First Order Partial Differential Equation, Part - 2: Non-linear Equation

PHOOLAN PRASAD

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF SCIENCE, BANGALORE

Teachers Enrichment Workshop
for teachers of engineering colleges

November 26 - December 1, 2018
Institute of Mathematical Sciences, Chennai

(Sponsored by National Centre for Mathematics and IMSc)

First Order Non-linear Equation

$$F(x, y, u, p, q) = 0,\ p = u_x,\ q = u_y$$
$$F \in C^2(D_3),\ \text{domain } D_3 \subset \mathbb{R}^5 \qquad (1)$$

- Example: $u_x^2 + u_y^2 = -1$, better we take $u_x^2 + u_y^2 = 1$.
- No directional derivative of u in (x, y)-plane for a general F.
- Take a known solution $u(x, y) \in C^2(D)$, $D \subset \mathbb{R}^2$

$$F(x, y, u(x, y), p(x, y), q(x, y)) = 0 \text{ in } D. \qquad (2)$$

Charpit Equqtions

- Taking x derivative of the PDE in (1)

$$F_x + F_u u_x + F_p p_x + F_q q_x = 0$$

- Using $q_x = (u_y)_x = (u_x)_y = p_y$

$$F_p p_x + F_q p_y = -F_x - pF_u \tag{3}$$

- Beautiful, p is differentiated in the direction (F_p, F_q). Similarly

$$F_p q_x + F_q q_y = -F_y - qF_u \tag{4}$$

Charpit Equations contd..

For a given solution $u(x, y)$ in (x, y)-plane, along one parameter family of curves given by

$$\frac{dx}{d\sigma} = F_p, \qquad \frac{dy}{d\sigma} = F_q \tag{5}$$

we have

$$\frac{dp}{d\sigma} = -F_x - pF_u \tag{6}$$

$$\frac{dq}{d\sigma} = -F_y - qF_u \tag{7}$$

Further (note also for future reference)

$$\frac{du}{d\sigma} = u_x\frac{dx}{d\sigma} + u_y\frac{dy}{d\sigma} = pF_p + qF_q \tag{8}$$

Derived for a given solution $u = u(x, y)$.

Charpit Equations contd..

These, 5 equations for 5 quantities x, y, u, p, q are complete irrespective of the solution $u(x, y)$.
They are **Charpit equations**.

Given values (u_0, p_0, q_0) at (x_0, y_0), such that $(x_0, y_0, u_0, p_0, q_0) \in D_3$
$\Rightarrow$ local unique solution of Charpit Equations with $(x_0, y_0, u_0, p_0, q_0)$ at $\sigma = 0$.

Autonomous system of five equations $\Rightarrow$ 4 parameter family of solutions.
$(x, y, u, p, q) = (x, y, u, p, q)(\sigma, c_1, c_2, c_3, c_4)$.

Charpit Equations contd..

Theorem Every solution of the Charpit's equations satisfies (it is a part of Charpit E)

$$\frac{du}{d\sigma} = p(\sigma)\frac{dx}{d\sigma} + q(\sigma)\frac{dy}{d\sigma} \tag{9}$$

Charpit Equations contd..

Theorem. The function F is constant for every solution of the Charpit's equations i.e.
$F(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma)) = C(c_1, c_2, c_3, c_4)$
is independent of σ.

Proof. Simple,

$$\frac{dF}{d\sigma} = \frac{dx}{d\sigma}F_x + \frac{dy}{d\sigma}F_y + \frac{du}{d\sigma}F_u + \frac{dp}{d\sigma}F_p + \frac{dq}{d\sigma}F_q = 0 \qquad (10)$$

when we use Charpit's equations.
Important: In order that solution of Charpit's equations satisfies the relation $F = 0$, choose c_1, c_2, c_3 and c_4 such that

$$C(c_1, c_2, c_3, c_4) = 0 \Rightarrow c_4 = c_4(c_1, c_2, c_3) \qquad (11)$$

Monge strip and characteristic curves

Monge strip is a solution of the Charpit's equations satisfying

$$F(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma)) = 0. \tag{12}$$

It is a 3 parameter family of functions

$$(x, y, u, p, q)(\sigma, c_1, c_2, c_3) = 0 \tag{13}$$

Characteristic curves: From the the first two of the Monge strips, take

$$x = x(\sigma, c_1, c_2, c_3), \quad y = y(\sigma, c_1, c_2, c_3),$$

which in (x, y)-plane form are 3-parameter family of characteristic curves.

Cauchy problem

- Find solution $u(x, y) : D \to \mathbb{R}$
- Datum curve $\gamma : (x = x_0(\eta), y = y_0(\eta))$ is curve in D.
- Cauchy data on γ, $\quad u_0(\eta) = u(x_0(\eta), y_0(\eta))$
- Choose a point P_0 on γ and find evolution of u along the characteristic stating for it.

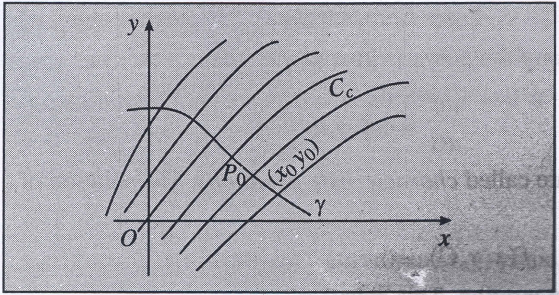


Fig. 1.1. *Solution of a Cauchy problem with the help of characteristic curves* C_c

Solution of a Cauchy problem

For a general nonlinear equation,value of u is carried along a characteristic not independently but together with values of p and q.

We need values of x_0, y_0, u_0, p_0 and q_0 at P_0 on datum curve as initial data for Charpit's equations at $\sigma = 0$.

Solution of a Cauchy problem contd..

How to get values of p_0 and q_0 from Cauchy data:

$$u = u_0(\eta) \ \text{ on } \ \gamma : x = x_0(\eta), y = y_0(\eta)? \tag{14}$$

- We have one equation

$$F(x_0(\eta), y_0(\eta), u_0(\eta), p_0(\eta), q_0(\eta)) = 0. \tag{15}$$

- $u_0(\eta) = u(x_0(\eta), y_0(\eta))$ gives another equation

$$u_0'(\eta) = p_0(\eta)x_0'(\eta) + q_0 y_0'(\eta). \tag{16}$$

- Solve $p_0(\eta)$ and $q_0(\eta)$ from these two to get Cauchy data for Charpit's ODEs

$$(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma)) \mid_{\sigma=0} = (x_0, y_0, u_0, p_0, q_0)(\eta). \tag{17}$$

Solution of a Cauchy problem contd..

Step 1: Solve the Charpit's equations

$$\frac{dx}{d\sigma} = F_p, \ \frac{dy}{d\sigma} = F_q, \ \frac{du}{d\sigma} = pF_p + qF_q \tag{18}$$

$$\frac{dp}{d\sigma} = -(F_x + pF_u), \ \frac{dq}{d\sigma} = -(F_y + qF_u) \tag{19}$$

with above initial conditions to get

$$x = x(\sigma, \eta), \ y = y(\sigma, \eta) \tag{20}$$

$$u = u(\sigma, \eta), \ p = p(\sigma, \eta), \ q = q(\sigma, \eta) \tag{21}$$

Step 2: Then solve $\sigma = \sigma(x, y), \eta = \eta(x, y)$.
Step 3: and get the solution

$$u(x, y) = u(\sigma(x, y), \eta(x, y)).$$

Question: We can also get $p(x, y), q(x, y)$ but are they u_x, u_y?

Solution of a Cauchy problem contd..

Complicated statement - just ignore & see next slide.

Theorem

- $F(x, y, u, p, q) \in C^2(D_3)$, domain $D_3 \subset \mathbb{R}^5$
- $x_0(\eta), y_0(\eta), u_0(\eta) \in C^2(I)$, $\eta \in I \subset \mathbb{R}$
- $(x_0(\eta), y_0(\eta), u_0(\eta) p_0(\eta), q_0(\eta)) \in D_3$, $\eta \in I$
- $p_0(\eta), q_0(\eta) \in C^1(I)$, $\eta \in I$
- $\frac{dx_0}{d\eta} F_q(x_0, y_0, u_0, p_0, q_0) - \frac{dy_0}{d\eta} F_p(x_0, y_0, u_0, p_0, q_0) \neq 0$

$\Rightarrow$ There exist a unique local solution of the Cauchy problem such that

$$u(x, y)|_\gamma = u_0,\ p(x, y)|_\gamma = p_0,\ q(x, y)|_\gamma = q_0 \tag{22}$$

Note that the theorem does not guarantee that u will be $\mathbb{C}^2$.

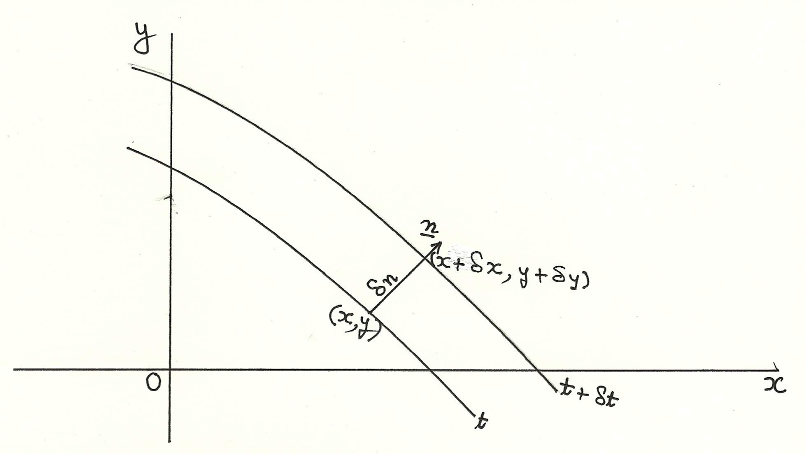
Why local?

Solution of a Cauchy problem contd..

- Solution guaranteed by above theorem is only local.
- Important point for the existence and uniqueness of the Cauchy problem is that the datum curve γ is no where tangential to a characteristic curve.
- If γ is a characteristic curve, the data $u_0(\eta)$ is to be restricted (i.e., the equations of p_0 and q_0 should also be satisfied)
- When above restriction is satisfied, the solution of the Cauchy problem is non-unique - infinity of solutions exist.

Isotropic wave motion with constant velocity

Consider an 2-D a wave moving into a uniform isotropic medium[1] (one assumption) with constant velocity c (another assuption). Example?



[1] An isotropic medium is one in which the velocity of wave at a point in the medium is same in all directions.

Isotropic wave motion with constant velocity contd..

Let a wavefront Ω_t in such a wave be represented by $u(x, y) = ct$. Then on $\Omega_{t+\delta t}$ we have $c(t + \delta t) = u(x + \delta x, y + \delta y)$.

Taylor expansion of u up to first order terms (using $ct = u(x, y)$)

$$\frac{c}{\sqrt{u_x^2 + u_y^2}}\delta t = \frac{u_x}{\sqrt{u_x^2 + u_y^2}}\delta x + \frac{u_y}{\sqrt{u_x^2 + u_y^2}}\delta y = n_1\delta x + n_2\delta y = \delta n \tag{23}$$

$$\frac{c}{\sqrt{u_x^2 + u_y^2}} = \frac{\delta n = \text{normal displacement}}{\delta t} = c \tag{24}$$

$$\Rightarrow p^2 + q^2 = 1; \quad p = u_x, q = u_y. \tag{25}$$

Isotropic wave motion with constant velocity contd..

Problem. Find successive positions of the wavefront $u(x, y) = ct$ when the initial position is

$$\alpha x + \beta y = 0, \quad \alpha^2 + \beta^2 = 1, \quad \text{where } u = 0. \tag{26}$$

What do you expect in this case?

Cauchy problem

$$F \equiv p^2 + q^2 = 1$$

$$\gamma : x_0 = \beta\eta, \; y_0 = -\alpha\eta$$

Cauchy data

$$u_0 = 0$$

Values of p_0 and q_0

$$\begin{aligned} p_0^2 + q_0^2 = 1, \; \beta p_0 - \alpha q_0 = 0 \\ \Rightarrow p_0 = \pm\alpha, \; q_0 = \pm\beta \end{aligned} \tag{27}$$

Isotropic wave motion with constant velocity contd..

Solution of Charpit's equations

$$\frac{dx}{d\sigma} = 2p, \ \frac{dy}{d\sigma} = 2q,$$
$$\frac{du}{d\sigma} = 2(p^2+q^2) = 2, \ \frac{dp}{d\sigma} = 0, \ \frac{dq}{d\sigma} = 0 \tag{28}$$

We can use the equation $p^2 + q^2 = 1$ in the Charpit equations here.Why?

Solution are

$$\begin{aligned} x = \pm 2\alpha\sigma + \beta\eta, \quad y = \pm 2\beta\sigma - \alpha\eta \\ u = 2\sigma, \ p = \pm\alpha, \quad q = \pm\beta \\ \Rightarrow \sigma = \pm\frac{1}{2}(\alpha x + \beta y) \\ \Rightarrow u = \pm(\alpha x + \beta y) \end{aligned} \tag{29}$$

Two solutions of 2 problems, but uniqueness theorem not violated.

Isotropic wave motion with constant velocity contd..

Wavefronts

$$\alpha x + \beta y = \pm ct. \tag{30}$$

$+$ sign for forward propagating wavefront.

$-$ sign for backward propagating wavefront.

Normal distance at time t from the initial position $= \pm ct$.

We have presented the theory of characteristics of first order PDEs briefly. It is based on the existence of characteristics curves in the (x, y)-plane.

Along each of these characteristics we derive a number of compatibility conditions, which are transport equations and which are sufficient to carry all necessary information from the datum curve in the Cauchy problem into a domain in which solution is determined.

In this sense every first order PDE is a hyperbolic equation.

We have omitted a special class of solutions known as complete integral, for which any standard text may be consulted.

Every solution of the PDE (1) can be obtained from a complete integral.

We can also solve a Cauchy problem with its help.

Complete integral plays an important role in Physics.

So far we have discussed only a genuine solution (except for two examples in the case of quasilinear equations), which is valid locally.

- We have seen that characteristic carry information about the solution.
- Characteristic curves are the only curve which can sustain discontinuities of certain types in the solution.
- For a linear equation the discontinuities can be in the solution and its derivatives, for a quasilinear equation the discontinuities can be in the first and higher order derivatives.
- for nonlinear equations the discontinuities can be in second and higher order derivatives.

1 Consider the partial differential equation

$$F \equiv u(p^2 + q^2) - 1 = 0.$$

(i) Show that the general solution of the Charpit's equations is a four parameter family of strips represented by

$$x = x_0 + \frac{2}{3}u_0(2\sigma)^{\frac{3}{2}}\cos\theta,\ y = y_0 + \frac{2}{3}u_0(2\sigma)^{\frac{3}{2}}\sin\theta,$$
$$u = 2u_0\sigma,\ p = \frac{\cos\theta}{\sqrt{2\sigma}},\ q = \frac{\sin\theta}{\sqrt{2\sigma}}$$

where x_0, y_0, u_0 and θ are the parameters.

(ii) Find the three parameter sub-family representing the totality of all Monge strips.

(iii) Show that the characteristic curves consist of all straight lines in the (x, y)-plane.

2 Solve the following Cauchy problems:

(i) $\frac{1}{2}(p^2+q^2)=u$ with Cauchy data prescribed on the circle $x^2+y^2=1$ by

$$u(\cos\theta, \sin\theta)=1, \ \ 0\le\theta\le 2\pi$$

(ii) $p^2+q^2+\left(p-\frac{1}{2}x\right)\left(q-\frac{1}{2}y\right)-u=0$ with Cauchy data prescribed on the x-axis by

$$u(x,0)=0$$

(iii) $2pq-u=0$ with Cauchy data prescribed on the y-axis by

$$u(0,y)=\frac{1}{2}y^2$$

(iv) $2p^2x+qy-u=0$ with Cauchy data on x-axis

$$u(x,1)=-\frac{1}{2}x.$$

3 Find a representation of Monge strips of the equation

$$2pqx^2y^2 - px - qy - u = 0 \qquad (1)$$

in the form

$$x = (2m_2 - m_3e^{\sigma})^{-1}, y = (2m_1 - m_4e^{\sigma})^{-1},$$
$$p = m_1(2m_2 - m_3e^{\sigma})^2, \; q = m_2(2m_1 - m_4e^{\sigma})^2,$$
$$u = -2m_1m_2 + (m_1m_3 + m_2m_4)e^{\sigma}, \qquad (2)$$

where one of the arbitrary constants $m_1, m_2, m_3,$ and m_4 can be absorbed in in a choice of σ.

Solution: Many important comments at the end. Charpit equations of (1) are

$$\frac{dx}{d\sigma} = 2qx^2y^2 - x \quad (3), \qquad \frac{dy}{d\sigma} = 2px^2y^2 - y \quad (4),$$
$$\frac{dp}{d\sigma} = 2pqxy^2 - x \quad (5), \qquad \frac{dq}{d\sigma} = 2pqx^2y - y \quad (6),$$
$$\frac{du}{d\sigma} = 2pqx^2y_2 - px - qy \quad (7).$$

3 ⋯ continues In the results below, $m_1, m_2, m_3,$ and m_4 are arbitrary constants.

$$x\frac{dp}{d\sigma} = -2(pqx^2y^2 - px) = -2(u + qy) \quad \textit{(using(1))}, \quad (8)$$

$$\textit{similarly} \quad p\frac{dx}{d\sigma} = u + qy. \quad (9)$$

$$\Rightarrow \frac{x}{p}\frac{dp}{dx} = -2 \quad \Rightarrow \quad px^2 = m_1 \quad (10), \quad \textit{similarly} \quad qy^2 = m_2. \quad (11)$$

From (3) using (11)

$$\frac{dx}{d\sigma} = (2m_2x - 1)x \qquad \Rightarrow \quad x = \frac{1}{2m_2 - m_3e^{\sigma}}, \quad (12)$$

$$\textit{similarly} \quad y = \frac{1}{2m_1 - m_4e^{\sigma}}. \quad (13)$$

From (10) and (12), and (11) and (13) we get the expressions for p and q. Using the pde (1),

$$u = 2pqx^2y^2 - px - qy = -2m_1m_2 + 9(m_1m_3 = m_2m_4)e^{\sigma} \quad (14).$$

3 ⋯ continues

- (2) is not the general solution of the Charpit equations (3)-(7), since we have used (1) at many steps, for example in derivation of (8). Note also that we have not used the equation (7). Instead we have used (1) to derive the expression (14) of u. Thus we have got a general form of equations of a Monge strip, which has four arbitrary constants, one more than what one should have.

- One of the four constants m_1, m_2, m_3, m_4 can be absorbed in the choice of the origin of σ. For example, we write (13) in the form $(2m_1y-1)/y = m_4e^{\sigma} = \exp(\sigma + ln(m_4))$ and set $\sigma' = \sigma + ln(m_4)$. In the derivation of (8), (9), (10), (11) and (12); the only integration with respect to σ appears in (12). This remains same when replace σ by σ'.

- Thus we have derived the equations representing the Monge strip in the question.

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Thank You!