An Example of Hyperbolic Consevation Law: Single Conservation Law

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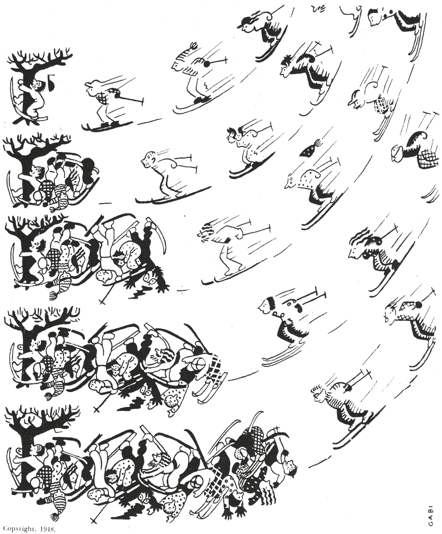
Introduction - Receding shock wave



RECEDING SHOCK WAVE

See Courant – Friedrichs, *Supersonic Flow and Shock Waves*, page 129

Introduction - Sonic boom experimental picture

Genuine Nonlinearity

- The theory of conservation laws is intimately related to wave propagation with genuine nonlinearity.
- In fact, the theory was developed because genuine nonlinearity led to a mathematical difficulty which could not be overcome without this theory.
- We shall explain genuine nonlinearity later.

Reference

Many books as text books, but at your level I mention only:

Phoolan Prasad and Renuka Ravindran, Partial Differential Equations, New Age International Publishers, section 3.5.

Phoolan Prasad, Nonlinear Hyperbolic Waves in Multi-dimensions, Chapman & Hall/CRC, 2001, Chapter 1.

Prasad, P. (1997) *Nonlinearity, Conservation Law and Shocks*, RESONANCE- Journal of Science Education by Indian Academy of Sciences, Bangalore,
Part I: Genuine Nonlinearity and Discontinuous Solutions, Vol-2, No.2, 8-18;
Part II: Stability Consideration and Examples, Vol-2, No.7, 8-19.

Soft copy of last three will be to be given to you.

J. D. Logan, An Introduction to Nonlinear PDE, Wiley, 2008 a book with many engineering applications.

Simplest wave equation

$$u_t + c\,u_x = 0\ ,\ \ c = \text{real constant} \tag{1}$$

Method of characteristics for first order PDE gives

$$u = \phi(x - ct),\ \ \phi : \mathbb{R} \to \mathbb{R} \tag{2}$$

When $\phi \ \in \ C^1(\mathbb{R}) \Rightarrow$ Genuine solution.

(2) represents a wave every point of which moves with the same constant velocity c.

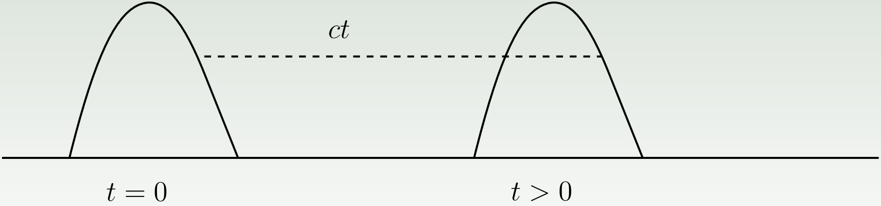


Figure: This figure does not reprent a genuine solution. Why?

Simplest wave equation contd..

When $\phi \notin C^1(\mathbb{R}) \Rightarrow$ Generalized or weak solution

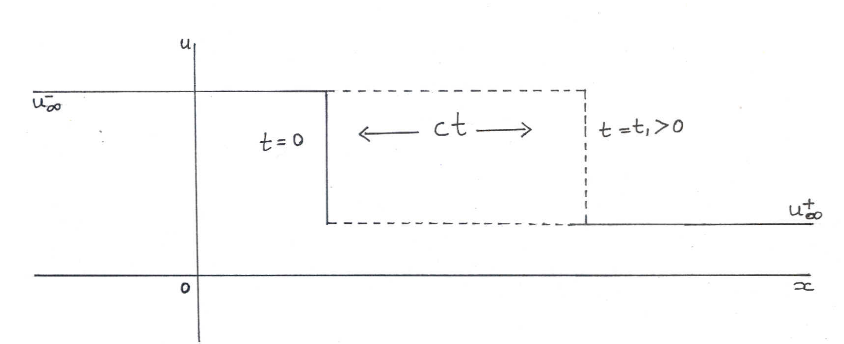


Figure: A generalized solution can even be discontinuous or a distribution like $\delta-$ function.

Genuine Nonlinearity

Airy or Euler equation[1] (Burgers equation is not an appropriate name)

$$u_t + uu_x = 0 \tag{3}$$

This quasi-linear equation contains *genuine nonlinearity*, a name given by Peter Lax. He precisely defined genuinely nonlinear characteristic field.

Physical interpretation: The velocity of a wave, in a particular mode of propagation containing genuine nonlinearity, depends on the amplitude of the wave.

Note[1] Jerry Bona told me about Airy's and Vladimir Arnold about Euler's point of view.

Genuine Nonlinearity

General solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$

$$u = f(x - ct) + g(x + ct)$$

shows that it has two modes of propagation.

Q.1 What are these modes?
Q.2 Are they genuinely nonlinear?

Genuine Nonlinearity

General solution of (3) is

$$u = f(x - ut)$$

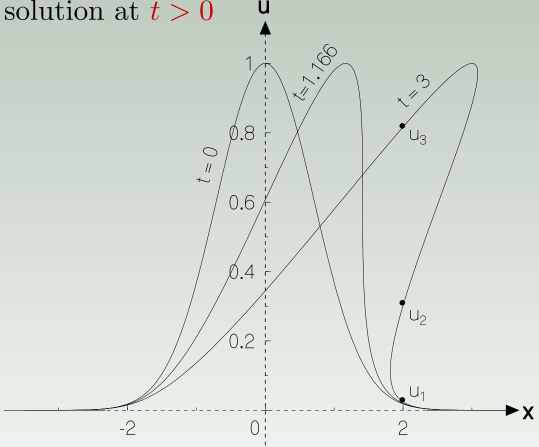
where the velocity of propagation is equal to the amplitude u of the wave.

Genuine nonlinearity is different from nonlinearity present in a semilinear equation i.e., $u_t + cu_x = u^2$. Already discussed in previous lecture.

Explicit solution of an initial value problem for (3) is very involved as pointed out my lecture on first order PDE.

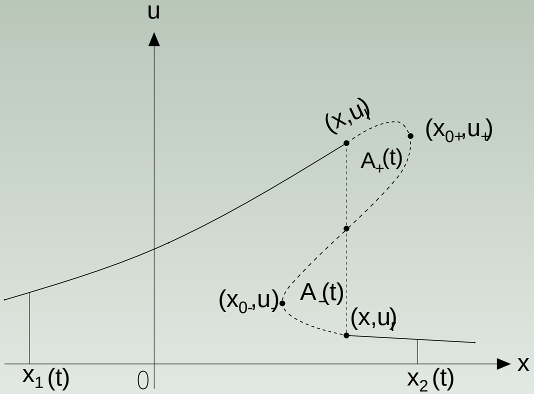
Nonlinear Deformation due to Genuine Nonlinearity

Graphs of the solution at $t > 0$



The pulse now deforms as t increases and at $t = 3$, and the solution becomes multi-valued after a critical time t_c (in this case $t_c \approx 1.166$) the graph does not represent any solution.

Nonlinear deformation contd..



When the graph folds at a large time, we need to interpret the solution as a weak solution with a discontinuity which is a shock.
We shall present a very simple theory (original one before general theory was developed in 1951 by a Fields medalist Lawrence Schwartz).

Nonlinear deformation contd..

EXAMPLE 1:

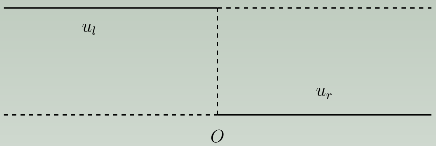


Figure: Initial data with a discontinuity at $x = 0$.

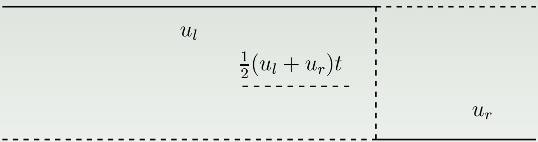


Figure: A solution containing shock. Solution, which is unique, stable and remains discontinuous. The discontinuity moves with velocity $S = \frac{1}{2}(u_l + u_r)$.

Diffusion: Burgers equation

- Simplest example containing diffusion is the one dimensional heat conduction equation (do you know that heat diffuses irreversibly?) equation

$$u_t = \nu u_{xx}, \;\; \nu = \text{real} \;\; \text{and} \;\; > 0. \tag{4}$$

As time increases, concentration of u diffuses.

- However, I shall mention here Burgers equation (1948)

$$u_t + u \; u_x = \nu u_{xx} \tag{5}$$

- In this both genuine nonlinearity and diffusion are present.

Diffusion: Burgers equation contd..

The discontinuous solution on slide 13 becomes

$$\begin{aligned} u(x,t) =& \frac{1}{2}(u_\infty^- + u_\infty^+) \\ & - \frac{1}{2}(u_\infty^- - u_\infty^+) \ \tanh \ \left[\frac{u_\infty^- - u_\infty^+}{4\nu}\left\{x - \frac{1}{2}(u_\infty^+ + u_\infty^-)t\right\}\right] \end{aligned}$$

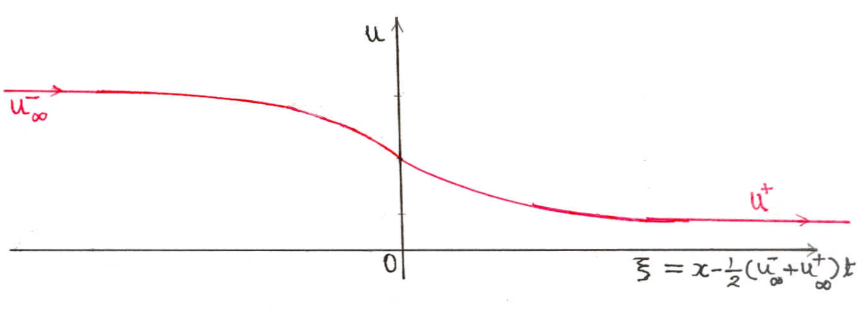


Figure: The genuine nonlinearity and diffusion balance each other and the discontinuous shock profile becomes a *steady* $C^\infty(\mathbb{R})$ solution.

Critical time t_c for the break down of a genuine solution in an equation having genuine nonlinearity

We have seen that a genuine solution of the initial value problem

$$u_t + uu_x = 0,\ u(x,0) = \phi(x)$$

may break down at a finite critical time.

The critical time t_c can be calculated easily in terms of largest negative value of the derivative of the initial data $\phi(x)$.

New mathematical formulation

- Physical phenomena governed by equations having genuine nonlinearity are many.

- It is observed that beyond t_c some state variables represented by $\mathbf{u}$ become discontinuous with finite values on the two sides of the discontinuity and hence are not differentiable.

- We have shown such a solution graphically on slide 11.

Conservation law in one space dimension

- Consider density $H(u)$ and flux $F(u)$ of the state $u(x,t)$ of a physical system. We shall take the conservation law in form

$$\frac{d}{dt}\int_{x_1}^{x_2} H(u(\xi,t))d\xi = F(u(x_1,t)) - F(u(x_2,t)) \qquad (6)$$

  for arbitrary points x_1 and x_2.

- This means that the time rate of change of total quantity $H(u)$ contained in the interval x_1 and x_2 is equal to the difference of flux $F(u(x_2,t)$ going out of the interval at the point x_2 and flux $F(u(x_1,t)$ entering at point x_1.

- Interpret this formulation for the heat density $H(u)$ and heat flux $F(u_x)$ in one-dimensional rod. We shall take up a simpler case when we have $F(u)$.

Conservation law in one space dimension contd...

Problem: If the functions H and F are differentiable and state u is also so, show that the conservation law implies

$$\frac{\partial H(u)}{\partial t} + \frac{\partial F(u)}{\partial x} = 0. \tag{7}$$

Further if

$$H'(u) \equiv \frac{dH}{du} \neq 0 \text{ and } \frac{F'(u)}{H'(u)} = u, \tag{8}$$

we get the equation with genuine nonlinearity.

$$u_t + uu_x = 0 \tag{9}$$

Definition: Equation (7) is called a **conservation law**, by which we mean the equation (6).

Generalized or weak solution of Conservation Law

Important Result stated in simple words: A function $f(x)$ with countable number of finite discontinuities in a closed interval $[x_1, x_2]$ is integrable and

$$\int_{x_1}^{x_2} H(u(\xi,t))d\xi \quad exists.$$

Definition A generalized or weak solution of the conservation law (7) is a bounded integrable function $u(x,t)$, which satisfies the integral equation (6) for all choices of x_1, x_2.

Theorem

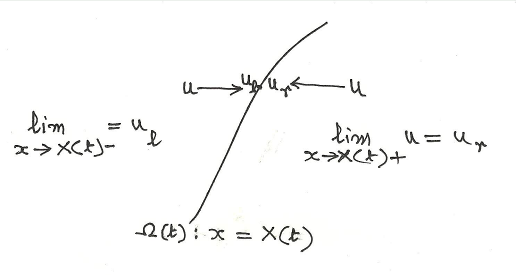
Every differentiable weak solution of (7) with $\{F'(u)/H'(u)\} = u$, is a genuine solution of the PDE $u_t + uu_x = 0$.

Nature of a discontinuous weak solution

- Let $u(x,t), u_x(x,t)$ and $u_t(x,t)$ be bounded and discontinuous across a single smooth curve

$$\Omega : x = X(t)$$

  and are C^1 in the rest of the (x,t) -plane.
- We assume that limiting values of u, u_x and u_y as we approach Ω from either side exist.
- u is a genuine solution of $u_t + uu_x = 0$ in $\mathbb{R}^2 \setminus \Omega$



Derivation of Jump relation

Let $x_1 < X(t) < x_2$ for $t \in$ an open interval.

$$\int_{x_1}^{x_2} H(u(\xi,t))d\xi = \int_{x_1}^{X(t)} H(u(\xi,t))d\xi + \int_{X(t)}^{x_2} H(u(\xi,t))d\xi \quad (10)$$

From (25)

$$\int_{x_1}^{X(t)} H' u_t(\xi,t)d\xi + \int_{X(t)}^{x_2} H' u_t(\xi,t))d\xi + \dot{X}(t))\big\{H(u(X(t)-,t)) \\ -H(u(X(t)+,t))\big\} = \{F(u(x_1,t)) - F(u(x_2,t))\} \quad (11)$$

First two terms on the left hand side tend to zero as $x_1 \to X(t)-$ and $x_2 \to X(t)+$ and (11) $\Rightarrow$ Jump relation:

$$\dot{X}(t)(H(u_l) - H(u_r)) = F(u_l) - F(u_r) \quad (12)$$

Jump relation or Rankine-Hugoniot condition

$$\dot{X}(t) = [F]/[H] \tag{13}$$

where

$$[f] = f(u_r) - f(u_l). \tag{14}$$

The Airy (inviscid Burgers) equation

$$u_t + uu_x = 0 \tag{15}$$

leads to the conservation law

$$u_t + (\frac{1}{2}u^2)_x = 0, \tag{16}$$

which is a particular case of

$$(u^n)_t + (\frac{n}{n+1}u^{n+1})_x = 0, \;\; n = +\text{ve integer}. \tag{17}$$

Jump relation or Rankine-Hugoniot condition contd...

Jump relations from these conservation laws give

$$\dot{X}(t) = \frac{1}{2}\frac{u_r^2 - u_l^2}{u_r - u_l} = \frac{1}{2}(u_r + u_l) \tag{18}$$

$$\dot{X}(t) = \frac{n}{n+1}\left(\sum_{i=0}^{n} u_r^{n-i} u_l^i\right) \Big/ \left(\sum_{i=0}^{n-1} u_r^{n-1-i} u_l^i\right). \tag{19}$$

For $n = 2$

$$\dot{X}(t) = \frac{2}{3}\ \frac{u_r^2 + u_r u_l + u_l^2}{u_r + u_l} \tag{20}$$

Same differential equation, but different conservation laws equivalent to it, give different jump relations.

Example 1

Consider conservation law (16), i.e.

$$u_t + (\frac{1}{2}u^2)_x = 0$$

with a discontinuous initial data

$$u(x,0) = \begin{cases} 0, & x \leq 0 \\ 1, & 0 < x. \end{cases} \tag{21}$$

Example 1: One Solution

One solution is discontinuous one on the line $x = \frac{1}{2}t$

$$u(x,t) = \begin{cases} 0 \quad , & x \leq 0 \\ 1 \quad , & \frac{1}{2}t < x \end{cases} \tag{22}$$

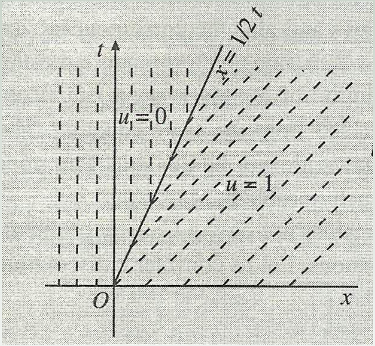


Figure: Two constant states are separated by a line of discontinuity on $x = \frac{1}{2}t$ satisfying the jump condition (18)

Example 1: One Continuous Solution

But you can have a continuous (not genuine) solution also - the initial discontinuity at $x = 0$ is immediately resolved

$$u(x,t) = \begin{cases} 0 \ , & x \le 0 \\ x/t \ , & 0 < x \le t \\ 1 \ , & t < x \end{cases} \tag{23}$$

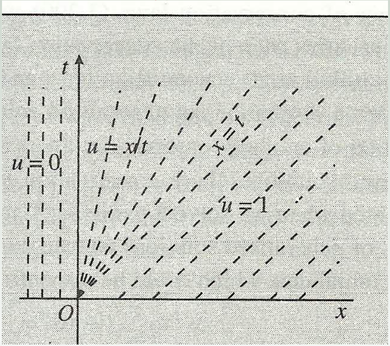


Figure: Two constant states are separated by a centred simple wave.

Example 1: Non-unique Solution Contd...

Are you surprised at two solutions? See now an infinity of discontinuous solutions

$$u(x,t) = \begin{cases} 0 \;, & x \le 0 \\ x/t \;, & 0 < x \le \alpha t \\ \alpha \;, & \alpha t < x \le \frac{1}{2}(1+\alpha)t \\ 1 \;, & \frac{1}{2}(1+\alpha)t < x \end{cases} \tag{24}$$

where α is a constant $0 \le \alpha \le 1$.

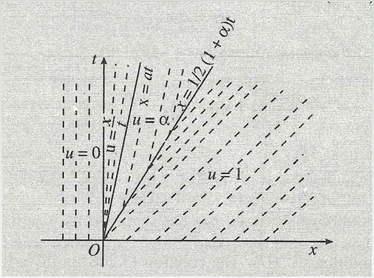


Figure: Three constant states are separated by a centred simple wave and a line of discontinuity on $x = \frac{1}{2}(1+\alpha)t$ satisfying the jump condition (18).

Example 1 contd...

- There is only one line of discontinuity in u for $0 \leq \alpha < 1$, with slope $\frac{x}{t} = \frac{1}{2}(1 + \alpha)$.
- For $\alpha = 1$, it is a continuous weak solution (26) of the conservation but not a genuine solution of the PDE $u_t + uu_x = 0$.
- Why?
- There is a centered fan $u = \frac{x}{t}$ in $0 < x < \alpha t$.

Example 2

Solve

$$u_t + (\frac{1}{2}u^2)_x = 0, \ (x,t) \in \mathbb{R} \times \mathbb{R}_+ \tag{25}$$

with

$$u(x,0) = \begin{cases} 1, & x \leq 0 \\ 0, & x > 0 \end{cases} \tag{26}$$

We get only one solution of this problem as

$$u(x,t) = \begin{cases} 1, & x - \frac{1}{2}t \leq 0 \\ 0, & x - \frac{1}{2}t > 0 \end{cases} \tag{27}$$

Example 3 ⋯ conti.

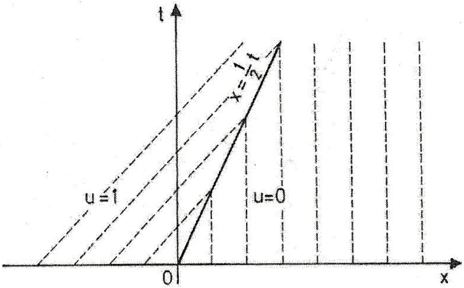


Fig. 1.4.3 : Solution (1.4.7) with characteristic curves shown by broken lines.

Mathematical criterion for uniqueness of a weak solution

- In Example 1, case $\alpha = 0$ a particular case of $\alpha < 1$, the characteristics diverge away from the points on the discontinuity line $x = \frac{1}{2}t$. These characteristic do not carry any information from the initial data at $t = 0$.

- In Example 1, case $\alpha = 1$, we have a discontinuous solution. In the centered fan, the information comes from the initial data, though just from one point.

- In Example 2, too much information comes from the initial data. The discontinuity curves $x = \frac{1}{2}t$ appears and avoids multi-valuedness of the solution.

Mathematical criterion for uniqueness of a weak solution contd...

Mathematical criterion is not to accept any discontinuity in u in a weak solution from where characteristic diverge as t increases.

- Thus all weak solutions in example 1 except for $\alpha = 1$ are inadmissible. We get a unique continuous (but not genuine) solution of the problem and it is for $\alpha = 1$
- In example 2, the only discontinuous solution is admissible.

Definition. A discontinuity in an admissible weak solution is called a **shock**.

For a shock we have

$$u_r < u_l. \tag{28}$$

This is a a necessary and sufficient condition for existence and uniqueness of a weak solution.

Entropy condition for existence and uniqueness of a weak solution

Entropy condition
In gas dynamics, an expansion shock was mathematically derived about 160 years back. But only in 2010 it was shown (Rayleigh) that an expansion shock violated the second law of thermodynamics and the only acceptable shock was a compression shock across which entropy of fluid elements increased.

The condition $u_r < u_l \Rightarrow$

$$u_r < S(t) < u_l \tag{29}$$

where velocity of discontinuity $\dot{X}(t)$ is denoted by $S(t)$.

Definition: (29) is called entropy condition of Lax[1].

[1] Peter Lax is the greatest living applied mathematician toady.

Non-equivalence of conservation laws for weak solutions

Solve the Cauchy problem (initial value problem)

$$(u^2)_t + \left(\frac{2}{3}u^3\right)_x = 0, \;\; (x,t) \in \mathbb{R} \times \mathbb{R} \tag{30}$$

$$u(x,0) = \begin{cases} 1, & x \leq 0 \\ 0, & x > 0. \end{cases} \tag{31}$$

Shock velocity

$$S(t) = \frac{2}{3} \quad \frac{u_l^2 + u_l u_r + u_r^2}{u_l + u_r} = \frac{2}{3} \tag{32}$$

Solution is

$$u(x,t) = \begin{cases} 1, & x \leq \frac{2}{3}t \\ 0, & \frac{2}{3}t < x \end{cases} \tag{33}$$

Non-equivalence of conservation laws for weak solutions contd...

- Though the Cauchy data is same as that in example 2, the solution is different.
- Note that differential forms of

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \quad \text{and} \quad (u^2)_t + \left(\frac{2}{3}u^3\right)_x = 0 \tag{34}$$

are same PDE

$$u_t + uu_x = 0 \tag{35}$$

- Genuine solutions of these two conservation laws with same initial data are same but we see that the two conservation laws are not equivalent for a weak solution.
- From now onwards in all examples we shall consider the only one conservation law

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \tag{36}$$

Irreversibility of a weak solution

For the initial data

$$u(x,0) = \begin{cases} 2\ , & x \le \frac{1}{4} \\ 0\ , & x > \frac{1}{4} \end{cases} \tag{37}$$

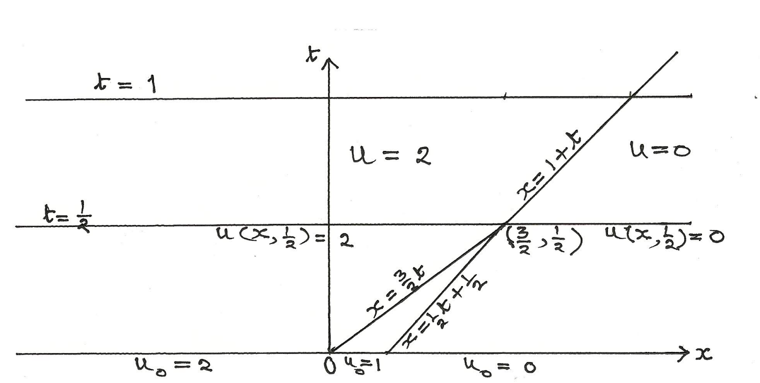
The solution is

$$u(x,t) = \begin{cases} 2\ , & x \le t + \frac{1}{4} \\ & \\ 0\ , & t + \frac{1}{4} < x \end{cases} \qquad \text{for } t > 0 \tag{38}$$

Irreversibility of a weak solution contd...

For the initial data

$$u(x,0) = \begin{cases} 2\ , & x \le 0 \\ 1\ , & 0 < x \le \frac{1}{2} \\ 0\ , & \frac{1}{2} < x \end{cases} \tag{39}$$



Irreversibility of a weak solution contd...

Solution, depicted in the figure is:

$$u(x,t) = \begin{cases} 2\,, & x \le \frac{3}{2}t \\ 1\,, & \frac{3}{2}t < x \le \frac{1}{2}t + \frac{1}{2}, \\ 0\,, & \frac{1}{2}t + \frac{1}{2} < x \end{cases} \qquad \text{for } \; 0 < t < \frac{1}{2} \tag{40}$$

$$u(x,t) = \begin{cases} 2\,, & x \le t + \frac{1}{4} \\ 0\,, & t + \frac{1}{4} < x. \end{cases} \qquad \text{for } t > \frac{1}{2}$$

Irreversibility of a weak solution contd...

- Solutions of the two initial data, though different for $0 < t < \frac{1}{2}$, are same for $t > \frac{1}{2}$
- Thus same state for any $t > \frac{1}{2}$ correspond to two different initial data and states between $0 < t < \frac{1}{2}$. Infact we can construct infinity of initial data leading to the same solution after some time in future.

This shows irreversibility - past can not be determined uniquely by future.

Example 3.

Initial data

$$u(x,t) = \begin{cases} 1\ , & x \leq -\frac{1}{2} \\ \frac{1}{2} - x\ , & -\frac{1}{2} < x \leq \frac{1}{2} \\ 0\ , & x > \frac{1}{2} \end{cases} \tag{41}$$

Solution remains continuous for $0 \leq t < 1$

$$u(x,t) = \begin{cases} 1\ , & x \leq -\frac{1}{2} + t \\ \frac{(1/2)-x}{1-t}\ , & -\frac{1}{2} + t < x \leq \frac{1}{2} \\ 0\ , & x > \frac{1}{2} \end{cases} \tag{42}$$

Example 3 contd...

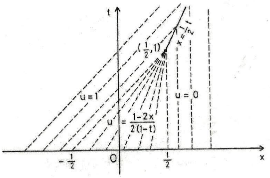


Fig. 1.5.1 : All characteristic curves starting from the points of the initial data between $-\frac{1}{2} \leq x \leq \frac{1}{2}$ meet at the point $\left(\frac{1}{2}, 1\right)$.

Data at $t = 1$

$$u(x,1) = \begin{cases} 1\ , & -\infty < x \leq \frac{1}{2} \\ 0\ , & \frac{1}{2} < x < \infty \end{cases} \tag{43}$$

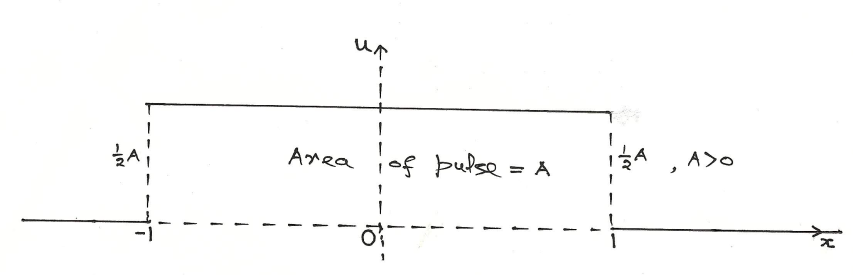
For $t \geq 1$, the solution has a single shock at

$$x = X(t) \equiv \frac{1}{2} t \tag{44}$$

Example 4.

Initial data

$$u(x,0) = \begin{cases} \frac{1}{2}A, & -1 < x \le 1,\ A > 0 \\ 0\ , & x \le -1 \text{ and } x > 1 \end{cases} \tag{45}$$



Example 4. contd...

Solution for $0 < t \leq \frac{8}{A}$

$$u(x,t) = \begin{cases} 0 \ , & x \leq -1 \\ \frac{x+1}{t} \ , & -1 < x \leq -1 + \frac{A}{2}t \\ \frac{1}{2}A \ , & -1 + \frac{A}{2}t < x \leq 1 + \frac{A}{4}t \\ 0 \ , & 1 < x \end{cases} \quad (46)$$

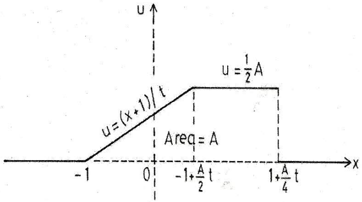
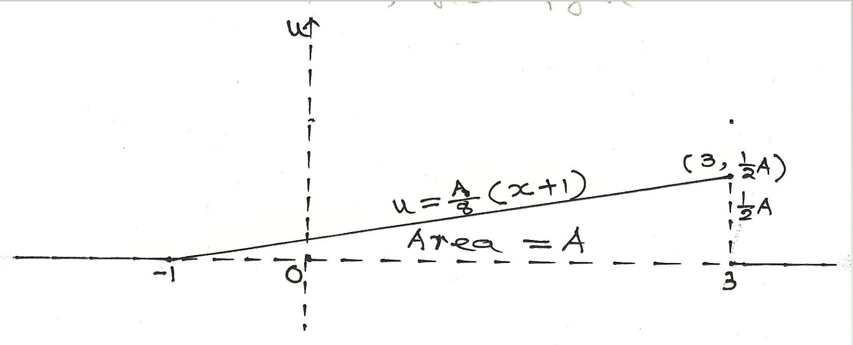


Fig. 1.5.2 : Graph of the solution with initial value (1.5.3) valid in the time interval $0 < t < \frac{8}{A}$.

Example 4. contd...

Solution at $t = 8/A$



The centered wave overtakes the shock at $x = 3$.

Example 4. contd...

For $t > \frac{8}{A}$, shock path $x = X(t)$ is given by

$$\frac{dX}{dt} = \frac{1}{2}(u_l(X(t)) + u_r(X(t)) = \frac{X+1}{2t}\,,\;\; X\left(t = \frac{8}{A}\right) = 3$$

which gives

$$X(t) = -1 + \sqrt{2At} \tag{47}$$

At $x = X(t)$, the shock strength $u_l - u_r = u_l$ is given by

$$u = \frac{x+1}{t}\Big|_{x=X(t)} = \sqrt{\frac{2A}{t}} \tag{48}$$

Example 4. contd...

For $t > \frac{8}{A}$, the pulse has a triangular shape

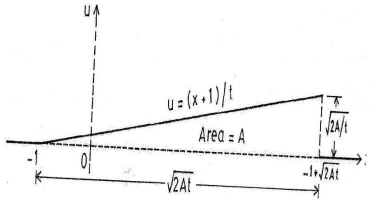


Fig. 1.5.3 : Graph of the solution with initial condition (1.5.3) valid from $t > \frac{8}{A}$.

Base length $=\sqrt{2At}$, Hight $= \sqrt{\frac{2A}{t}}$.

The solution u and shock strength decay to zero asymptotically as $\sim \frac{1}{\sqrt{t}}$.

Example 4. contd...

- For $t > \frac{8}{A}$, area of the pulse in (x, u)-plane

$$= \frac{1}{2}\sqrt{2At}.\sqrt{\frac{2A}{t}} = A$$

- This **important result** agrees with the general property of the conservation law $u_t + (\frac{1}{2}u^2)_x = 0$ i.e.,

$$\frac{d}{dt}\int_{\infty}^{\infty} u(\xi, t)d\xi = \frac{1}{2}u^2(-\infty, t) - \frac{1}{2}u^2(\infty, t) = 0, \qquad (49)$$

which is the initial area of the pulse.

- When the solution vanishes outside a closed bounded interval of x-axis, $\int_{\infty}^{\infty} u(\xi, t)d\xi$ is independent of t.

Example 5.

$$u(x,0) = \begin{cases} 0\ , & -\infty < x < -1 \\ -x-1\ , & -1 < x \le -\frac{1}{2} \\ x\ , & -\frac{1}{2} < x \le 1 \\ -x+2\ , & 1 < x \le 2 \\ 0\ , & 2 < x < \infty \end{cases} \tag{50}$$

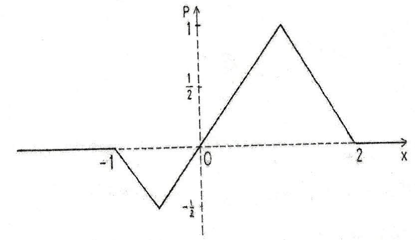


Fig. 1.5.4 : Graph of the initial data (1.5.7).

Example 5 contd...

For $t < 1$

- Solutions remains continuous. Explain Why?
- The points $-1,\ 0$ and 2 on x-axis remain fixed as pulse evolves
- Solution is zero outside the interval (-1,2).
- At $t = 1$, shocks appear at $x = -1$ and $x = 2$.
- Solution can be easily written.

Example 5 contd...

For $t > 1$,

$$u(x,t) = \begin{cases} 0\ , & -\infty < x \leq -\sqrt{\frac{1}{2}(1+t)} \\ \frac{x}{1+t}\ , & -\sqrt{\frac{1}{2}(1+t)} < x < \sqrt{2(1+t)} \\ 0\ , & \sqrt{2(1+t)} < x < \infty \end{cases} \tag{51}$$

with shocks at leading and trailing ends and shock strengths $\sqrt{\frac{2}{1+t}}$ and $\frac{1}{\sqrt{2(1+t)}}$ respectively.

Example 5 contd...

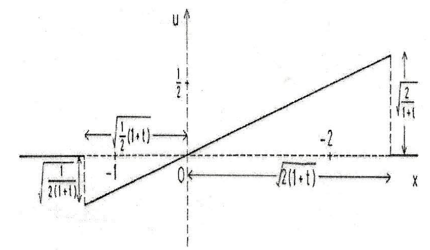


Fig. 1.5.5 : A N-wave solution (1.5.8) at $t = 3$.

Area of positive pulse $= \frac{1}{2}\sqrt{2(1+t)}\sqrt{\frac{2}{1+t}} = 1$

Area of negative pulse $= \frac{1}{2}\sqrt{\frac{1+t}{2}}\frac{1}{\sqrt{2(1+t)}} = \frac{1}{4}$

Areas on two sides of $x = 0$ are preserved. Why?

The figure represents an N-wave.

The solution u and shock strength decay to zero asymptotically $\sim \frac{1}{\sqrt{t}}$

Example 6

$$u(x,0) = \begin{cases} 0\ , & -\infty < x \le -\lambda \\ 2a(x+\lambda)/\lambda\ , & -\lambda < x \le -\frac{1}{2}\lambda \\ -2ax/\lambda\ , & -\frac{1}{2}\lambda < x \le \frac{1}{2}\lambda \\ 2a(x-\lambda)/\lambda & \frac{1}{2}\lambda < x \le \lambda \\ 0\ , & \lambda < x < \infty \end{cases} \tag{52}$$

where $\lambda > 0$ and $a > 0$.

Value of u at $x = -\frac{1}{2}\lambda$ is a and that at $x = \frac{1}{2}\lambda$ is $-a$.

Example 6 contd...

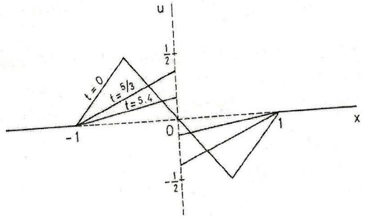


Fig. 1.5.6 : The initial pulse is given by (1.5.9) with $\lambda = 1, a = \frac{1}{2}$. The solution develops a shock at the origin which decays to zero with time as $0(\frac{1}{t})$.

$u(x,0)$ is drawn with $\lambda = 1$ and $a = \frac{1}{2}$.

The solution remains continuous for $0 < t < \frac{\lambda}{2a}$. Why?

At $t = \frac{\lambda}{2a}$, a shock with strength $2a$ appears at $x = 0$.

Example 6 contd...

- Solution in the interval $-\lambda < x \leq -\frac{1}{2}\lambda + at$ and in $-\lambda < x \leq 0$ for $t > \frac{\lambda}{2a}$ is given by

$$u(x,t) = \frac{x+\lambda}{t+\frac{\lambda}{2a}} \tag{53}$$

- Similar expression of $u(x,t)$ for $0 < x \leq \lambda$.
- For $t > \frac{\lambda}{2a},\ u_r(t) = -u_l(t) = \frac{\lambda}{t+\frac{\lambda}{2a}}$.
- Shock strength at $x = 0$ for $t > \frac{\lambda}{2a}$ is

$$u_l - u_r = 2\lambda/(t + \frac{\lambda}{2a}) \tag{54}$$

The solution u and shock strength decay to zero asymptotically as $\sim \frac{1}{t}$.

Example 6 contd...

- The solution remembers the length 2λ of the interval $(-\lambda, \lambda)$ of the x-axis. Why?
- The solution decays for large t as

$$u(x,t) \sim \begin{cases} \frac{x+\lambda}{t}, & -\lambda < x \leq 0 \\ \frac{x-\lambda}{t}, & 0 < x < \lambda \end{cases} \tag{55}$$

- If the initial data in $(-\lambda, \lambda)$ is periodically extended in $(-\infty, -\lambda)$ and (∞, λ), the solution evolves independently in each period $((2n-1)\lambda, (2n+1)\lambda)$.
- This gives a hope for finding asymptotic shape of the solution from a general periodic shape.

Example 7

$$u(x,0) = -a\sin\frac{\pi x}{\lambda};\ \lambda > 0, a > 0 \tag{56}$$

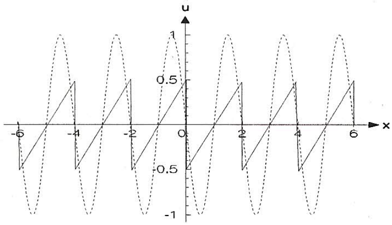


Fig. 1.5.7 : The saw-tooth solution arising from a periodic initial data shown by a dotted line.

Example 8 contd...

- We can not solve this problem in terms of known functions.
- From the example 6 we conclude that the solution remains periodic and the period 2λ is preserved.
- The asymptotic form of the solution in period $-\lambda < x < \lambda$ is

$$u(x,t) = \begin{cases} (x+\lambda)/t \ , & -\lambda < x \leq 0 \\ (x-\lambda)/t \ , & 0 < x \leq \lambda \end{cases} \tag{57}$$

General remarks contd...

- We have given a mathematically very simple definition of weak solution and entropy condition. The mathematical theory of hyperbolic conservation laws is very sophisticated today.
- The subject has many open research problems of great practical important but they are too difficult mathematically.
- The theory of conservation laws in multi-space-dimensions is fascinating.

Problems Solve $u_t + \left(\frac{1}{2}u^2\right)_x = 0$ **with following initial conditions**

1.
$$u(x,0) = \begin{cases} 1 & , \quad -\infty < x \le 0 \\ 2 & , \quad 0 < x \le 1 \\ 0 & , \quad x > 1 \end{cases}$$

2.
$$u(x,0) = \begin{cases} 0 & , \quad |x| \ge 1 \\ -1 & , \quad -1 < x < 0 \\ 1 & , \quad 0 < x < 1 \end{cases}$$

3.
$$u(x,0) = \begin{cases} 0 & , \quad -\infty < x \le -1 \\ 1 & , \quad -1 < x \le 0 \\ 2 & , \quad 0 < x \le 1 \\ 0 & , \quad x > 1 \end{cases}$$

4.
$$u(x,0) = \begin{cases} 2 & , \quad x \le 0 \\ 0 & , \quad 0 < x \le 1 \\ 1 & , \quad x > 1 \end{cases}$$

Thank You!