The Wave Equation

A Simple Presentation for Teachers of Engineering Colleges

Phoolan Prasad

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF SCIENCE, BANGALORE

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Syllabus for Teaching of PDE

I have noticed that participants to this workshop are also teachers teaching mathematics at M.Sc. level. There are also some research students.

What should be suitable syllabus for teaching differential equations?

There was a very detailed discussion on this in two discussion meeting, sponsored by Indian Academy of Sciences and RMS in February, 2018.

Report, syllabus and some lectures are available at
http://web-japps.ias.ac.in:8080/SEP/pdffiles/discussion_meeting.pdf

33 professors from IISERS, IISc, IITs, Universities and TIFR-CAM took part in the two meetings, spread over 6 days. There were also lectures on some equations of mathematical physics as mathematicians and as physicists see them.

Wave Equation

When we talk of waves, in general we think of waves goverened by The Wave Equation.

But most of the waves are not governed by “the wave equation”.

We have already seen this in the lecture on “Single Conservation Law”.

We not mentioned one of a very important wave equation “Korteweg de Vries (KdV) Equation”, which led to some of the most important developments in mathematics and physics in the last century (with some finest applications to engineering).

KdV Equation and Solitons

KdV equation led to the concept of soliton.

See a popular, very simple and beatiful article by Alex Kasman
"A brief history of solitons and the KdV equation"
Current Science, Volume 115 - Issue 8 : 25 October 2018:
http://www.currentscience.ac.in/Volumes/115/08/1486.pdf

See also a review of Kasman's book by PP in a forthcoming issue of Current Science.

Simplest wave equation - A Part of the Wave Equation Waves

$$u_t + c\, u_x = 0\ ,\ c = \text{real constant} \tag{1}$$

Method of characteristics for first order PDE gives

$$u = f(x - ct),\ f : \mathbb{R} \to \mathbb{R} \tag{2}$$

When $f\ \in C^1(\mathbb{R}) \Rightarrow$ Genuine solution

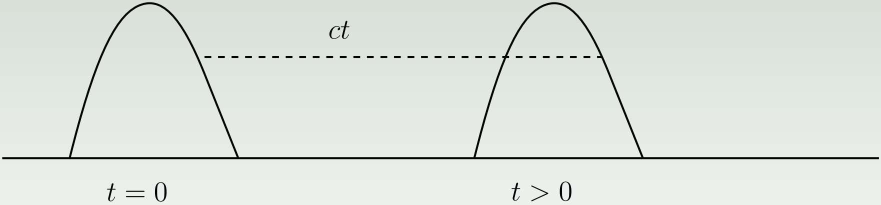


Figure: Wave of translation. This figure does not reprent a genuine solution. Why?

The One-dimensional Wave Equation

$$u_{tt} - c^2 \; u_{xx} = 0, \;\; c = \text{constant} \;\; > 0 \tag{3}$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0$$

LHS has two units of simplest wave operators.
Set

$$\xi = x - ct, \;\; \eta = x + ct$$
$$(15) \Rightarrow \frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \tag{4}$$

Shows general solution (genuine solution)

$$u = f(x - ct) + g(x + ct) \;;\;\; f, \;\; g \in C^2(\mathbb{R}) \tag{5}$$

$\Rightarrow$ two waves of translations, one moves with velocity c and another with velocity $-c$.

The One-dimensional Wave Equation contd..

The curves

$$x - ct = \text{ constant}, \; x + ct = \text{ constant}$$

are called characteristic curves

In any problem in physics, a PDE comes with additional conditions to completely determine the solution.

Cauchy Problem - here as an initial value problem.

$$u_{tt} - c^2 \; u_{xx} = 0, \; (x, t) \in \mathbb{R}^2 \tag{6}$$

$$u(x, 0) = u_0(x), \; u_t(x, 0) = u_1(x) \tag{7}$$

The One-dimensional Wave Equation contd..

For a genuine solution, we need

$$u_0(x) \in C^2(\mathbb{R}),\ u_1(x) \in C^1(\mathbb{R}) \tag{8}$$

Substitute $u = f(x - ct) + g(x + ct)$ in (7), and after one

integration of the second result and solving for f and g, we get

$$f(x) = \frac{1}{2}u_0(x) + \frac{1}{2c}\int_0^x u_1(\tau)d\tau + \frac{\delta}{2c} \tag{9}$$

$$g(x) = \frac{1}{2}u_0(x) - \frac{1}{2c}\int_0^x u_1(\tau)d\tau - \frac{\delta}{2c} \tag{10}$$

where δ is the constant of integration.

Now we get, D'Alembert's solution (year 1747)

$$u(x,t) = \frac{1}{2}\left[u_0(x+ct) + u_0(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} u_1(\tau)d\tau\right] \tag{11}$$

Shows:

- We need $u_0 \in C^2(\mathbb{R}),\ u_1 \in C^1(\mathbb{R})$
- the (genuine) solution exists and is unique.
- How uniqueness?

Solution depends continuouly on initial data. There are three steps:

1). Let $u_i(x)$ and $\bar{u}_i(x), i = 1, 2$ agree every where except in the interval $[a, b]$ and let given ε

$$|\bar{u}_0 - u_0| \leq \frac{\varepsilon}{4}, \quad |\bar{u}_1 - u_0| \leq \frac{\varepsilon c}{2 + (b - a)} \text{ for } x \in [a, b] \tag{12}$$

2). D'Alembert's solution for u and $\bar{u}$ gives

$$|\bar{u}(x,t) - u(x,t)| \leq \frac{1}{2}\left[|\bar{u}_0(x+ct) - u_0(x+ct)| + |\bar{u}_0(x-ct) - u_0(x-ct)|\right] + \left[\frac{1}{c}\int_{x-ct}^{x+ct} |\bar{u}_1(\tau) - u_1(\tau)| d\tau\right]$$

$$\Rightarrow \quad \leq \frac{1}{2}\left[\frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon + \frac{\varepsilon}{2 + (b-a)}(b-a)\right] \leq \frac{\varepsilon}{2}\left[\frac{1}{4} + \frac{1}{4} + \frac{1}{2}\right]$$

3). Thus we have shown that

$$|\bar{u}(x,t) - u(x,t)| \leq \varepsilon, \tag{13}$$

which proves the result.

The One-dimensional Wave Equation contd..

- Solution depends continuously on initial data up to any time.
- The initial value problem is well posed in the sense of Hadamard.
- Ill posed problems today are quite relevant but are more difficult.

One-dimensional wave equation contd..

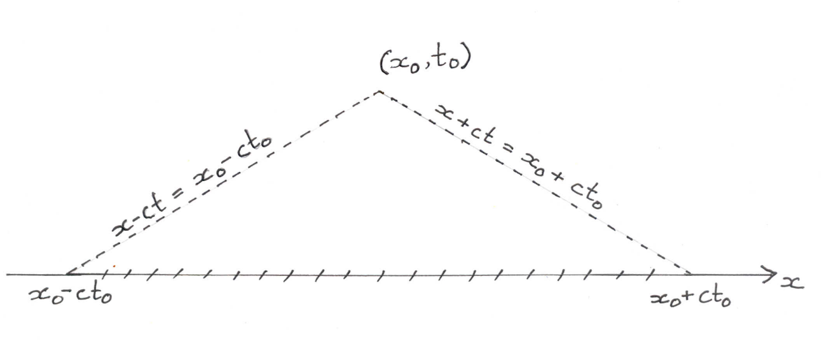


Figure: Domain of dependence of a point (x_0, t_0) is $[x_0 - ct_0, x_0 + ct_0]$.

One-dimensional wave equation contd..

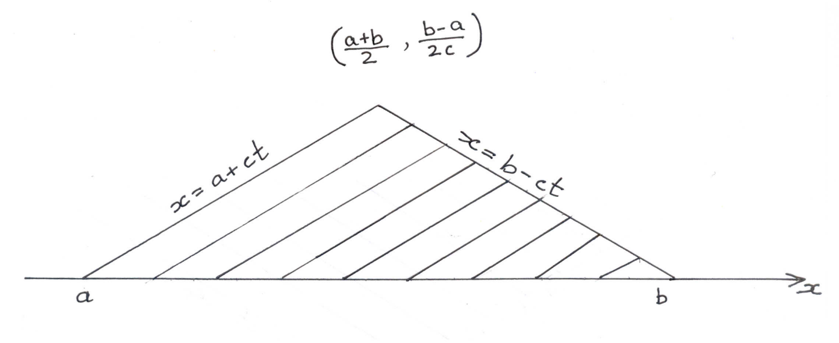


Figure: Domain of determinancy of an interval $[a, b]$ on the $x\!-\!axis$

is the closed region$= \{(x, t) : a + ct \le x \le b - ct,\ 0 < t \le \frac{b-a}{2c}\}$

One-dimensional wave equation contd..

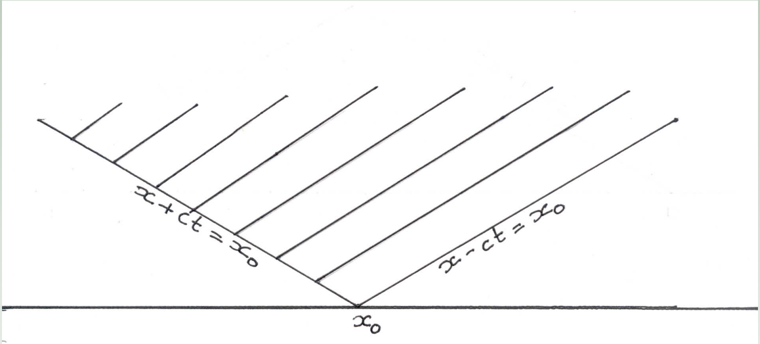


Figure: Range of influence of a point x_0 on the $x-axis$ is $\{(x,t) : x_0 - ct \leq x \leq x_0 + ct\}$

One-dimensional wave equation contd..

- A solution of the wave equation represents a phenomenon with finite speed of propagation.
- This is an important property of all hyperbolic PDEs.
- Inclusion of dispersion and diffusion in the equations make the speed of propagation to be infinite.

One-dimensional wave equation contd..

D'Alembert's solution as a generalized solution, when $u_0 \notin C^2(\mathbb{R})$ and $u_1 \notin C^1(\mathbb{R})$.

Example: Solution of the one-dimensional wave equation with

$$u_0(x) = \begin{cases} x^2 & \text{for } x > 0 \\ -x^2 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \end{cases} \tag{14}$$

$$u_1(x) = 0, \ x \in \mathbb{R}.$$

Note, using [ ] as a symbol for jump from left to right, we find the jump in the second derivative at $x = 0$ as

$$[u_0''(0)] = u_0''(0+) - u''(0-) = 4. \tag{15}$$

One-dimensional wave equation contd..

The D'Alembert's solution gives

$$u(x,t) = \begin{cases} x^2 + c^2t^2, \ x > ct \\ 2ctx, \ -ct < x < ct \\ -(x^2 + c^2t^2), \ -ct < x \end{cases} \tag{16}$$

$u(x,t) \in C^2(\mathbb{R} \times \mathbb{R}^+)$ everywhere except the characteristics through the initial point of discontinuity:

$$[u_{xx}(-ct,t)] = 2 = [u_{xx}(ct,t)] \tag{17}$$

Note that initial discontinuity breaks into discontinuities, each of the half the magnitude 4.

One-dimensional wave equation contd..

Result on the propagation of discontinuities of second and higher order derivatives shows that discontinuities in these derivatives can exist only across characteristic curves.

What about discontinuities in the function u and its first derivatives?

The answer lies in the definition of a *weak* or *generalized* solution. Theorem on the next slide motivates a definition.

One-dimensional wave equation contd..

Theorem $u(x,t) \in C^2(\mathbb{R})$ is a genuine solution of the one-dimensional wave equation iff

$$u(A) + u(C) = u(B) + u(D) \tag{18}$$

where A, B, C, D are vertices of any characteristic parallelogram

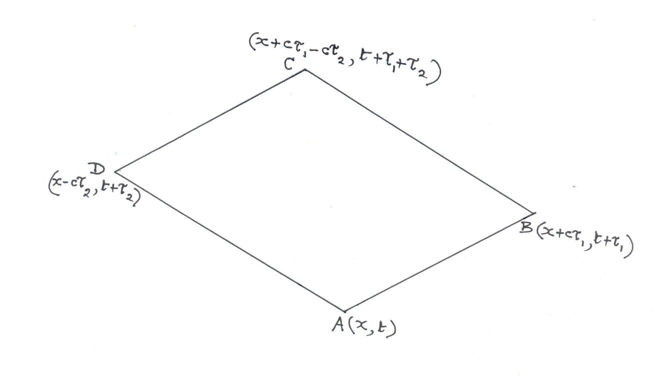


Figure: A characteristic parallelogram

Proof: Hint

use $u = f(x+ct) + g(x-ct)$ to derive (30).

Use finite Taylor expansion to derive the wave equation from (30).

Problems and Methods for Engineering Teachers and Students

- So far we have described essential and basic part of 1-D wave equation, which every one should know.
- Teaching only methods of solutions for some problems without basic concepts (like the above) will leave students ill prepared, a common complain we receive from both academic and industrial employers.
- I shall just mention some common problems and methods of solution. You will find them in all engineering mathematics books or some methods of mathematical physics.
- These problems are too many and I need not discuss them in just two lectures.

Problems and Methods for Engineering Teachers and Students

1. Solve the initial - boundary value problem

PDE:

$$u_{tt} - c^2 u_{xx} = 0;\ 0 < x < L,\ t > 0 \tag{19}$$

Initial conditions:

$$u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x),\ 0 < x < L \tag{20}$$

Boundary conditions:

$$u(0, t) = h_1(t),\ u(L, t) = h_1(t),\ t \geqslant 0 \tag{21}$$

This problem has been beautifully solved using (18) in [1] on page 108.

Problems and Methods for Engineering Teachers and Students $\cdots$ conti.

- But in most engineering books, for $u_0(x) = 0$ and $u_1(x) = 0$ it is solve by method of separation of variables and then using Fourier series, for example see [2] page 109. There are many more problems for engineering courses in the book.
- I shall suggest all pages 109-129 of [2] for further problems, including solution of of the Neumann problem for the inhomogeneous wave equation (Neumann was an engineer in Germany):
$$u_{tt} - c^2 u_{xx} = g(x,t);\ 0 < x < L,\ t > 0. \tag{22}$$
Initial conditions:
$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ 0 < x < L. \tag{23}$$
Boundary conditions (this makes it Neumann problem):
$$u_x(0,t) = h_1(t),\ u_x(L,t) = h_1(t),\ t \geqslant 0. \tag{24}$$
- In [2] you will also find proof of uniqueness of solution using energy method, which is quite simple.

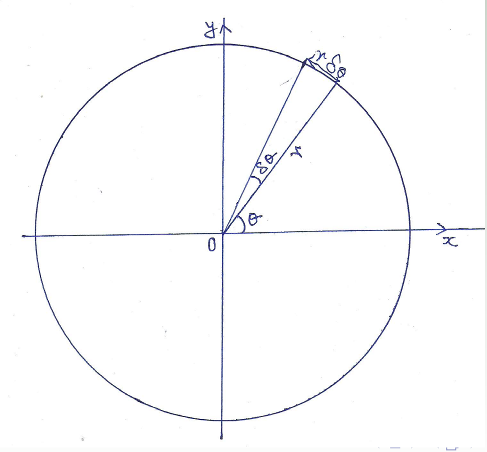
A Circle

A circle of radius r and centre $(0,0)$ is given by

$$x^2 + y^2 = r^2, \tag{25}$$

we write its parametric representation as

$$x = r\cos\theta, y = r\sin\theta, \quad 0 \le \theta < 2\pi. \tag{26}$$



Circle and Unit Circle

Generally small increment δs and differential ds are different, and in limit $\delta s \to ds$. In this case δs is not an approximation. Element of its arc length of the circle (26) of radius r is $rd\theta$.

For unit circle, which is is a circle with radius 1 and centre origin $(0, 0)$:

$$x^2 + y^2 = 1 \tag{27}$$

with parametric representation $x = \cos\theta, y = \sin\theta$.

Arc length element $= d\theta$.

Coordinates on a Sphere

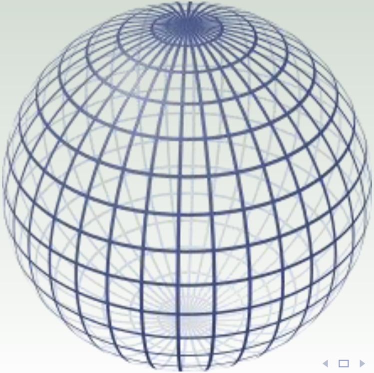
A sphere of radius r and centre $(0, 0)$ is given by

$$x^2 + y^2 + z^2 = r^2, \tag{28}$$

we write its parametric representation as

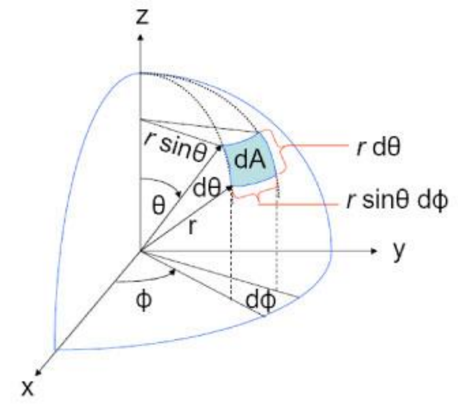
$$x = r \sin\theta \cos\phi, y = r \sin\theta \sin\phi, z = r \cos\theta, \tag{29}$$

where $0 \le \theta \le \pi, 0 \le \phi < 2\pi$. $\theta = 0$ at the north pole. Interpret these 3 inequalities and 1 equality. ϕ is longitude and $|\pi - \theta| N$ or $|\pi - \theta| S$ is latitude.



Solid Angle & Unit Sphere

A surface element of a sphere is given by $dA = r^2 \sin\theta d\theta d\phi$.



For unit sphere, $r = 1$ the surface element is $\sin\theta d\theta d\phi$, which called solid angle and I denote it by $d\omega$.

$$d\omega = \sin\theta d\theta d\phi \tag{30}$$

Problem: Write the expression for $d\omega_m$ in m-D.

Answer: In m-D $\omega_m = 2\pi^{\frac{m}{2}}\Gamma(\frac{m}{2})$, where Γ is gamma function.

IVP for the wave equation in multi-dimensions

Let me pass on to some basic results for the wave equation in multi-space dimensions.

$$\mathbf{x} = (x_1, x_2, \cdots x_m), \;\; \triangle_m = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}$$

$$u_{tt} - c^2 \triangle_m u = 0, \;\; (\mathbf{x} \in \mathbb{R}^m, t > 0) \tag{31}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \;\; \mathbf{x} \in \mathbb{R}^m \tag{32}$$

$$u_t(\mathbf{x}, 0) = u_1(\mathbf{x}), \;\; \mathbf{x} \in \mathbb{R}^m \tag{33}$$

Mean Value of Function on Surface of Sphere in 3-D

We define

$$(M(t)u)(\mathbf{x}) = \frac{1}{4\pi}\int_{|\boldsymbol{\nu}|=1} u(\mathbf{x}+\boldsymbol{\nu}ct)d\omega, \tag{34}$$

and ω is the surface area of unit sphere in 3-D and $\boldsymbol{\nu}$ is unit normal to the surface of the sphere with centre at $\mathbf{x}$ and radius ct.

Note that $u(\mathbf{x}+\boldsymbol{\nu}ct)$ is the value of u at a point on the surface of sphere with centre at $\mathbf{x}$ and of radius ct directed in direction $\boldsymbol{\nu}$ from its centre $\mathbf{x}$.

Another expression $M(t)u$ is

$$M(t)u = \frac{1}{4\pi c^2 t^2}\int_{|\mathbf{y}-\mathbf{x}|=\mathbf{ct}} u(\mathbf{x}+\mathbf{y})dS_y, \tag{35}$$

where $\mathbf{y}$ is a point on the surface of sphere with center $\mathbf{x}$ and radius ct and dS_y is its surface element.

The wave equation in multi-dimensions contd..

Theorem: If $u_0(\mathbf{x}) \in C^3$ and $u_1(\mathbf{x}) \in C^2$, for $\mathbf{x} \in \mathbb{R}^3$, the solution for the Cauchy problem (36)-(38) for $m = 3$ is

$$u(\mathbf{x}, t) = tM(t)u_1 + \frac{\partial}{\partial t}[tM(t)u_0] \tag{36}$$

We omit the proof.

In 1-space-D,

$$tM(t)u_1 = t\frac{1}{2ct}\int_{x-ct}^{x+ct} u_1(\tau)d\tau, \tag{37}$$

$$\frac{\partial}{\partial t}[tM(t)u_0] = \frac{\partial}{\partial t}\frac{1}{2c}\int_{x-ct}^{x+ct} u_0(\tau)d\tau = \frac{1}{2}\left\{u_0(x+ct) - u_0(x-ct)\right\}. \tag{38}$$

Thus we have derived the D'Alembert's solution (11) from expression (36), which is also valid for space of any odd dimensions.

The Wave Equation: Solution in $(2n+1)$-D and $2n$-D

- Expression 9360 is valid also for any odd space dimensions $m = 2n + 1$.
  It can be derived by a beautiful method of spherical means (Fritz John[1], 1955).

- For an even space dimensions $m = 2n$, we deduce the solution from the solution for $m = 2n + 1$ by Hadamard's method of descent.

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1. His popular book on PDE was based on his lectures at IISc. Later, when he went back to Counrant Institute, New York, there was some correspondence with him and he wrote that he remembered me as most active person during his lectures

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[1] 1

For $m = 2$, let $\bar{\mathbf{x}} = (x_1, x_2)$
Theorem: If $u_0 \in C^3(\mathbb{R}^2)$ and $u_1 \in C^2(\mathbb{R}^2)$, then the solution of (36)-(38) for $m = 2$ is

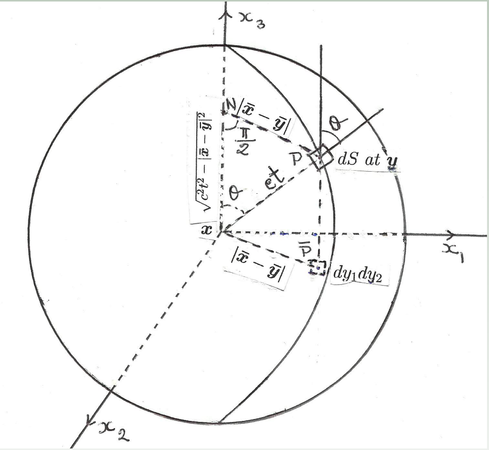
$$u(\bar{x}, t) = \{\bar{M}(t)\}u_1 + \frac{\partial}{\partial t}(\{\bar{M}(t)\}u_0) \tag{39}$$

where

$$\left(\{\bar{M}(t)\}u_i\right)(\mathbf{x}) = \frac{1}{2\pi c}\int_{|\bar{\mathbf{y}}-\bar{\mathbf{x}}|\le ct} \frac{u_i(\bar{\mathbf{y}})}{\{c^2t^2 - |\bar{\mathbf{y}} - \bar{\mathbf{x}}|^2\}^{1/2}} d\bar{\mathbf{y}} \tag{40}$$

Note that in $(M(t)u)(\mathbf{x})$ the integration is over the surface of the sphere and in $\left(\{\bar{M}(t)\}u\right)(\mathbf{x})$ integration is over the closed region inside and on the boundary of the circle.

Geometry the Method of Descent: sphere with center $\mathbf{x}$ and radius ct, $\mathbf{y}$ is point on the surface



Distance of N from $\mathbf{x}$ and that of P from $\bar{P} = \sqrt{c^2t^2 - |\bar{\boldsymbol{x}} - \bar{\boldsymbol{y}}|^2}$.

$\mathbf{x} = (x_1, x_2, x_3)$, $\bar{\mathbf{x}} = (x_1, x_2)$, $\cos\theta = \frac{\sqrt{c^2t^2-|\bar{\boldsymbol{x}}-\bar{\boldsymbol{y}}|^2}}{ct}$, $dy_1dy_2 = dS\cos\theta$.

Domain of dependence

$m = 1$:

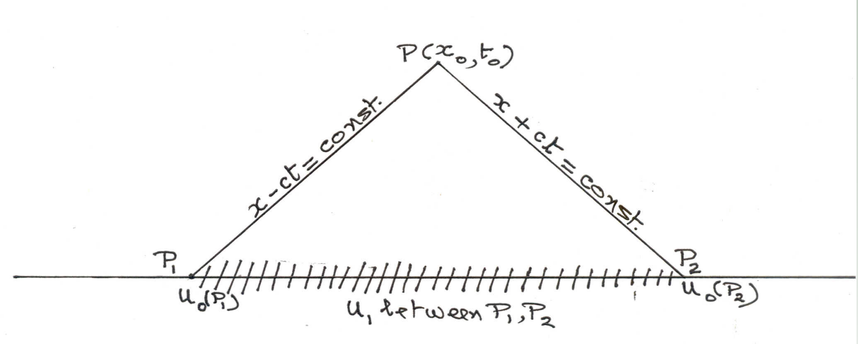


Figure: Solution at (x_0, t_0) depends on values of u_0 at $P_1(x_0 - ct_0)$ and $P_2(x_0 - ct_0)$ and u_1 from P_1 to P_2.

Domain of dependence contd..

$m = 2$:

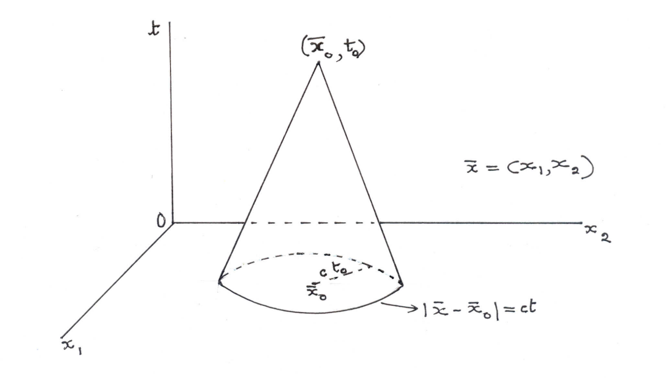


Figure: Solution at (x_0, t_0) depends on the boundary of the circle $|x - x_0| = ct_0$ and its interior.

Domain of dependence contd..

$m = 3:$

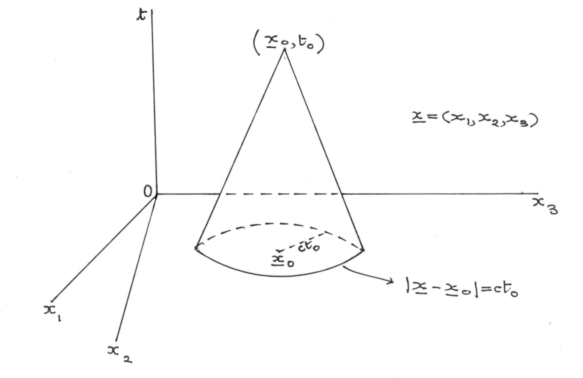


Figure: Solution at (x_0, t_0) depends only on the values of u_0, u_1 and the first order derivatives of u_0 on the surface S of the sphere.

Propagation of confined initial disturbances and reverberation

Initial data $u_0(\mathbf{x})$ and $u_1(\mathbf{x})$ such that

$$\text{Supports of } u_0 \text{ and } u_1 \subset \{\mathbf{x} : |\mathbf{x}| < \delta\} \tag{41}$$

What is the meaning of support?

3-Space Dimensions

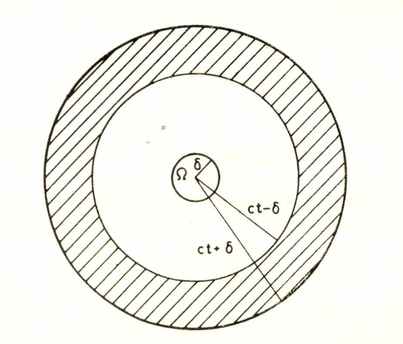


Figure: The solution at $t > \delta/c$ is non-zero only in a spherical shell of outer and inner radii $ct + \delta$ and $ct - \delta$.

Propagation of confined initial disturbances and reverberation contd..

At a fixed point $\mathbf{x}, |\mathbf{x}| > \delta,\ u(\mathbf{x}, t) \neq 0$ for a time interval $\frac{|\mathbf{x}|-\delta}{c} < t < \frac{|\mathbf{x}|+\delta}{c}$.

This shows that in 3-D transmission of signals in the form of waves has sharp leading and trailing fronts.

Propagation of confined initial disturbances and reverberation contd..

2-Space Dimensions: Consider a point $\bar{\boldsymbol{y}}$ outside the confined circular domain of radius δ, i.e. $|\bar{\boldsymbol{y}}| > \delta$.

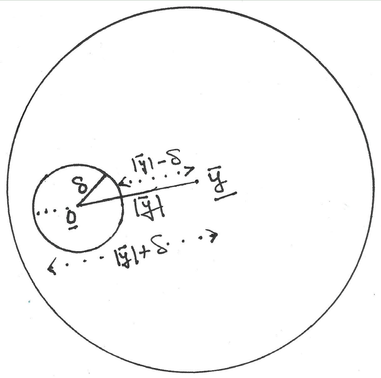


Figure: Signal first reaches $\bar{\boldsymbol{y}}$ at time $\frac{|\bar{\boldsymbol{y}}|-\delta}{c}$. After time $\frac{|\bar{\boldsymbol{y}}|+\delta}{c}$, all points of $|\bar{\boldsymbol{x}}| < \delta$ are always in the domain of dependence of a point $\bar{\boldsymbol{y}}$ and $\bar{\boldsymbol{y}}$ keeps on getting signal.

Propagation of confined initial disturbances and reverberation contd..

2-Space Dimensions. Geometrical interpretation.

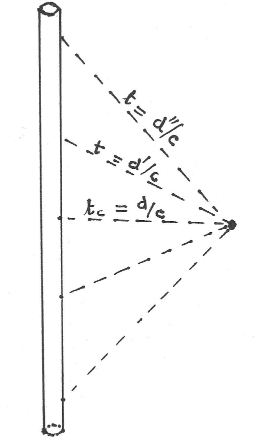


Figure: Reverberation phenomenon in 2-space-D. A sudden flash of light from infinite tube light reaches a point at distance d in time $\frac{d}{c}$ but the point continues to get light all time.

Propagation of confined initial disturbances and reverberation contd..

1-Space Dimensions. It is a very interesting case.

Part $\frac{1}{2}[u_0(x+ct)+u_0(x-ct)]$ shows that if u_0 has compact support and $u_1(x)=0$ we have transmission of signals with sharp leading and trailing edges.

The part $\frac{1}{2c}\int_{x-ct}^{x+ct} u_1(\tau)d\tau$ shows that if $u_1(x)\neq 0$, the signal has sharp leading edge but then it continues indefinitely $\Rightarrow$ reverberetion.

EXERCISE

1. In the theory of acoustics (linearised theory of sound with small disturbances about an equilibrium state) the velocity components u and v, pressure p and density ρ satisfy the following equations :

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} + \rho_0 \frac{\partial v}{\partial y} = 0$$

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0, \rho_0 \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = 0, p - \rho_0 = a_0^2(\rho - \rho_0)$$

where the subscript 0 denotes the equilibrium state. Show that each of the quantities u, v, p, ρ satisfy the wave equation with velocity of propagation a_0. Find the solution in the one-dimensional case (when $v = 0$ and all quantities are independent of y) given that initially $u(x, 0) = f(x), p(x) = 0.$

EXERCISE contd..

2. Using an energy integral $I(t) = \int_a^b (u_x^2 + u_t^2) dx$, show that the solution to the mixed initial and boundary value problem of the one-dimensional wave equation

$$u_{tt} - c^2 \ u_{xx} = 0, \quad a < x < b, t < 0$$

with $u(x,0) = \varphi(x), u_t(x,0) = \psi(x), u(a,t) = \chi(t), u(b,t) = \lambda(t)$ is unique.

3. If $u(x,t)$ satisfies the wave equation

$$u_{tt} - c^2 \ u_{xx} = 0$$

show that

$$\int_C \{u_\tau(\xi,\tau) d\xi + c^2 u_\xi(\xi,\tau) d\tau\} = 0$$

around any simple closed curve. Deduce that the solution which satisfies the initial conditions $u = u_0(x), u_t = u_1(x)$ when $t = 0$, is given by (2.8)

4. Find the solution of the equation $u_{tt} - c^2 u_{xx} = 0$, given that on $t = 0, u = \sin\ \pi x/c$ when $0 \leq x \leq c, \quad u = 0$ when $x > c$ and $x < 0, u_t = 0$ for all x. Examine the continuity of u and its derivatives.

5. A flexible string of length l is fastened at the ends $x = 0$ and $x = l$ and is in equilibrium under a uniform tension. It is displaced at $x = \frac{1}{2}l$ to an elevation h and then released. Find subsequent displacement for all times, assuming that the motion is governed by the wave equation.

Hint : Use the method of separation of variables.

6. Find the deflection $u(x,t)$ of a taut string which was at rest at time $t=0$, if it is fastened at the end point $x=l$ and subjected at the other end point $x=0$ to a motion represented by $u(0,t)=f(t)$.

7. Determine the generalised or weak solution of the equation

$$u_{xx} - \frac{1}{c^2}u_{tt} = 0$$

given that

$$u(x,0) = \begin{cases} x^2+5, & x>0 \\ 5, & x=0 \\ -x^2+5, & x<0 \end{cases}$$

$$u_t(x,0) = 0$$

Verify that the discontinuity in the second order derivatives will propagate along characteristics.

8. Determine the weak solution of the equation

$$u_{xx} - \frac{1}{c^2}u_{tt} + \frac{2}{c}u_t - u_x = 0$$

given that $u_t(x,0) = 0$ and

$$u(x,0) = \begin{cases} x^2, & x > 0 \\ 0, & x = 0 \\ -x^2, & x < 0 \end{cases}$$

Examine how the discontinuity in the second order derivatives will propagate.

Phoolan Prasad and Renuka Ravindran, *Partial Differential Equations*, Wiley Eastern Ltd and John Wiley & Sons, 1984.

Yehuda Pinchover and Jacob Rubinstein, *Introduction to Partial Differential Equations*, Cambridge University Press, 2005.

Thank You!