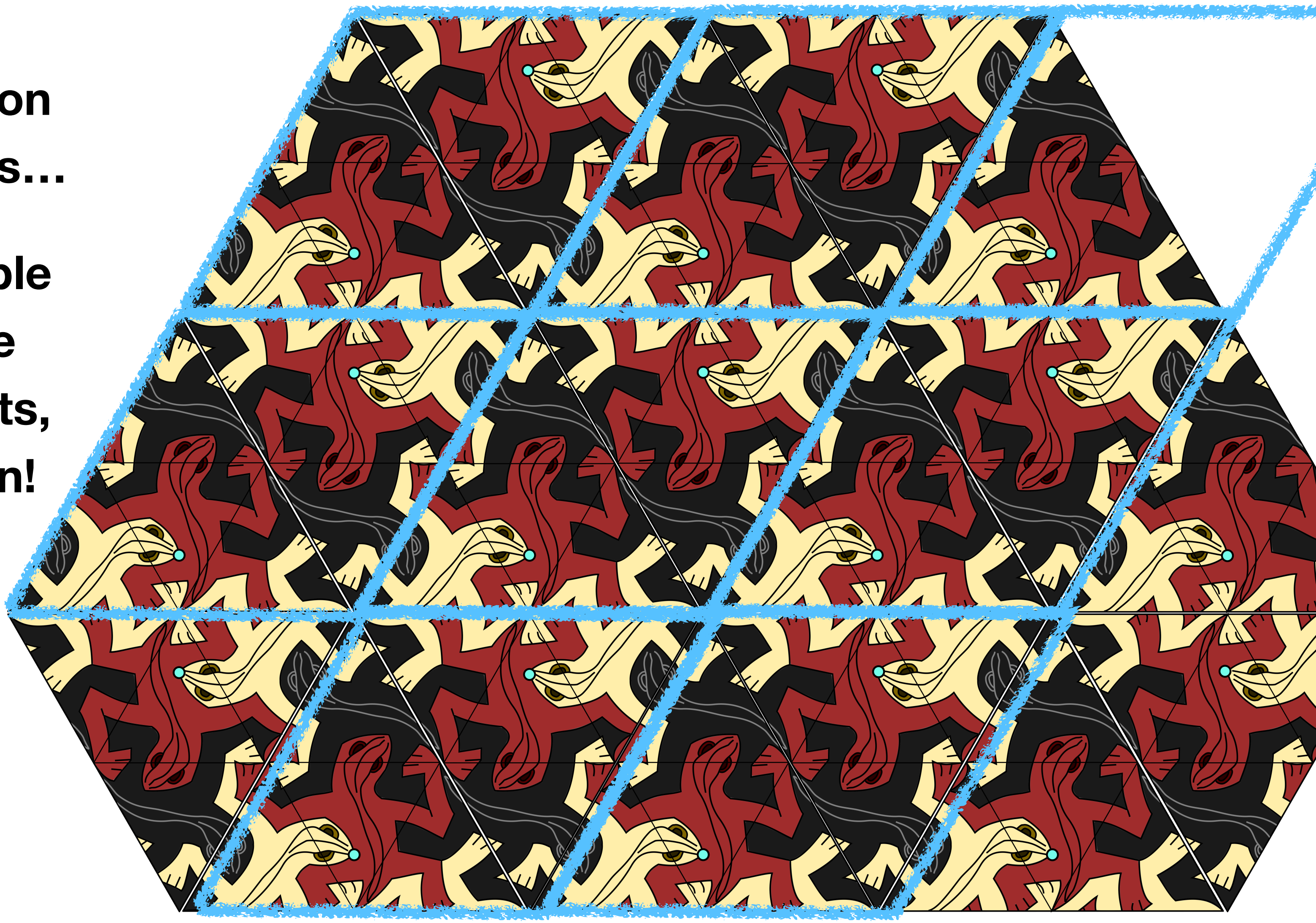


Notice that this rhombus tessellation quotients to a torus…



Notice that this rhombus tessellation quotients to a torus…

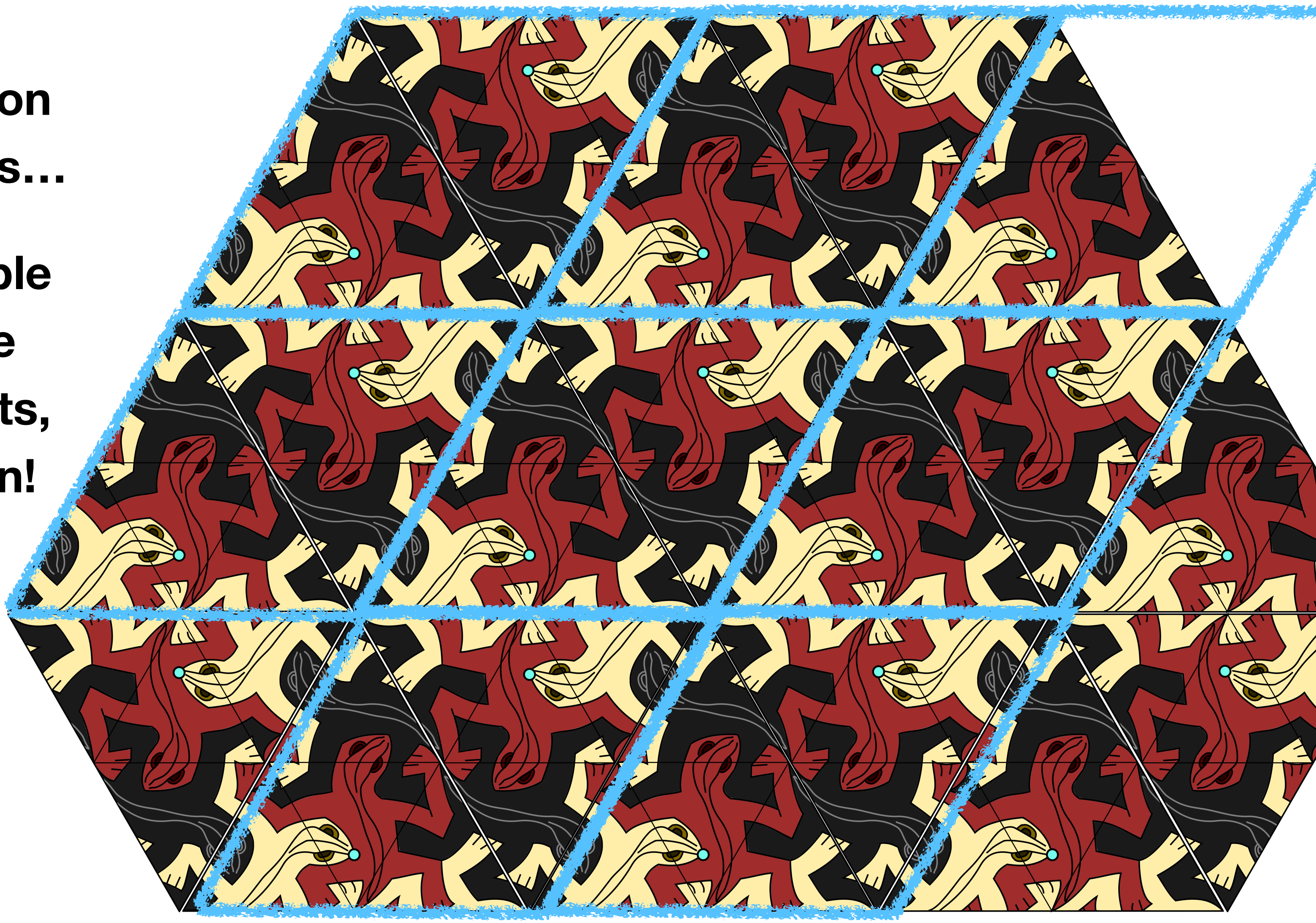
The torus is a double cover of the sphere with 4 branch points, aka the tetrahedron!



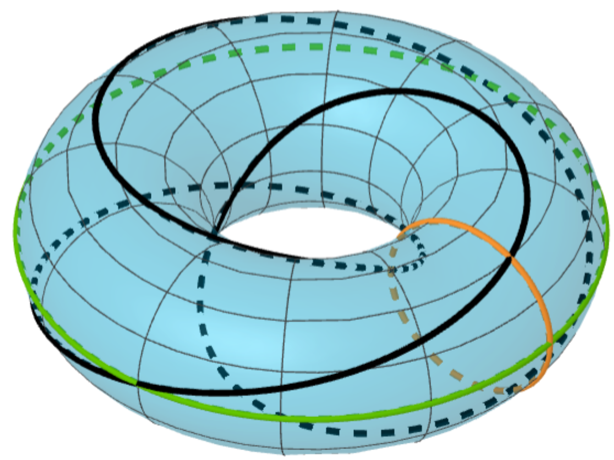
Notice that this rhombus tessellation quotients to a torus…

The torus is a double cover of the sphere with 4 branch points, aka the tetrahedron!

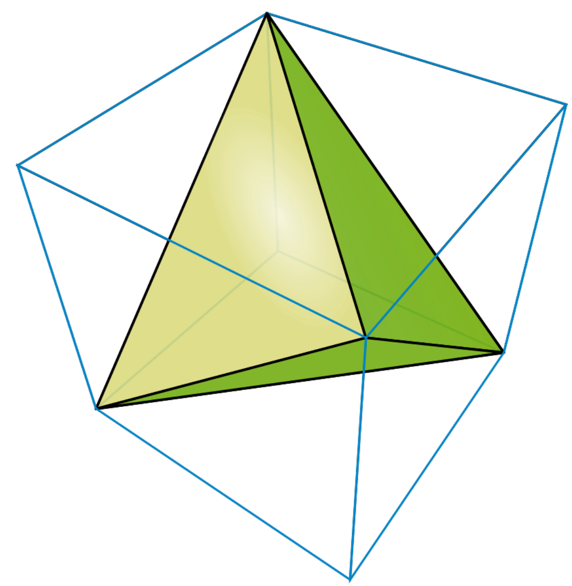
Moreover, the closed geodesics on the torus descend to closed vertex-avoiding geodesics on the tetrahedron!



The torus is a double cover of the sphere with 4 branch points, aka the tetrahedron!

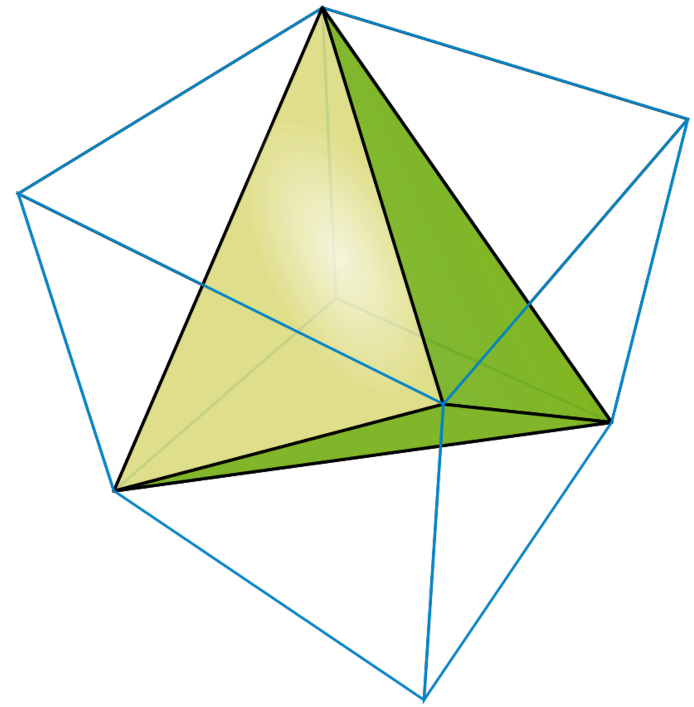


Moreover, the closed geodesics on the torus descend to closed vertex-avoiding geodesics on the tetrahedron!



One last remark:

What happens to initially parallel geodesics on the tetrahedron?



Looking at the infinite sheeted covering, they ought to remain parallel forever…

But it's worth verifying this with your physical tetrahedron!

The tetrahedron is another example of an orbifold: although it has cone point singularities, they are very well behaved.

Original painting by MC Escher:
Lizards, 1942

PLATE 39

Original painting by MC Escher:
***Lizards*, 1942**

There was a (very tiny) amount of human error, since it was created entirely by hand in 1942.

So I made a slightly modified version in Inkscape.

PLATE 39

Original painting by MC Escher: *Lizards*, 1942

There was a (very tiny) amount of human error, since it was created entirely by hand in 1942.

So I made a slightly modified version in Inkscape.

Thank you!

For more on these topics:

Geometry of Surfaces by John Stillwell

Experiencing Geometry by David Henderson

Visual Differential Geometry by Tristan Needham

Symmetries of Things by Chaim Goodman-Strauss, Heidi Burgiel, and John Horton Conway