First Order Partial Differential Equations, Part - 1: Single Linear and Quasilinear First Order Equations

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General Comments

- First order PDE is simplest and historically oldest (a general class of) PDE with almost a complete theory and beautiful mathematical structure.
- Yet students find its theory mysterious and more difficult than unstructured theory of higher order equations.
- Classical theory of first order PDE started in about 1760 with Euler and D'Alembert and ended in about 1890 with the work of Lie.
- In intervening period Lagrange, Charpit, Monge, Pfaff, Cauchy, Jacobi and Hamilton made deep and important contributions to it and mechanics.

General Comments . . . contd

- Wave equation first appeared in print in 1747 (a little before the theory of FOPDE) by Lagrange and Laplace equation in 1784 by Laplace.

- But they did not give the general theory.

General Comments

- Complete integral of FOPDE played a very important role mechanics.
- But the theory of "complete integrals", is no longer treated as essential for study in a basic course in PDE (see Evan's book).
- I shall also skip complete integrals, while dealing with nonlinear equations.

Definition

First order PDE for a function $u(x, y)$ of two independent variables is a relation

$$F(x, y; u; u_x, u_y) = 0, \quad F \text{ a known \textbf{real} function from } D_3 \subset \mathbb{R}^5 \to \mathbb{R}. \tag{1}$$

In this lecture we denote

- by D a domain in $\mathbb{R}^2$ where a solution u is defined.
- We shall define other domains when needed.

Classification

Linear equation (nonhomogeneous):

$$a(x, y)u_x + b(x, y)u_y = c_1(x, y)u + c_2(x, y) \tag{2}$$

Nonlinear equation: All other equations with subclasses:

1). Semilinear equation:

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u) \tag{3}$$

2). Quasilinear equation:

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{4}$$

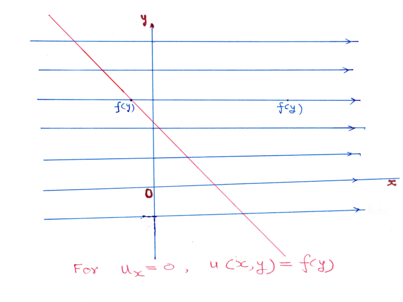
3). Nonlinear equation: $F(x, y; u; u_x, u_y) = 0$ where F is not linear in u_x, u_y.

Properties of solutions of all 4 classes of equations are quite different.

Example 1a: Simplest PDE

$$u_x = 0$$

General solution in $D = \mathbb{R}^2$ is $u = f(y)$, where f is an arbitrary $\mathbb{C}^1$ function.



Solution u is uniquely determined if it is prescribed on any curve no where parallel to x-axis.

On a line parallel to x-axis, we can not prescribe u arbitrarily. Why?

Example 1a: Nonhomogeneous equation

Consider PDE

$$u_x = c(x, y), \quad c(x, y) \; a \; known \; function \tag{5}$$

with condition

$$u(0, y) = f(y), \quad f(y) \; a \; function \; prescribed \; on \; the \; y \; axis. \tag{6}$$

- The unique and stable solution of this problem is

$$u = \int_0^x c(\sigma, y) d\sigma + f(y). \tag{7}$$

- Is solution of a first order equation so simple?
- Yes, it is for a linear equation provided we understand the role of characteristic curves. See the article provided to you.

Example 2: Preliminaries through an example

- A transport equation in two independent variables:

$$u_y + cu_x = 0, \; c = \text{real constant.} \tag{8}$$

- Introduce a variable $\eta = x - cy$. For a fixed η, $x - cy = \eta$ is a straight line with slope $\frac{1}{c}$ in (x, y) plane.
- Along this straight line

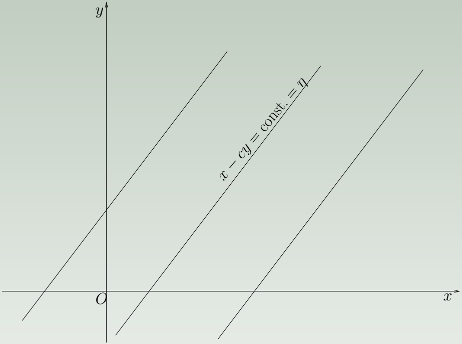
$$x(y) = cy + \eta \tag{9}$$

the derivative of a solution $u(x, y) = u(x(y), y)$ on this line is

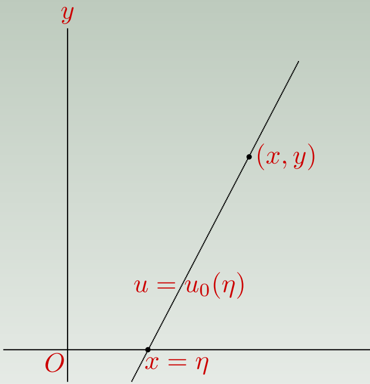
$$\begin{aligned} \frac{d}{dy}u(x(y), y) &= u_x \frac{dx}{dy} + u_y \\ &= cu_x + u_y \\ &= 0. \end{aligned}$$

- Thus, the solution u is constant along curves $x - cy = \eta$. These lines are characteristic curves of (8). PDE See figure on next slide.

Example 2: Preliminaries through an example ... conti...



(a)



(b)

Characteristic curves of $u_y + cu_x = 0$.
(a) Characteristics form a one parameter family of straight lines $x - cy = \eta$, where η is the parameter.
(b) u is constant along a characteristic curve.

- Consider an initial value problem (which is a Cauchy problem) of the equation (8) in which

$$u(x,0) = u_0(x). \tag{10}$$

- To find solution at (x, y), draw the characteristic through (x, y) and let it meet the x-axis at $x = \eta$. Then, u is constant on $x - cy = \eta$, i.e,

$$\begin{aligned} u(x,y) &= u(\eta,0) \\ &= u_0(\eta). \end{aligned}$$

Hence, as $\eta = x - cy$,

$$u(x,y) = u_0(x - cy). \tag{11}$$

For the simplest PDE (10) and Real u_0

Theorem

If c is real and $u_0 \in C^1(\mathbb{R})$, there is a unique solution $u \in C^1(\mathbb{R}^2)$ to the initial value problem (8), (10). The solution is given by the formula $u(x,y) = u_0(x - cy)$.

- The solution is $C^1(\mathbb{R})$. Here $D = \mathbb{R}^2$
- Solution is stable for small changes in Cauchy data u_0

parametric representation of a curve

Do you know a parametric representation of a curve?

A parametric representation of circle $x^2 + y^2 = 1$ is
$x = \cos\eta,\ y = \sin\eta;\quad 0 \le \eta < \pi$.

Problem: Write a parametric representation of

$$y^2 = x.$$

Now we prescribe the Cauchy data
for the PDE (8): $u_y + cu_x = 0,\ c = \text{real constant}$
on curve shown in the Figure 2(a): $\gamma : x = x_0(y),\ x_0 \in \mathcal{C}^1(I)$, written parametrically as $\gamma : x = x_0(\eta), y = \eta = y_0(\eta),\ say$.
u is prescribed on γ as $u(x_0(\eta), y_0(\eta) \equiv u(x_0(\eta), \eta) = u_0(\eta)$.

Problem: Find u in a neighbourhood of γ.

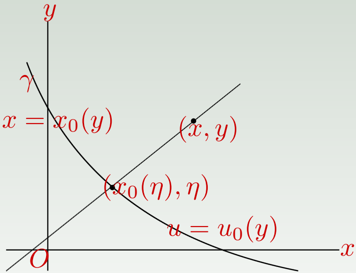


Figure: 2(a) - The datum curve $x = x_0(y)$, i.e. $x = x_0(\eta), y = y_0(\eta) = \eta$, is nowhere tangential to a characteristic curve.

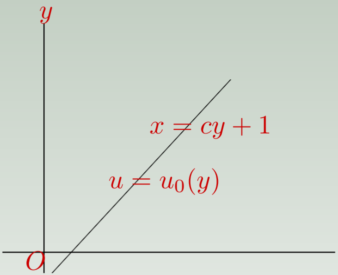


Figure: 2(b) - The datum curve is a characteristic curve $x = cy + 1$.

Solution for the case 2(a).

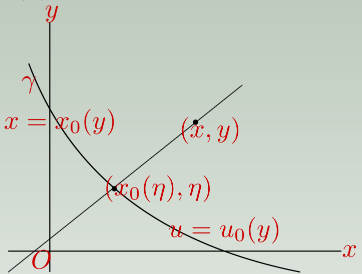


Figure: 2(a) - The datum curve $x = x_0(y)$, i.e. $x = x_0(\eta), y = y_0(\eta) = \eta$, is nowhere tangential to a characteristic curve.

If the characteristic through (x, y) meets γ at $(x_0(\eta), \eta)$, then

$$x - cy = x_0(\eta) - c\eta. \tag{12}$$

But on γ, $u = u_0(\eta)$. How to get $u(x, y)$?

Implicit function theorem: It roughly says, you can solve $f(x, y) = 0$ for y at (x_0, y_0), if $f_y(x_0, y_0) \neq 0$. Try $x^2 + y^2 - 1 = 0$ at $(1, 0)$ and $(0, 1)$.

- Suppose γ is nowhere tangential to a characteristic curve, then $x_0'(\eta) \neq c$, and using implicit function theorem, we can solve (14) for η locally in a neighourhood of *each point* of γ in the form

$$\eta = g(x - cy), \quad g \in \mathcal{C}^1(I_1), \quad I_1 \subset I. \tag{13}$$

- The solution of this noncharacteristic Cauchy problem in a domain containing the curve γ is

$$\begin{aligned} u(x, y) &= u_0(\eta) \\ &= u_0\left(g(x - cy)\right). \end{aligned} \tag{14}$$

Example 2: Preliminaries through an example ... conti ...

- In Figure 2(b), the datum curve is a characteristic curve $x = cy + 1$.

- The data $u_0(y)$ prescribed on this line must be a constant, say $u_0(y) = a$.
  Why?. For answer, see slide 10.

- Now we can verify that the solution is given by

$$u(x, y) = a + (x - cy - 1)h(x - cy), \tag{15}$$

  where $h(\eta)$ is an *arbitrary* $\mathcal{C}^1$ function of just one argument.

- **This verifies a general property that the solution of a characteristic Cauchy problem, when it exists, it is not unique.**

We have done a lot with just one example.

Directional derivative

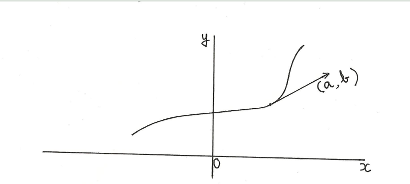
$u_x = 0$ means rate of change of u in direction $(1, 0)$ parallel to $x-$axis is zero

i.e. $(1, 0).(u_x, u_y) = 0$

We say $u_x = 0$ is a directional derivative in the direction $(1, 0)$.

Consider a curve with parametric representation $x = x(\sigma), y = y(\sigma)$ given by ODE

$$\frac{dx}{d\sigma} = a(x, y), \quad \frac{dy}{d\sigma} = b(x, y) \tag{16}$$



Tangent direction of the curve at (x, y):

$$(a(x, y), b(x, y)) \tag{17}$$

Directional derivative contd..

Rate of change of $u(x,y)$ with σ as we move along this curve is

$$\begin{aligned} \frac{du}{d\sigma} &= u_x \frac{dx}{d\sigma} + u_y \frac{dy}{d\sigma} \\ &= a(x,y)u_x + b(x,y)u_y \end{aligned} \tag{18}$$

which is a directional derivative in the direction (a,b) at (x,y).

If u satisfies PDE $au_x + bu_y = c(x,y,u)$ then

$$\frac{du}{d\sigma} = c(x,y,u) \tag{19}$$

Characteristic equation and compatibility condition

For the PDE

$$a(x,y)u_x + b(x,y)u_y = c(x,y) \tag{20}$$

Characteristic equations

$$\frac{dx}{d\sigma} = a(x,y), \frac{dy}{d\sigma} = b(x,y) \tag{21}$$

and compatibility condition

$$\frac{du}{d\sigma} = c(x,y). \tag{22}$$

(21) and (22) with $a(x,y) \neq 0$ give another form of characteristic equation and compatibility condition

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} \tag{23}$$

$$\frac{du}{dx} = \frac{c(x,y)}{a(x,y)}. \tag{24}$$

Characteristic PDE

Characteristic curves of linear and semilinear equations form a one parameter family of curves.

How?

Please note the difference between parametric representation and one or two or multi-parameter family of curves.

Example 4

$$yu_x - xu_y = 0 \tag{25}$$

Characteristic equations are

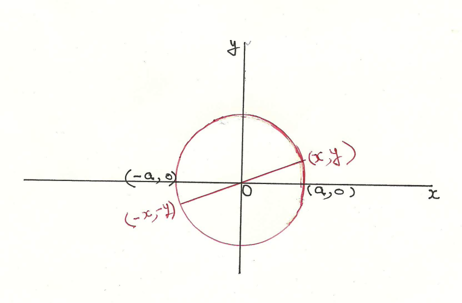
$$\frac{dx}{d\sigma} = y, \ \frac{dy}{d\sigma} = -x \tag{26}$$

or

$$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow y\ dy + x\ dx = 0 \Rightarrow x^2 + y^2 = \text{constant.} \tag{27}$$

The *characteristic curves* form a one parameter family of curves, which are circles with centre at (0,0).

Example 4 contd..



- Compatibility conditions along these curves are

$$\frac{du}{d\sigma} = 0 \Rightarrow u = \text{constant}. \tag{28}$$

- Hence value of u at (x, y) = value of u at $(-x, -y)$.
- u is an even function of x and also of y.

Example 4 contd..

- Will this even function be of the form $u = f(x^2 + y^4)$?
- The information
  $u =$ constant on the circles $x^2 + y^2 =$ constant
  $\Rightarrow u = f(x^2 + y^2)$

  where $f \in C^1(\mathbb{R})$ is arbitrary.
- Every solution is of this form.
- u is an even function of x and y but of a special form.

Nonhomogeneous linear first order PDE

$$a(x,y)u_x + b(x,y)u_y = c_1(x,y)u + c_2(x,y) \tag{29}$$

- Let $w(x,y)$ be any solution of the nonhomogeneous equation (29). Set $u = v + w(x,y)$
  $\Rightarrow v$ satisfies the homogeneous equation

$$a(x,y)v_x + b(x,y)v_y = c_1(x,y)v \tag{30}$$

- Let $f(x,y)$ be a general solution of (30)

$$\Rightarrow u = f(x,y) + w(x,y) \tag{31}$$

Example 5: Equation with constant coefficients

$$au_x + bu_y = c; \quad a, b, c \text{ are constants} \tag{32}$$

For the homogeneous equation, $c = 0$, characteristic equation (with $a \neq 0$)

$$\frac{dy}{dx} = \frac{b}{a} \Rightarrow ay - bx = \text{constant} \tag{33}$$

Along these

$$\frac{du}{dx} = 0 \Rightarrow u = \text{constant} \tag{34}$$

Hence $u = f(ay - bx)$ is general solution of the homogeneous equation.

Example 5: Equation with constant coefficients contd..

- For the nonhomogeneous equation, the compatibility condition

$$\frac{du}{dx} = \frac{c}{a} \Rightarrow u = \text{const} + \frac{c}{a}x \tag{35}$$

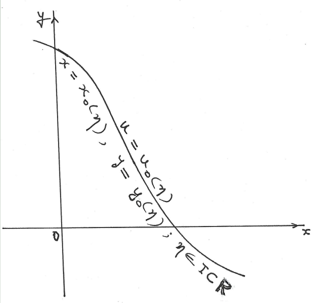
The constant here is constant along the characteristics $ay - bx = \text{const.}$
Hence general solution

$$u = f(ay - bx) + \frac{c}{a}x. \tag{36}$$

Alternatively $u = \frac{c}{a}x$ is a particular solution. Hence the result.

- Solution of a PDE contains arbitrary elements. For a first order PDE, it is an arbitrary function.
- In applications - additional condition $\Rightarrow$ Cauchy problem.

The Cauchy Problem for $F(x, y; u; u_x, u_y) = 0$



- Cauchy data $u_0(\eta)$ is prescribed on curve $\gamma : x = x_0(\eta),\ y = y_0(\eta),\ \eta \in I \subset \mathbb{R}$.
- Find a solution $u(x, y)$ in a neighbourhood of γ such that the solution takes the prescribed value $u_0(\eta)$ on γ, i.e.

$$u(x_0(\eta), y_0(\eta)) = u_0(\eta) \tag{37}$$

Existence and uniqueness of solution of a Cauchy problem requires restrictions on γ, the function F and the Cauchy data $u_0(\eta)$.

Method of Solution of Cauchy Problem - shown geometrically

Characteristics carry the solution.

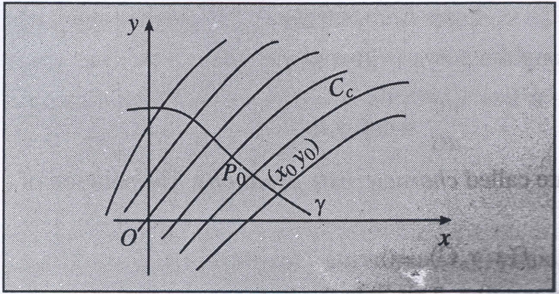


Fig. 1.1. *Solution of a Cauchy problem with the help of characteristic curves C_c*

Example 6a

Solve $yu_x - xu_y = 0$ in $\mathbb{R}^2$
with $u(x,0) = x, x \in \mathbb{R}$

- The solution must be an even function of x and y.

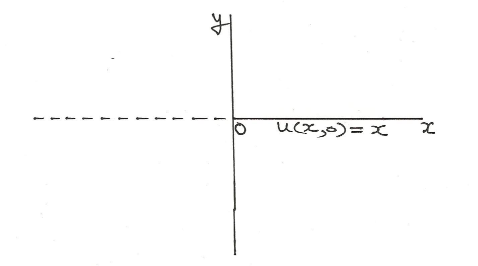
  But the Cauchy data is an odd function.
- Does the solution exist?

Example 6b

Solve $yu_x - xu_y = 0$ in a domain D

$$u(x,0) = x, \quad x \in \mathbb{R}_+ \tag{38}$$

Example 6b conti....



- Solution is $u(x, y) = (x^2 + y^2)^{1/2}$, verify with partial derivatives

$$u_x = \frac{x}{(x^2 + y^2)^{1/2}}, u_y = \frac{y}{(x^2 + y^2)^{1/2}} \tag{39}$$

- Solution is determined in $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Algorithm to Solve A Cauchy Problem

1. Write the Cauchy data as
$$x = x_0(\eta), \quad y = y_0(\eta) \qquad (A); \qquad u = u_0(\eta). \qquad (B)$$

2. Solve Characteristic equations and compatibility condition
$$\frac{dx}{d\sigma} = a(x, y), \quad \frac{dy}{d\sigma} = b(x, y); \qquad \frac{du}{d\sigma} = c(x, y)$$
with initial data at $\sigma = 0$ as in (A) and (B), i.e.
$$(x, y, u)|_{\sigma=0} = (x_0(\eta), y_0(\eta), u_0(\eta)).$$

3. We get
$$\begin{aligned} x &= x(\sigma, x_0(\eta), y_0(\eta), u_0(\eta)) \equiv X(\sigma, \eta) \\ y &= y(\sigma, x_0(\eta), y_0(\eta), u_0(\eta)) \equiv Y(\sigma, \eta) \\ u &= u(\sigma, x_0(\eta), y_0(\eta), u_0(\eta)) \equiv U(\sigma, \eta) \end{aligned}$$

4. Solving the first two for $\sigma = \sigma(x, y), \eta = \eta(x, y)$ we get the solution $u = U(\sigma(x, y), \eta(x, y)) \equiv u(x, y)$.

Example 7

Cauchy problem: Solve

$$2u_x + 3u_y = 1$$

with data $u|_\gamma = u(\alpha\eta, \beta\eta) = u_0(\eta)$ on

$$\gamma : \beta x - \alpha y = 0; \;\; \alpha, \beta = \text{constant}$$

A parametric representation of Cauchy data is

$$x|_\gamma = \alpha\eta = x_0(\eta), \;\; say, \;\; y|_\gamma = \beta\eta = y_0(\eta), \;\; say; \quad u|_\gamma = u_0(\eta)$$

where $u_0(\eta)$ is a given function.

Example 7 contd..

- Solution of characteristic equations

$$\frac{dx}{d\sigma} = 2, \ \frac{dy}{d\sigma} = 3$$

satisfying $x(\sigma = 0) = \alpha\eta, \ \ y(\sigma = 0) = \beta\eta$ are

$$x = \alpha\eta + 2\sigma, \ y = \beta\eta + 3\sigma, \ \eta = \text{const}, \ \sigma \text{ varies.} \tag{40}$$

These are characteristic curves staring from the points $x(\sigma = 0) = \alpha\eta, \ \ y(\sigma = 0) = \beta\eta$ of γ.

- Solution of the compatibility condition

$$\frac{du}{d\sigma} = 1$$

satisfying $u(\sigma = 0) = u_0(\eta)$ is

$$u = u_0(\eta) + \sigma \tag{41}$$

Example 7 contd..

To get the solution of the Cauchy problem, we

- first solve $x = \alpha\eta + 2\sigma,\ y = \beta\eta + 3\sigma$ for σ and η

$$\sigma = \frac{\beta x - \alpha y}{2\beta - 3\alpha},\ \eta = \frac{2y - 3x}{2\beta - 3\alpha} \tag{42}$$

- and then substitute in expression $u_0(\eta) + \sigma$ for u

$$u(x,y) = \frac{\beta x - \alpha y}{2\beta - 3\alpha} + u_0\left(\frac{2y - 3x}{2\beta - 3\alpha}\right) \tag{43}$$

Example 7 contd.. Existence and Uniqueness

The solution exists as long as $2\beta - 3\alpha \neq 0$ i.e., the datum curve is not a characteristic curve

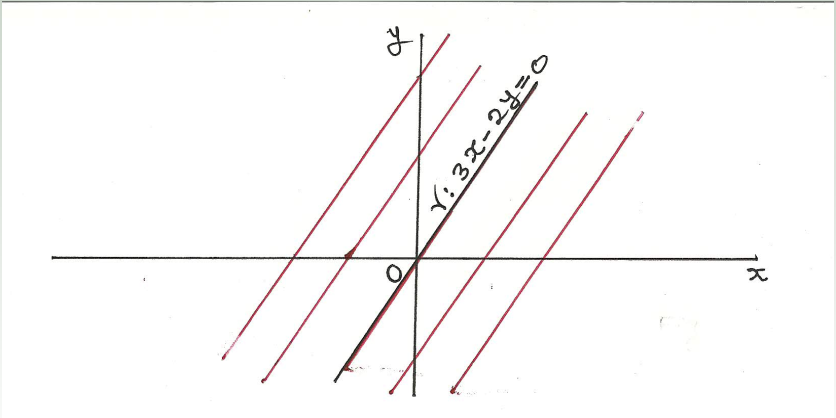
Uniqueness:
Compatibility condition carries information on the variation of u along a characteristic in unique way. This leads to uniqueness.

What happens when $2\beta - 3\alpha = 0$?

Example 8: Characteristic Cauchy problem

$2\beta - 3\alpha = 0 \Rightarrow$ datum curve is a characteristic curve.

Choose $\alpha = 2,\ \beta = 3 \Rightarrow x = 2\eta, y = 3\eta$.
Check with (40) with $\sigma = 0$.



Example 8: Characteristic Cauchy problem contd...

- The characteristic Cauchy problem: Solve

$$2u_x + 3u_y = 1$$

with data

$$u(2\eta, 3\eta) = u_0(\eta)$$

- Since

$$\frac{du_0(\eta)}{d\eta} = \frac{d}{d\eta}u(2\eta, 3\eta) = 2u_x + 3u_y = 1, \text{ using PDE,} \tag{44}$$

- the Cauchy data u_0 cannot be prescribed arbitrarily on γ.
$u_0(\eta) = \eta = \frac{1}{2}x$, ignoring constant of integration.

Example 8 contd..

- $u = \frac{1}{2}x$ is a particular solution satisfying the Cauchy data and $g(3x - 2y)$ is solution of the homogeneous equation.
- Hence

$$u = \frac{1}{2}x + g(3x - 2y), \; g \in C^1 \text{ and } g(0) = 0 \tag{45}$$

  is a solution of the Cauchy problem.
- Since g is any C^1 function with $g(0) = 0$, solution of the Characteristic Cauchy problem is not unique.
- We verify an important theorem "**in general, solution of a characteristic Cauchy problem does not exist and if exists, it is not unique**".

Quasilinear equation

Consider the equation

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{46}$$

- Since a and b depend on u, it is not possible to interpret

$$a(x, y, u)\frac{\partial}{\partial x} + b(x, y, u)\frac{\partial}{\partial y} \tag{47}$$

  as a directional derivative in (x, y)-plane.
- We substitute a known solution $u(x, y)$ for u in a and b, then at any point (x, y), it represents directional derivative $\frac{\partial}{\partial \sigma}$ in the direction given by

$$\frac{dx}{d\sigma} = a(x, y, u(x, y)), \frac{dy}{d\sigma} = b(x, y, u(x, y)) \tag{48}$$

- Along characteristic curves, given by (48), we get compatibility condition

$$\frac{du}{d\sigma} = c(x, y, u(x, y)) \tag{49}$$

Quasilinear equation contd..

- (49) is true for every solution $u(x, y)$. The **Characteristic equations**

$$\frac{dx}{d\sigma} = a(x, y, u), \ \frac{dy}{d\sigma} = b(x, y, u) \tag{50}$$

along with the **Compatibility condition**

$$\frac{du}{d\sigma} = c(x, y, u)$$

forms closed system.

Method of solution of a Cauchy problem

Solve

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \tag{51}$$

in a domain D containing

$$\gamma : x = x_0(\eta),\ y = y_0(\eta)$$

with Cauchy data

$$u(x_0(\eta), y_0(\eta)) = u_0(\eta) \tag{52}$$

Method of solution of a Cauchy problem contd..

Solve

$$\frac{dx}{d\sigma} = a(x,y,u), \ \ \frac{dy}{d\sigma} = b(x,y,u), \ \ \frac{du}{d\sigma} = c(x,y,u) \tag{53}$$

$$(x,y,u)|_{\sigma=0} = (x_0(\eta), y_0(\eta), u_0(\eta)) \tag{54}$$

$$\Rightarrow \quad \begin{aligned} x &= x(\sigma, x_0(\eta), y_0(\eta), u_0(\eta)) \equiv X(\sigma,\eta) \\ y &= y(\sigma, x_0(\eta), y_0(\eta), u_0(\eta)) \equiv Y(\sigma,\eta) \\ u &= u(\sigma, x_0(\eta), y_0(\eta), u_0(\eta)) \equiv U(\sigma,\eta) \end{aligned} \tag{55}$$

Solving the first two for $\sigma = \sigma(x,y), \eta = \eta(x,y)$ we get the solution

$$u = U(\sigma(x,y), \eta(x,y)) \equiv u(x,y) \tag{56}$$

Method of solution of a Cauchy problem contd..

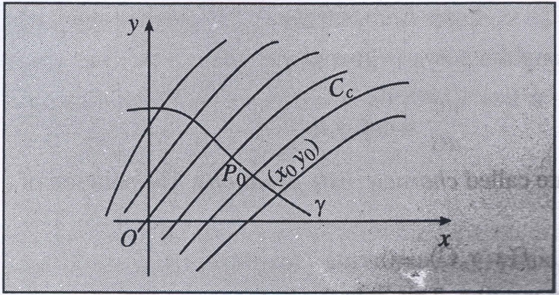


Fig. 1.1. *Solution of a Cauchy problem with the help of characteristic curves* C_c

Quasilinear equations conti...

Do not worry about the complex statement in the theorem below. As long as the datum curve γ is not tangential to a characteristic curve and the functions involved are smooth, the solution exist locally and is unique (see previous slide).

Theorem:

1. $x_0(\eta), y_0(\eta), u_0(\eta) \in C^1(I)$, say $I = (0, 1)$
2. $a(x, y, u), b(x, y, u), c(x, y, u) \in C^1(D_2)$, where D_2 is a domain in $(x, y, u)-$ space
3. D_2 contains curve Γ in (x, y, u)-space $\Gamma : x = x_0(\eta), y = y_0(\eta), u = u_0(\eta), \ \eta \in I$
4. $\frac{dy_0}{d\eta} a(x_0(\eta), y_0(\eta), u_0(\eta)) - \frac{dx_0}{d\eta} b(x_0(\eta), y_0(\eta), u_0(\eta)) \neq 0, \ \eta \in I$

There exists a unique solution of the Cauchy problem in a domain D containing I.

- **Note 1:** Condition 4 rules out that datum curve $\gamma : (x_0(\eta), y_0(\eta))$ is a characteristic curve.

Example 9

Cauchy problem

$$u_x + u_y = u$$
$$u(x,0) = 1 \Rightarrow$$
$$x_0 = \eta,\ y_0 = 0,\ u_0 = 1 \tag{57}$$

Step 1. Characteristic curves

$$\frac{dx}{d\sigma} = 1 \Rightarrow x = \sigma + \eta$$
$$\frac{dy}{d\sigma} = 1 \Rightarrow y = \sigma \tag{58}$$

Step 2. Therefore $\sigma = y, \quad \eta = x - y$

Step 3. Compatibility condition

$$\frac{du}{d\sigma} = u \Rightarrow u = u_0(\eta)e^{\sigma} = e^{\sigma}$$

Step 4. Solution $u = e^y$

exists on $D = \mathbb{R}^2$

Example 10

Cauchy problem (same as problem 9 with a small change on the RHS of the PDE)

$$\begin{aligned} u_x + u_y &= u^2 \\ u(x,0) &= 1 \Rightarrow \\ x_0 = \eta,\ y_0 = 0,\ u_0 &= 1 \end{aligned} \tag{59}$$

Step 1. Characteristic equations give

$$x = \sigma + \eta,\ y = \sigma$$

Step 2. Compatibility condition gives

$$\frac{du}{d\sigma} = u^2 \Rightarrow u = \frac{1}{u_0(\eta) - \sigma}$$

Step 3. Solution $u = \frac{1}{1-y}$
exists locally on the domain $D = y < 1$ and $u \to +\infty$ as $y \to 1-$.

Example 11

Cauchy problem

$$\begin{gathered} uu_x + u_y = 0 \\ u(x,0) = x, \ 0 \le x \le 1 \end{gathered} \tag{60}$$

$$\Rightarrow x = \eta, \ y = 0, \ u = \eta, \ 0 \le \eta \le 1 \text{ at } \sigma = 0$$

Step 1. Characteristic equations and compatibility condition

$$\frac{dx}{d\sigma} = u, \ \frac{dy}{d\sigma} = 1, \ \frac{du}{d\sigma} = 0 \tag{61}$$

Step 2. Quasilinear equations, characteristics depend on the solution

$$u = \eta$$

Example 11 contd..

Step 3. Substituting $u = \eta$ in (61) we get

$$x = \eta(\sigma + 1), \; y = \sigma$$

Step 4. From solution of characteristic equations $\sigma = y$ and $\eta = \frac{x}{y+1}$

Step 5. Solution is $u = \frac{x}{y+1}$, but what is domain D of the solution ?

Step 6. Characteristic curves are straight lines

$$\frac{x}{y+1} = \eta, \quad 0 \leq \eta \leq |$$

which meet at the point $(-1, 0)$.

Step 7. u is constant on these characteristics (see next slide).

Example 11 contd..

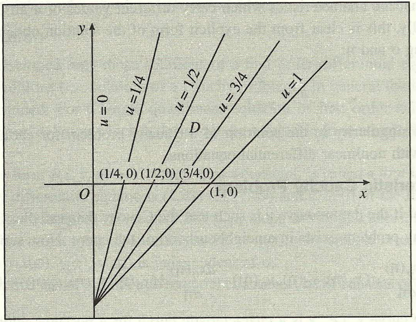


Fig. 1.2. *The solution is determined in the wedge shaped region D of the (x, y)-plane*

Figure: Solution is determined in a wedged shaped region in (x, y)-plane including the lines $y = 0$ and $y = x + 1$.

We note

$$u(0, -1) \tag{62}$$

is not defined.

Example 12

Cauchy problem

$$\begin{aligned} uu_x + u_y = 0 \\ u(x,0) = \tfrac{1}{2},\ 0 \le x \le 1 \end{aligned} \tag{63}$$

Step 1. Parametrization of Cauchy data

$$\Rightarrow x_0 = \eta,\ y_0 = 0,\ u_0 = \frac{1}{2},\ 0 \le \eta \le 1.$$

Step 2. The compatibility condition along characteristic curves gives

$$u = constant = \frac{1}{2}.$$

Step 3. The characteristic curves are

$$y - 2x = -2\eta, \quad 0 \le \eta \le 1. \tag{64}$$

on which solution has the same value $u = \frac{1}{2}$.

Example 12 contd..

Step 4. The solution $u = \frac{1}{2}$ of the Cauchy problem is determined in an infinite strip $2x - 2 \leq y \leq 2x$ in (x, y)-plane.

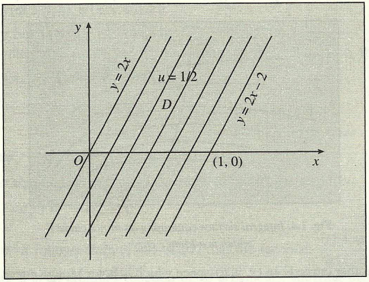


Fig. 1.3. *The domain D when the Cauchy data is u (x, 0) = 1/ 2 for 0 ≤ x ≤ 1*

Important: From examples 11 and 12, we notice that the domain, where solution of a Cauchy problem for a quasilinear equation is determined, depends on the Cauchy data.

Example 13

Consider initial data for $uu_x + u_y = 0$:

$$u(x,t) = \begin{cases} 1\ , & x \le 0 \\ 1 - x\ , & 0 < x \le 1 \\ 0\ , & x > 1. \end{cases} \tag{65}$$

Solution remains continuous for $0 \le y < 1$

$$u(x,y) \ = \begin{cases} 1\ , & x \le y \\ \frac{1-x}{1-y}\ , & y < x \le 1 \\ 0\ , & x > 1 \end{cases} \tag{66}$$

Solution is not valid at $y = 1$ but data at $y = 1$

$$u(x,1) \ = \begin{cases} 1\ , & -\infty < x \le \frac{1}{2} \\ 0\ , & 1 < x < \infty \end{cases} \tag{67}$$

Draw the figure in (x,y)-plane.

Example 14

Cauchy problem

$$uu_x + u_y = 0$$

$u(x,0) = 0, x < 0,$
$u(x,0) = x, 0 \le x \le 1,$
$u(x,0) = 1, x > 1.$

Initial data is continuous but solution (given below) is not a genuine solution - why?

$$u(x,y) = 0, \quad x < 0; \quad u(x,y) = 1, \quad x > 1 + y;$$

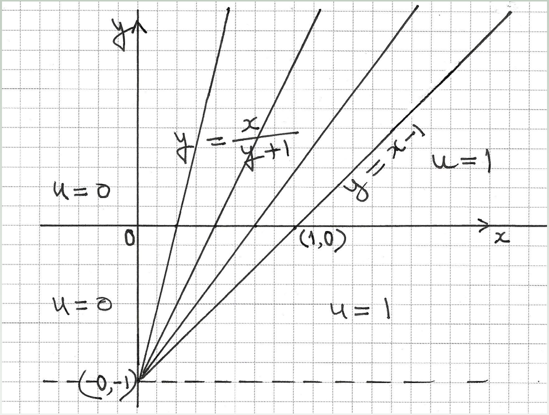
$$u(x,y) = \frac{x}{y+1}, \quad 0 \le \frac{x}{y+1} \le 1, \quad y > -1. \tag{68}$$

Solution as $y \longrightarrow (-1)+$ is

$$u(x,-1) = 0,\ x < 0; \qquad u(x,-1) = 1,\ x > 0. \tag{69}$$

Solution is shown graphically on next slide.

Example 14 ··· conti.



Example 15

For the Cauchy problem

$$u u_x + u_y = 0$$
$$u(x,0) = x,\ 0 \le x \le 1/2, \quad u(x,0) = \tfrac{1}{2},\ 1/2 \le x \le 1. \qquad (70)$$

- Find the solution,
- find the domain of the solution,
- draw characteristic curves and
- note that the solution is continuous but not a genuine solution.
- Why is it not a genuine solution?

General solution

General solution of a first order PDE contains an arbitrary function.

Theorem : If $\phi(x,y,u) = C_1$ and $\psi(x,y,u) = C_2$ be two independent first integrals of the ODEs

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)} \tag{71}$$

and $\phi_u^2 + \psi_u^2 \neq 0$, the general solution of the PDE $au_x + bu_y = c$ is given by

$$h(\phi(x,y,u),\ \psi(x,y,u)) = 0 \tag{72}$$

where h is an arbitrary function.

For proof see PP-RR PDE.

Example 16

$$uu_x + u_y = 0 \tag{73}$$

$$\frac{dx}{u} = \frac{dy}{1} = \frac{du}{0} \tag{74}$$

Note 0 appearing in a denominator to be properly interpreted

$$\begin{aligned} \Rightarrow u &= C_1 \\ x - C_1 y &= C_2 \\ \Rightarrow x - uy = C_2 &\Rightarrow \end{aligned} \tag{75}$$

General solution is given by

$$\begin{aligned} \phi(u, x - uy) &= 0 \\ \text{or } u &= f(x - uy) \end{aligned} \tag{76}$$

where h and f arbitrary functions.
Note : Solution of this nonlinear equation may be very difficult. Numerical method is generally used.

Example 17

Consider the differential equation

$$(y + 2ux)u_x - (x + 2uy)u_y = \frac{1}{2}(x^2 - y^2) \tag{77}$$

The characteristic equations and the compatibility condition are

$$\frac{dx}{y + 2ux} = \frac{dy}{-(x + 2uy)} = \frac{du}{\frac{1}{2}(x^2 - y^2)} \tag{78}$$

To get one first integral we derive from these

$$\frac{xdx + ydy}{2u(x^2 - y^2)} = \frac{2du}{x^2 - y^2} \tag{79}$$

which immediately leads to

$$\varphi(x, y, u) \equiv x^2 + y^2 - 4u^2 = C_1 \tag{80}$$

Example 17 contd..

For another independent first integral we derive a second combination

$$\frac{ydx + xdy}{y^2 - x^2} = \frac{2du}{x^2 - y^2} \tag{81}$$

which leads to

$$\psi(x, y, u) \equiv xy + 2u = C_2 \tag{82}$$

The general integral of the equation (55)is given by

$$\begin{aligned} h(x^2 + y^2 - 4u^2,\ xy + 2u) = 0 \\ x^2 + y^2 - 4u^2 = f(xy + 2u) \end{aligned} \tag{83}$$

where h or f are arbitrary functions of their arguments.

We can use a general solution to solve a Cuachy problem. See next slide.

Example 17 contd..

Consider a Cauchy problem for equation (77) with Cauchy data $u = 0$ on $x - y = 0$

$$\Rightarrow \quad x = \eta, y = \eta, u = 0$$

- From (58) and (60) we get $2\eta^2 = C_1$ and $\eta^2 = C^2$ which gives a relation between constants in (80) and (82): $C_1 = 2C_2$.
- Therefore, the solution of the Cauchy problem is obtained, when we take $h(\varphi, \psi) = \varphi - 2\psi$.
- This gives, taking only the suitable one,

$$u = \frac{1}{2}\left\{\sqrt{(x-y)^2+1} - 1\right\}. \quad (84)$$

We note that the solution of the Cauchy problem is determined uniquely at all points in the (x, y)-plane.

Two Important References

- In 1992 I gave a lecture at The Larmor Society, which is the Natural Sciences Society, St Johns College at Cambridge.
- The lecture was meant for undergraduate students and hence I used the language of physics without any mathematical equations.
- Based on the idea in this lecture, I wrote a popular article Nonlinearity, Conservation Laws and Shocks in two parts and it was published in 1997 in Resonance. See reference [4].
- But a reader has to pause and think a lot to understand the mathematical concepts.

Exercise

1. Show that all the characteristic curves of the partial differential equation

$$(2x+u)u_x + (2y+u)u_y = u$$

through the point (1,1) are given by the same straight line $x - y = 0$

2. Discuss the solution of the differential equation

$$uu_x + u_y = 0, \;\; y > 0, \;\; -\infty < x < \infty$$

with Cauchy data

$$u(x,0) = \begin{cases} \alpha^2 - x^2 & \text{for } |x| \le \alpha \\ 0 & \text{for } |x| > \alpha. \end{cases} \qquad (85)$$

Exercise contd..

3. Find the solution of the differential equation

$$\left(1 - \frac{m}{r}u\right)u_x - mMu_y = 0$$

satisfying

$$u(0, y) = \frac{My}{\rho - y}$$

where m, r, ρ, M are constants, in a neighbourhood of the point $x = 0, y = 0$.

4. Find the general integral of the equation

$$(2x - y)y^2u_x + 8(y - 2x)x^2u_y = 2(4x^2 + y^2)u$$

and deduce the solution of the Cauchy problem when the $u(x, 0) = \frac{1}{2x}$ on a portion of the x-axis.

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Thank You!