

Measuring the Earth and the Moon with a straw and a string

ABHIRUP CHATTERJEE

Trigonometry as similar triangles — and how our ancestors used it to weigh up a hill, a planet, and its satellite.

Everything in these notes rests on one idea: in a right-angled triangle, the ratio of any two sides depends only on the angles, never on the size of the triangle. Give that ratio a name and you have trigonometry. Add a homemade angle-measuring device — a protractor, a drinking straw, a piece of string and a small weight — and you can go on to measure the height of a building you cannot reach, the height of a mountain whose base you cannot walk to, the radius of the Earth, the circumference of the Earth, the distance to the Moon, and the size of the Moon.

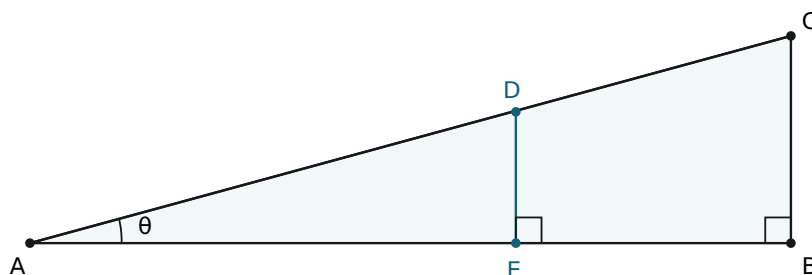
The people who first did this — Aristotle, Aristarchus, al-Ma'mūn's surveyors, al-Bīrūnī — had no better equipment than the class does. Even then, their answers were remarkably accurate.

RUNNING ORDER

- | | |
|--|---------------------------------------|
| 1 Trigonometry is similar triangles | 8 Latitude, and how to measure yours |
| 2 How do we measure an angle? | 9 The circumference of the Earth |
| 3 The height of a building | 10 Leaving Earth: is the Moon a ball? |
| 4 When the distance cannot be measured | 11 Aristarchus: how far is the Moon? |
| 5 Al-Bīrūnī at Nandana Fort | 12 How big is the Moon? |
| 6 Interlude: is the Earth round? | 13 Scoreboard |
| 7 The radius of the Earth | |

01 Trigonometry is similar triangles

Take a right-angled triangle ABC , with the right angle at B and the angle at A equal to θ . Choose *any* point E on AB and draw the perpendicular to AB through E . Say it cuts AC at D .



Any perpendicular dropped onto AB cuts off a smaller copy of the same triangle.

Then $\triangle ADE \sim \triangle ABC$, because the two triangles share the angle θ and both have a right angle. Similar triangles have proportional sides, so

$$\frac{AB}{BC} = \frac{AE}{ED}$$

Notice what this says. It did not matter where on AB we put E ; every choice gives the same number. The ratio does not care how big the triangle is. It depends on one thing only — the angle θ . So it deserves a name of its own, and we call it $\cot \theta$. In exactly the same way,

$$\frac{AB}{AC} = \frac{AE}{AD} = \cos \theta$$

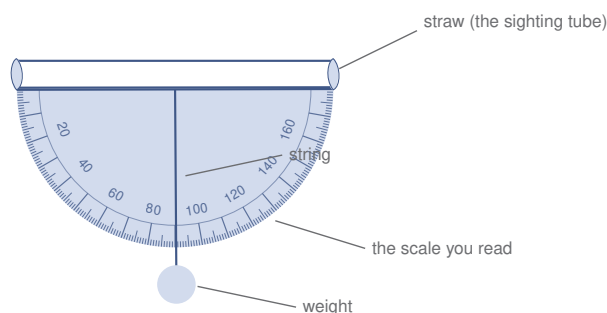
...and so on for the other four ratios. That is the whole of the machinery: an angle fixes the ratios, and a ratio fixes the angle. If I know one angle of a right triangle and one of its sides, I know all of it.

Now — how does this help?

02 How do we measure an angle?

Before answering that, let us look at how we measure angles in the first place. The first answer that comes to mind is the protractor, which is excellent for measuring the angle between two lines drawn on a flat page. It is much less useful when what I want to know is the angle I am tilting my head through to look at a bird's nest in a tree.

The good news is that a protractor can be modified into exactly that kind of device.



The protractor, a straw, a string and a weight: a [DIY inclinometer](#), shown here lying level.

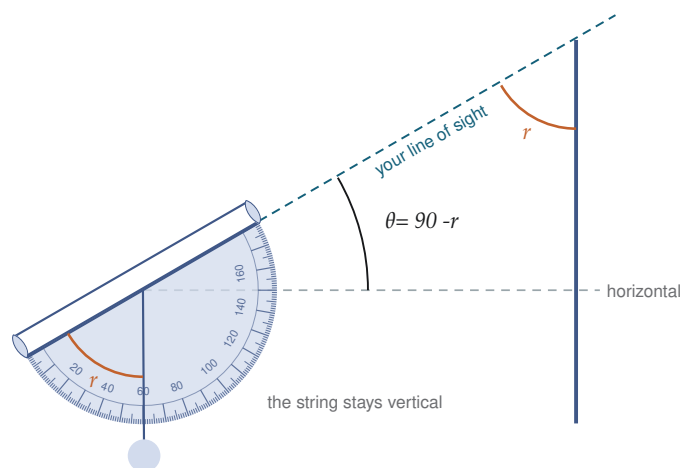
How to make it

1. **Attach the straw.** Tape the straw along the straight edge (the diameter) of the protractor. The straw is the sighting tube — you will look through it.
2. **Attach the string.** Tie one end of the string through the hole at the centre of the protractor.
3. **Attach the weight.** Tie the weight to the other end, so the string hangs freely across the scale.

This device is called an *inclinometer*. On that note, there are plenty of other angle-measuring instruments that people of the past relied on — the sextant, the astrolabe — each with its own advantages and disadvantages. For our purposes this DIY inclinometer is good enough.

How to use it

Sight the object through the straw and wait for the weight to settle. The straw now lies along the line from your eye to the object; the string lies along the line from the centre of the protractor straight down to the centre of the Earth. The number r where the string crosses the scale is the angle between those two lines — the tilt of your sight line away from the vertical, not from the horizontal. Since the vertical and the horizontal are themselves at right angles, the angle of elevation is



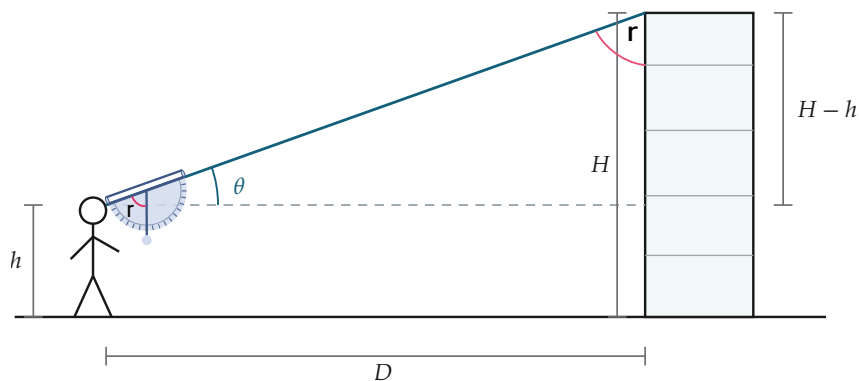
Tilt the instrument until the object sits in the straw. The scale records r , the angle between the line of sight and the string; the elevation θ is what is left over from a right angle. Since the string and the far vertical are parallel, the same r turns up again at the object.

$$\theta = 90^\circ - r$$

Every measurement in the rest of these notes is made with this one instrument.

03 The height of a building

So now, if we want to measure the angle at which I need to tilt my head to look at the top of a building, we can measure it. Here is the picture. I stand at a distance D from the building; my eye is at height h above the ground; the building has height H . Sighting the top through the straw, the string reads r , so my angle of elevation is $\theta = 90^\circ - r$.



Eye level splits the height into h (below) and $H - h$ (above). The reading r reappears at the top of the building, between the sight line and the wall.

The dashed eye-level line cuts the building at height h , so the part of the building above it is $H - h$, and it sits opposite the angle θ in a right triangle whose other side is D . That is exactly the ratio we named in §1:

$$\frac{D}{H - h} = \cot \theta$$

HEIGHT OF A REACHABLE OBJECT

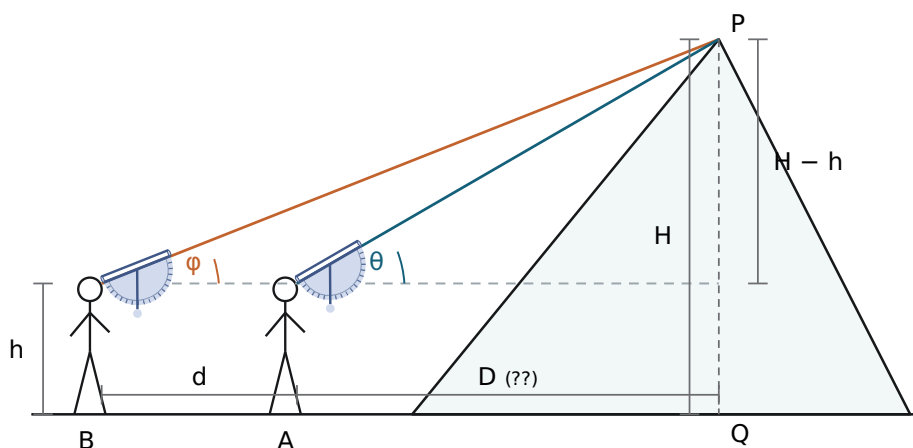
$$H = \frac{D}{\cot \theta} + h$$

So if I know the distance from myself to the building, I can work out the height of the building. No ladder, no tape measure running up the wall.

But what if I cannot measure D ?

04 When the distance cannot be measured

Say I want to measure the height H of a mountain peak P . The trouble is that the point Q directly below the peak is deep inside the mountain. I cannot measure my distance to it — not without digging a tunnel through the mountain.



Two stations, **A** and **B**, a known distance d apart. The unknown D never has to be measured.

So? Do we bring out our hammers and chisels and start digging away? Fortunately, mathematicians

are very lazy. We can find a way to calculate H without any of that manual labour.

Presenter cue. Pose the problem and give them a minute to think before showing the solution.

The solution: take two readings

Let us say that at A the angle of elevation of the peak is θ . Now walk straight forward or straight backward to B , along the same line, with the distance d between A and B measured or already known. Measure the angle of elevation of the peak from B ; say it is φ . Applying §3 twice,

$$\frac{D}{H-h} = \cot \theta \qquad \frac{D+d}{H-h} = \cot \varphi$$

Two equations, two unknowns (D and H), with d , h , $\cot \theta$ and $\cot \varphi$ all known. Subtracting the first from the second makes D disappear:

$$\frac{d}{H-h} = \cot \varphi - \cot \theta$$

HEIGHT OF AN UNREACHABLE PEAK

$$H = \frac{d}{\cot \varphi - \cot \theta} + h$$

Everything on the right-hand side is something I can measure standing on flat ground, with a protractor and a measuring tape. And once H is known, the first equation hands me D as a bonus — the distance to a point I can never reach.

05 Al-Bīrūnī at Nandana Fort

Abū Rayhān al-Bīrūnī

973 – c. 1048 CE · KATH, KHWAREZM (MODERN UZBEKISTAN)

A polymath of almost absurd range — astronomy, mathematics, geography, mineralogy, pharmacology, chronology, history — credited with something like 140 works. He spent his later career at the court of Mahmūd of Ghazni, travelled with it into north-western India, learned Sanskrit, and wrote the *Kitāb al-Hind*, a study of Indian learning, religion and custom so careful and so free of contempt that it is still read as a model of how to describe somebody else's civilisation.

His astronomical masterwork is the *al-Qānūn al-Mas'ūdī*. The hill measurement below comes from a different book, his treatise on fixing the coordinates of cities, written in the same years.

Al-Bīrūnī used this very method. Around 1025 CE he measured the height of Nandana hill, where Nandana Fort stands, in present-day Punjab, Pakistan. He obtained a height of roughly **300 m**.

But he did not stop there. He then walked up to the top of the hill and measured one more angle — the *dip angle of the horizon* — and from it worked out the radius of the Earth with stunning accuracy.

06 Interlude: how do we know the Earth is round?

Before we go further we should be honest about an assumption we are about to use. How did we know that the Earth is round — a sphere, a ball — in the first place?

There are countless ways in which people of the past concluded that the Earth must be round. The argument I want to present is one of Aristotle's, from about **350 BCE**.

Aristotle

384 – 322 BCE · STAGIRA AND ATHENS

Twenty years a student at Plato's Academy, then tutor to the young Alexander of Macedon, then founder of his own school in Athens, the Lyceum. He wrote on logic, physics, biology, ethics, poetry and politics, and for the better part of two thousand years he was simply *the* authority — which is a mixed blessing, since some of his physics was wrong and his reputation kept it alive.

The argument below is from *On the Heavens*, where he offers several reasons for a spherical Earth: the shadow at every lunar eclipse; the way a ship's hull vanishes over the horizon before its mast does; and the way the stars overhead change as you travel north or south — which also told him the Earth could not be very large.

Aristotle noticed that during *any* lunar eclipse, the Earth's shadow on the Moon is circular. Not sometimes circular — always, no matter the time of night or the season, and therefore no matter which way the Earth happens to be turned. Now, the only solid that casts a circular shadow in every orientation is a sphere. A disc would sometimes show its edge and cast a line; a cylinder would sometimes cast a rectangle. So the Earth must be a ball.



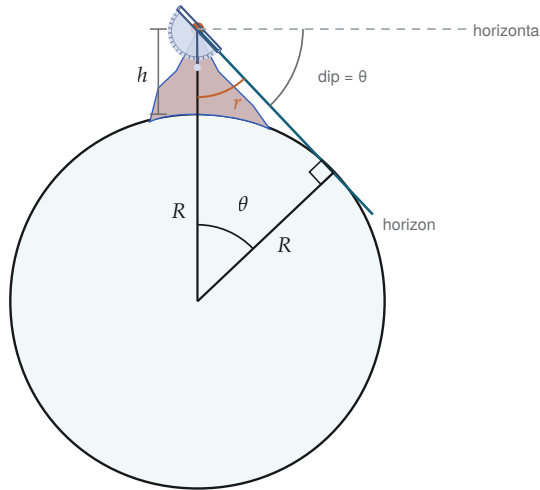
The evidence, photographed. At every stage of the eclipse the bite taken out of the Moon is bounded by an **arc of a circle** — and it is a circular arc at every eclipse, in every season, whichever way the Earth happens to be turned.

Photograph by Reddit user u/_wanderloots.

07 The radius of the Earth

Coming back to al-Bīrūnī. Standing on top of the hill, he sighted the horizon and read the angle by which his line of sight dipped *below* the horizontal. Call the reading on the instrument r as before.

The key geometrical fact is that the line of sight to the horizon just grazes the Earth. It is a tangent, and a tangent is perpendicular to the radius drawn to the point where it touches. That perpendicular hands us a right triangle: the observer's eye, the centre of the Earth, and the horizon point.



The hill height h is wildly exaggerated. The sight line to the horizon is a tangent, hence the right angle; r is what the instrument actually reads, the angle between the sight line and the string.

In that right triangle the hypotenuse is $R + h$ (centre to eye) and the side adjacent to the central angle θ is R . The string of the instrument lies along the line from the observer to the centre, so the reading r is the angle at the top of that triangle, and $\theta = 90^\circ - r$ as always. One bonus falls out of the same picture: the angle by which the sight line dips below the horizontal is exactly θ too. So

RADIUS OF THE EARTH FROM THE DIP OF THE HORIZON

$$\frac{R}{R + h} = \cos \theta \quad R = \frac{h \cos \theta}{1 - \cos \theta}$$

With $h = 300$ m and a dip of $\theta \approx 0.57^\circ$, this gives

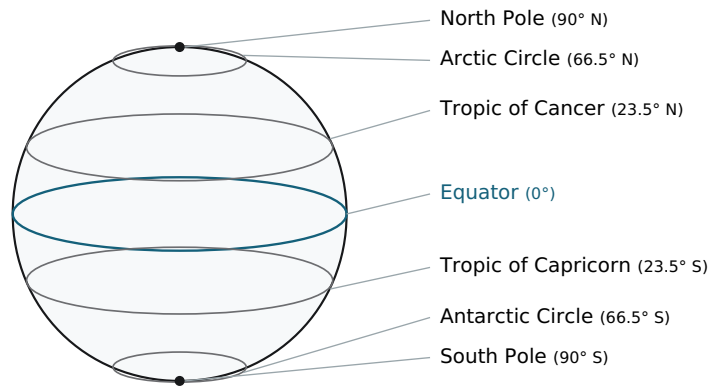
$$R \approx 6339 \text{ km}$$

The modern value is **6371 km**. He was remarkably accurate — less than 1 % error, from a hilltop, with a sighting instrument and a table of cosines.

CHECK THE ARITHMETIC 6339 km is al-Bīrūnī's own reported result, and the formula is very unforgiving near $\theta = 0$, because $1 - \cos \theta$ is tiny there. Substituting the rounded classroom values $h = 300$ m and $\theta = 0.57^\circ$ actually returns about 6060 km; you need a dip of about $0^\circ 33'$ (0.557°) to land back on 6339 km. Worth knowing in case a sharp student puts the numbers into a calculator — and worth making the point that here a hundredth of a degree moves the answer by hundreds of kilometres.

08 Latitude, and how to measure yours

Next we want an even earlier measurement: the *circumference* of the Earth. (And knowing the circumference and knowing the radius are essentially the same thing, since $C = 2\pi R$.) Before that, we need to talk about latitude, and how to measure our own with the DIY inclinometer.



The equator and the principal parallels.

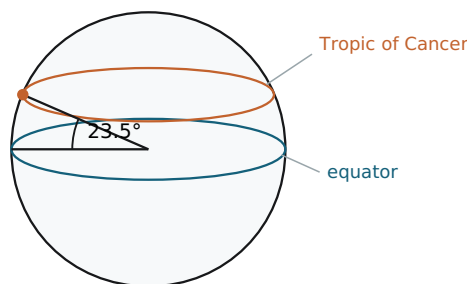
We have the *equator*, the imaginary circle that divides the Earth into two equal parts. Then there are the *parallels*: circles parallel to the equator, wrapped around the Earth to the north and to the south of it. Examples are the Tropic of Cancer, the Tropic of Capricorn, the Arctic Circle and the Antarctic Circle.

Presenter cue. Ask the teachers to translate these terms into Tamil before moving on.

There are certain numbers attached to these parallels — 0° to the equator, 23.5° N to the Tropic of Cancer, and so on. These numbers are called *latitudes*.

Q. Why is there a degree sign? Does latitude have anything to do with angles — or with temperatures?

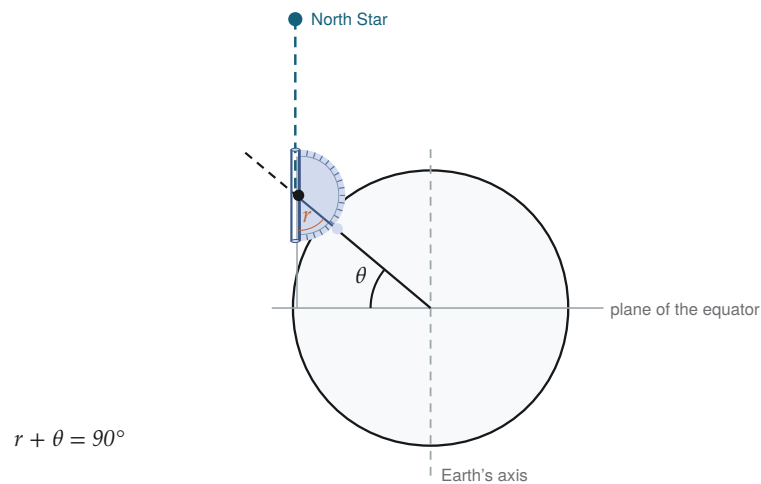
A. It is an angle, and here is the angle it is. Take any point on the parallel of latitude θ , and draw a straight line from it to the centre of the Earth. That line makes an angle θ with the plane of the equator. That is all latitude means.



Latitude is an angle at the centre of the Earth, measured up from the plane of the equator.

How can we measure our latitude?

At night, sight the Pole Star — the North Star, Polaris — through the straw and let the weight settle. Remember what the instrument reads: the straw lies along the line from your eye to the star, the string lies along the line from the centre of the protractor to the centre of the Earth, and r is the angle between those two lines.



What the instrument reads is r , between the star and the line to the centre of the Earth. The latitude θ is the angle at the centre — and the two are complementary.

Two facts turn that reading into your latitude. First, a hanging weight defines the local vertical, and the local vertical continued downwards passes through the centre of the Earth: the string lies along *your own radius*. Second, the Pole Star sits almost exactly on the Earth's axis, and it is so far away that its direction is the same from everywhere on Earth — so the straw ends up parallel to the axis.

The angle between your radius and the axis is therefore r . But your radius makes an angle θ with the plane of the equator, and the axis is perpendicular to that plane, so those two angles must add to a right angle:

$$r + \theta = 90^\circ \qquad \theta = 90^\circ - r$$

Read the scale, subtract from ninety, and you have your latitude. No instruments, no satellites, no permission required.

09 The circumference of the Earth

Two centuries before al-Bīrūnī climbed his hill, the same question had been attacked from flat ground, and on a much larger budget.

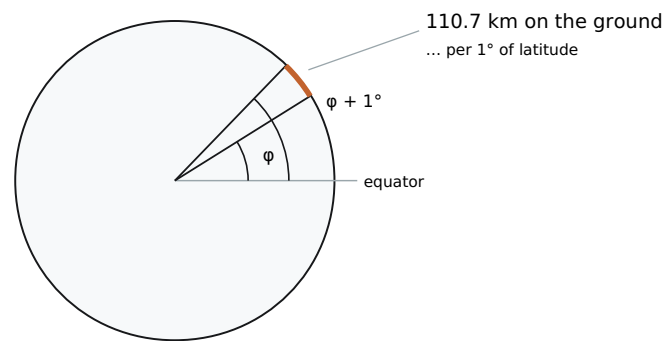
The caliph al-Ma'mūn

786 – 833 CE · BAGHDAD

Seventh of the Abbasid caliphs and a son of Hārūn al-Rashīd, he ruled from 813 until his death. What keeps his name in a mathematics talk is not his reign but what he paid for: the great translation movement that carried Greek, Persian and Indian science into Arabic, and the circle of scholars gathered around the Bayt al-Ḥikma, the House of Wisdom. Al-Khwārizmī worked in that circle — the man whose name became the word *algorithm* and the title of whose book became the word *algebra*.

Al-Ma'mūn's instinct was to check the Greeks rather than copy them. He had new astronomical tables compiled from fresh observations, and he sent out a survey expedition with a single instruction: go and measure the size of the Earth.

The method needed nothing but the latitude trick and a measuring rope. Two parties set out from the same point on a flat plain in northern Mesopotamia, one due north and one due south along the meridian, each measuring the ground distance covered, until the Pole Star stood exactly one degree higher or lower than it had at the start.



One degree of latitude, measured on the ground.

Their answer: one degree of latitude corresponds to about **110.7 km** of ground. So a full turn of 360° must be

$$110.7 \times 360 = 39\,852 \text{ km}$$

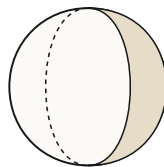
The true circumference of the Earth is **40 075 km** — an error of about half a per cent, obtained by walking.

WHERE 110.7 KM COMES FROM The surveyors' own result is recorded in Arabic miles, and converting it depends on the length of that mile, which is not known for certain; published conversions land between roughly 111 and 113 km to the degree. A degree of latitude today runs from about 110.6 km near the equator to 111.7 km near the poles, so every one of those figures is within a per cent of the truth. Any of them makes the point; 110.7 is the one used here.

10 Leaving Earth: is the Moon a ball?

Now let us finally leave the Earth and go to the Moon.

Q. The Moon definitely *looks* round. But how do we know it is a sphere and not a flat circular disc facing us?



A crescent: the terminator is an **ellipse**, not a straight line.

A. The lunar phases. A flat disc lit from the side would simply go dark along a straight line, or vanish edge-on. If the Moon were not a ball we would never get crescent-shaped shadows at all — that graceful curved boundary is the shadow of a sphere on itself.

Think about it!

11 Aristarchus: how far away is the Moon?

Aristarchus of Samos

C. 310 – C. 230 BCE · SAMOS AND ALEXANDRIA

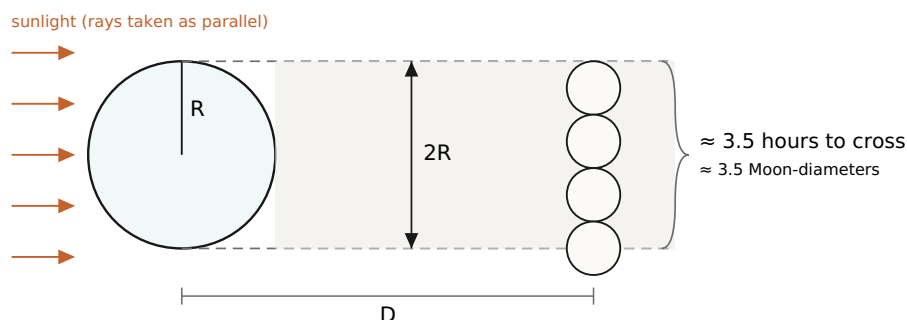
One work of his survives, *On the Sizes and Distances of the Sun and Moon*, in which he catches the Moon at the moment it is exactly half lit — the Sun, Earth and Moon then form a right angle — and measures the angle between Moon and Sun to get the ratio of their distances. His answer, that the Sun is about nineteen times further off than the Moon, is badly wrong (it is nearer four hundred), because that angle is desperately hard to measure by eye. The method is perfectly sound; only the instrument failed him.

For the thing he is most famous for we have to go to Archimedes, who reports in *The Sand Reckoner* that Aristarchus put the Sun, not the Earth, at the centre of things — some eighteen centuries before Copernicus.

His tool here is again the lunar eclipse. During a lunar eclipse the Moon passes through the shadow of the Earth. Aristarchus noticed that the maximum time the Moon takes to cross that shadow is around **3.5 hours**. The maximum matters: those are the eclipses in which the Moon passes straight through the middle of the shadow, along its full width, rather than clipping a corner of it.



The Moon photographed at intervals through one eclipse, laid out along its path: it enters the Earth's shadow at one side, passes through the middle of it (where it turns red), and leaves at the other. The width of that shadow and the time taken to cross it are the whole of the measurement.



The Earth's shadow, drawn as a cylinder of width $2R$, and the Moon crossing it.

If we take the Sun's rays to be parallel, the shadow is a cylinder as wide as the Earth, so the distance the Moon covers in that time is roughly $2R$. Hence the speed of the Moon is $2R/3.5$ per hour.

But we also know that the Moon goes once around the Earth in about 28 days — one full Moon to the next full Moon. In that time it covers a distance $2\pi D$, where D is its distance from us. So its speed is also $2\pi D / (28 \times 24)$ per hour. One speed, two expressions:

$$\frac{2\pi D}{28 \times 24} = \frac{2R}{3.5}$$

DISTANCE TO THE MOON

$$D = 61R \quad (\text{actual: } 60.3R)$$

So even with all the approximations — parallel rays, a cylindrical shadow, a circular orbit, 28 days flat — he was quite accurate.

12 How big is the Moon?

Two methods, both Greek, both using nothing more than a clock.

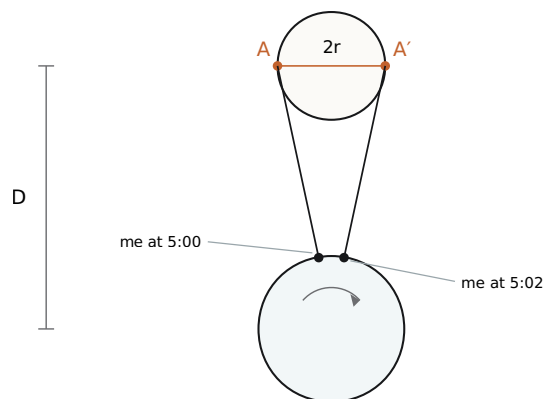
Method 1 — the eclipse again

Look back at the eclipse picture. Not only does the Moon take about 3.5 hours to cross the shadow, it takes about one hour to move its own diameter: roughly three and a half Moons would fit across the Earth's shadow. So

$$\frac{\text{Earth's diameter}}{\text{Moon's diameter}} \approx 3.5 \quad r \approx \frac{R}{3.5} \approx 1820 \text{ km}$$

Method 2 — two minutes of moonrise

Greek astronomers noticed that it takes about **2 minutes** for the Moon to rise fully: at 5:00 p.m. the first sliver appears over the hills, and by 5:02 p.m. the whole disc is clear of them. Now, this apparent movement of the Moon is due mostly to the rotation of the Earth, not to the speed of the Moon itself. Which means:



Earth's rotation carries my line of sight across the Moon from A to A' in two minutes.

The point A at which my line of sight meets the Moon scans across the whole Moon in 2 minutes, so in that time A travels a distance $2r$, where r is the Moon's radius. But we also know that the point A comes back to the same place in 24 hours, and in doing so it covers the distance $2\pi D$. Therefore

$$\text{speed of } A = \frac{2r}{2} = \frac{2\pi D}{24 \times 60}$$

RADIUS OF THE MOON

$$r = \frac{2\pi D}{24 \times 60} \approx 1700\text{--}1800 \text{ km} \quad (\text{actual: } 1737 \text{ km})$$

CHECK THE ARITHMETIC With $D = 61R$, Method 2 gives $r \approx 1700$ km; Method 1 gives $r \approx 1820$ km. The true value, 1737 km, sits neatly between them, so it is worth quoting both and letting the class see the two independent routes bracket the answer.

13 Scoreboard

MEASURED	THEIR ANSWER	MODERN VALUE	ERROR
Radius of the Earth al-Bīrūnī, c. 1025 CE	6339 km	6371 km	0.5 %
Circumference of the Earth al-Ma'mūn's surveyors, 9th c. CE	39 852 km	40 075 km	0.6 %
Distance to the Moon Aristarchus, 3rd c. BCE	61 R	60.3 R	1.2 %
Radius of the Moon Greek astronomers	1700–1820 km	1737 km	< 5 %

Every number in that table came out of the same small kit: a right triangle whose side ratios depend only on its angles; one length that can actually be walked or measured; and one or two angles read off a protractor with a straw taped to it. Everything else was the willingness to be lazy in the right way — to look for the measurement you *can* make instead of the one you cannot.