First Order Partial Differential Equations, Part - 1: Single Linear and Quasilinear First Order Equations

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General Comments

- First order PDE is simplest and historically oldest (a general class of) PDE with almost a complete theory and beautiful mathematical structure.

- Yet students find its theory mysterious and more difficult than unstructured theory of higher order equations.

- Classical theory of first order PDE started in about 1760 with Euler and D’Alembert and ended in about 1890 with the work of Lie.

- In intervening period Lagrange, Charpit, Monge, Pfaff, Cauchy, Jacobi and Hamilton made deep and important contributions to it and mechanics.
Wave equation first appeared in print in 1747 (a little before the theory of FOPDE) by Lagrange and Laplace equation in 1784 by Laplace.

But they did not give the general theory.
General Comments

- Complete integral of FOPDE played a very important role in mechanics.

- But the theory of “complete integrals”, is no longer treated as essential for study in a basic course in PDE (see Evan’s book).

- I shall also skip complete integrals, while dealing with nonlinear equations.
Definition

First order PDE for a function $u(x, y)$ of two independent variables is a relation

$$F(x, y; u; u_x, u_y) = 0,$$

$F$ a known **real** function from $D_3 \subset \mathbb{R}^5 \to \mathbb{R}$. \hfill (1)

In this lecture we denote

- by $D$ a domain in $\mathbb{R}^2$ where a solution $u$ is defined.
- We shall define other domains when needed.
Classification

Linear equation (nonhomogeneous):

\[ a(x, y)u_x + b(x, y)u_y = c_1(x, y)u + c_2(x, y) \]  \hspace{1cm} (2)

Nonlinear equation: All other equations with subclasses:

1). Semilinear equation:

\[ a(x, y)u_x + b(x, y)u_y = c(x, y, u) \]  \hspace{1cm} (3)

2). Quasilinear equation:

\[ a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \]  \hspace{1cm} (4)

3). Nonlinear equation: \( F(x, y; u; u_x, u_y) = 0 \) where \( F \) is not linear in \( u_x, u_y \).

Properties of solutions of all 4 classes of equations are quite different.
Example 1a: Simplest PDE

\[ u_x = 0 \]

General solution in \( D = \mathbb{R}^2 \) is \( u = f(y) \), where \( f \) is an arbitrary \( C^1 \) function.

Solution \( u \) is uniquely determined if it is prescribed on any curve nowhere parallel to \( x \)-axis.

On a line parallel to \( x \)-axis, we cannot prescribe \( u \) arbitrarily. Why?
Example 1a: Nonhomogeneous equation

Consider PDE

\[ u_x = c(x, y), \quad c(x, y) \text{ a known function} \quad (5) \]

with condition

\[ u(0, y) = f(y), \quad f(y) \text{ a function prescribed on the y axis.} \quad (6) \]

- The unique and stable solution of this problem is

\[ u = \int_0^x c(\sigma, y) d\sigma + f(y). \quad (7) \]

- Is solution of a first order equation so simple?
- Yes, it is for a linear equation provided we understand the role of characteristic curves. See the article provided to you.
Example 2: Preliminaries through an example

- A transport equation in two independent variables:
  \[ u_y + cu_x = 0, \quad c = \text{real constant}. \]  
  \( (8) \)

- Introduce a variable \( \eta = x - cy \). For a fixed \( \eta \), \( x - cy = \eta \) is a straight line with slope \( \frac{1}{c} \) in \((x, y)\) plane.

- Along this straight line

  \[ x(y) = cy + \eta \]  
  \( (9) \)

  the derivative of a solution \( u(x, y) = u(x(y), y) \) on this line is

  \[ \frac{d}{dy} u(x(y), y) = u_x \frac{dx}{dy} + u_y \]

  \[ = cu_x + u_y \]

  \[ = 0. \]

- Thus, the solution \( u \) is constant along curves \( x - cy = \eta \). These lines are characteristic curves of (8). PDE See figure on next slide.
Example 2: Preliminaries through an example ... conti...

(a) Characteristics form a one parameter family of straight lines $x - cy = \eta$, where $\eta$ is the parameter.

(b) $u$ is constant along a characteristic curve.

Characteristic curves of $u_y + cu_x = 0$. 

(a) Characteristics form a one parameter family of straight lines $x - cy = \eta$, where $\eta$ is the parameter.

(b) $u$ is constant along a characteristic curve.
Example 2: Preliminaries through an example ... conti...

- Consider an initial value problem (which is a Cauchy problem) of the equation (8) in which

\[ u(x, 0) = u_0(x). \] (10)

- To find solution at \((x, y)\), draw the characteristic through \((x, y)\) and let it meet the \(x\)-axis at \(x = \eta\). Then, \(u\) is constant on \(x - cy = \eta\), i.e,

\[ u(x, y) = u(\eta, 0) = u_0(\eta). \]

Hence, as \(\eta = x - cy\),

\[ u(x, y) = u_0(x - cy). \] (11)
For the simplest PDE (10) and Real $u_0$

Theorem

If $c$ is real and $u_0 \in C^1(\mathbb{R})$, there is a unique solution $u \in C^1(\mathbb{R}^2)$ to the initial value problem (8), (10). The solution is given by the formula $u(x, y) = u_0(x - cy)$.

- The solution is $C^1(\mathbb{R})$. Here $D = \mathbb{R}^2$
- Solution is stable for small changes in Cauchy data $u_0$
Do you know a parametric representation of a curve?

A parametric representation of circle $x^2 + y^2 = 1$ is $x = \cos \eta, \ y = \sin \eta; \ 0 \leq \eta < \pi$.

**Problem:** Write a parametric representation of $y^2 = x$. 
Example 2: Preliminaries through an example ... conti...

Now we prescribe the Cauchy data for the PDE (8): \( u_y + cu_x = 0, \ c = \text{real constant} \)
on curve shown in the Figure 2(a): \( \gamma: x = x_0(y), \ x_0 \in C^1(I), \) written parametrically as \( \gamma: x = x_0(\eta), \ y = \eta = y_0(\eta), \) say.
\( u \) is prescribed on \( \gamma \) as \( u(x_0(\eta), y_0(\eta)) \equiv u(x_0(\eta), \eta) = u_0(\eta). \)

**Problem:** Find \( u \) in a neighbourhood of \( \gamma. \)

![Graph](image)

**Figure:** 2(a) - The datum curve \( x = x_0(y), \) i.e. \( x = x_0(\eta), y = y_0(\eta) = \eta, \) is nowhere tangential to a characteristic curve.
Example 2: Preliminaries through an example ... conti...

\[ u = u_0(y) \]

\[ x = cy + 1 \]

Figure: 2(b) - The datum curve is a characteristic curve \( x = cy + 1 \).
Example 2: Preliminaries through an example ... conti ...

Solution for the case 2(a).

Figure: 2(a) - The datum curve \( x = x_0(y) \), i.e. \( x = x_0(\eta), y = y_0(\eta) = \eta \), is nowhere tangential to a characteristic curve.

If the characteristic through \((x, y)\) meets \(\gamma\) at \((x_0(\eta), \eta)\), then

\[
x - cy = x_0(\eta) - c\eta.
\]  

But on \(\gamma\), \( u = u_0(\eta) \). How to get \( u(x, y) \)?
Example 2: Preliminaries through an example ... conti ...

**Implicit function theorem:** It *roughly* says, you can solve $f(x, y) = 0$ for $y$ at $(x_0, y_0)$, if $f_y(x_0, y_0) \neq 0$. Try $x^2 + y^2 - 1 = 0$ at $(1, 0)$ and $(0, 1)$.

- Suppose $\gamma$ is nowhere tangential to a characteristic curve, then $x'_0(\eta) \neq c$, and using implicit function theorem, we can solve (14) for $\eta$ locally in a neighbourhood of *each point* of $\gamma$ in the form

$$
\eta = g(x - cy), \quad g \in C^1(I_1), \quad I_1 \subset I.
$$

(13)

- The solution of this noncharacteristic Cauchy problem in a domain containing the curve $\gamma$ is

$$
u(x, y) = u_0(\eta) = u_0\left(g(x - cy)\right).
$$

(14)
Example 2: Preliminaries through an example ... conti ...

- In Figure 2(b), the datum curve is a characteristic curve \( x = cy + 1 \).

- The data \( u_0(y) \) prescribed on this line must be a constant, say \( u_0(y) = a \).
  
  Why?. For answer, see slide 10.

- Now we can verify that the solution is given by

\[
 u(x, y) = a + (x - cy - 1)h(x - cy),
\]

(15)

where \( h(\eta) \) is an arbitrary \( C^1 \) function of just one argument.

- This verifies a general property that the solution of a characteristic Cauchy problem, when it exists, is not unique.

We have done a lot with just one example.
Directional derivative

\( u_x = 0 \) means rate of change of \( u \) in direction \((1, 0)\) parallel to \( x-\)axis is zero

i.e. \((1, 0). (u_x, u_y) = 0\)

We say \( u_x = 0 \) is a directional derivative in the direction \((1, 0)\).

Consider a curve with parametric representation \( x = x(\sigma), y = y(\sigma) \)

given by \( ODE \)

\[
\frac{dx}{d\sigma} = a(x, y), \quad \frac{dy}{d\sigma} = b(x, y)
\]  \hspace{1cm} (16)

Tangent direction of the curve at \((x, y)\):

\[(a(x, y), b(x, y))\]  \hspace{1cm} (17)
Directional derivative contd..

Rate of change of $u(x, y)$ with $\sigma$ as we move along this curve is

$$\frac{du}{d\sigma} = u_x \frac{dx}{d\sigma} + u_y \frac{dy}{d\sigma}$$

$$= a(x, y)u_x + b(x, y)u_y$$

which is a directional derivative in the direction $(a, b)$ at $(x, y)$.

If $u$ satisfies PDE $au_x + bu_y = c(x, y, u)$ then

$$\frac{du}{d\sigma} = c(x, y, u)$$
Characteristic equation and compatibility condition

For the PDE

\[ a(x, y)u_x + b(x, y)u_y = c(x, y) \]  \hspace{1cm} (20)

Characteristic equations

\[ \frac{dx}{d\sigma} = a(x, y), \quad \frac{dy}{d\sigma} = b(x, y) \]  \hspace{1cm} (21)

and compatibility condition

\[ \frac{du}{d\sigma} = c(x, y). \]  \hspace{1cm} (22)

(21) and (22) with \( a(x, y) \neq 0 \) give another form of characteristic equation and compatibility condition

\[ \frac{dy}{dx} = \frac{b(x, y)}{a(x, y)} \]  \hspace{1cm} (23)

\[ \frac{du}{dx} = \frac{c(x, y)}{a(x, y)}. \]  \hspace{1cm} (24)
Characteristic curves of linear and semilinear equations form a one parameter family of curves.

How?

Please note the difference between parametric representation and one or two or multi-parameter family of curves.
Example 4

\[ yu_x - xu_y = 0 \quad \text{(25)} \]

Characteristic equations are

\[ \frac{dx}{d\sigma} = y, \quad \frac{dy}{d\sigma} = -x \quad \text{(26)} \]

or

\[ \frac{dy}{dx} = -\frac{x}{y} \Rightarrow y \, dy + x \, dx = 0 \Rightarrow x^2 + y^2 = \text{constant}. \quad \text{(27)} \]

The characteristic curves form a one parameter family of curves, which are circles with centre at (0,0).
Compatibility conditions along these curves are

\[ \frac{du}{d\sigma} = 0 \Rightarrow u = \text{constant}. \]  \hspace{1cm} (28)

Hence value of \( u \) at \((x, y)\) = value of \( u \) at \((-x, -y)\).

\( u \) is an even function of \( x \) and also of \( y \).
Will this even function be of the form \( u = f(x^2 + y^4) \)?

The information

\[
u = \text{constant on the circles } x^2 + y^2 = \text{constant} \Rightarrow u = f(x^2 + y^2)
\]

where \( f \in C^1(\mathbb{R}) \) is arbitrary.

Every solution is of this form.

\( u \) is an even function of \( x \) and \( y \) but of a special form.
Nonhomogeneous linear first order PDE

\[ a(x, y)u_x + b(x, y)u_y = c_1(x, y)u + c_2(x, y) \]  

(29)

Let \( w(x, y) \) be any solution of the nonhomogeneous equation (29). Set \( u = v + w(x, y) \)

\[ \Rightarrow v \text{ satisfies the homogeneous equation} \]

\[ a(x, y)v_x + b(x, y)v_y = c_1(x, y)v \]  

(30)

Let \( f(x, y) \) be a general solution of (30)

\[ \Rightarrow u = f(x, y) + w(x, y) \]  

(31)
Example 5: Equation with constant coefficients

\[ au_x + bu_y = c; \quad a, b, c \text{ are constants} \quad (32) \]

For the homogeneous equation, \( c = 0 \), characteristic equation (with \( a \neq 0 \))

\[ \frac{dy}{dx} = \frac{b}{a} \Rightarrow ay - bx = \text{constant} \quad (33) \]

Along these

\[ \frac{du}{dx} = 0 \Rightarrow u = \text{constant} \quad (34) \]

Hence \( u = f(ay - bx) \) is general solution of the homogeneous equation.
Example 5: Equation with constant coefficients contd..

- For the nonhomogeneous equation, the compatibility condition

\[ \frac{du}{dx} = \frac{c}{a} \Rightarrow u = \text{const} + \frac{c}{a} x \]  \hspace{1cm} (35)

The constant here is constant along the characteristics \( ay - bx = \text{const} \).

Hence general solution

\[ u = f(ay - bx) + \frac{c}{a} x. \]  \hspace{1cm} (36)

Alternatively \( u = \frac{c}{a} x \) is a particular solution. Hence the result.

- Solution of a PDE contains arbitrary elements. For a first order PDE, it is an arbitrary function.

- In applications - additional condition \( \Rightarrow \) Cauchy problem.
The Cauchy Problem for \( F(x, y; u; u_x, u_y) = 0 \)

- Cauchy data \( u_0(\eta) \) is prescribed on curve
  \( \gamma : x = x_0(\eta), \ y = y_0(\eta), \ \eta \in I \subset \mathbb{R} \).
- Find a solution \( u(x, y) \) in a neighbourhood of \( \gamma \) such that the solution takes the prescribed value \( u_0(\eta) \) on \( \gamma \), i.e.
  \[
  u(x_0(\eta), y_0(\eta)) = u_0(\eta) \quad (37)
  \]

Existence and uniqueness of solution of a Cauchy problem requires restrictions on \( \gamma \), the function \( F \) and the Cauchy data \( u_0(\eta) \).
Method of Solution of Cauchy Problem - shown geometrically

Characteristics carry the solution.

Fig. 1.1. Solution of a Cauchy problem with the help of characteristic curves $C_c$
Example 6a

Solve \( yu_x - xu_y = 0 \) in \( \mathbb{R}^2 \)
with \( u(x, 0) = x, x \in \mathbb{R} \)

- The solution must be an even function of \( x \) and \( y \).
  But the Cauchy data is an odd function.
- Does the solution exist?

Example 6b

Solve \( yu_x - xu_y = 0 \) in a domain \( D \)

\[
u(x, 0) = x, \quad x \in \mathbb{R}_+ \quad (38)
\]
Solution is \( u(x, y) = (x^2 + y^2)^{1/2} \), verify with partial derivatives

\[
\begin{align*}
  u_x &= \frac{x}{(x^2 + y^2)^{1/2}}, \\
  u_y &= \frac{y}{(x^2 + y^2)^{1/2}}
\end{align*}
\]  

\[ (39) \]

Solution is determined in \( \mathbb{R}^2 \setminus \{(0, 0)\} \).
Algorithm to Solve A Cauchy Problem

1. Write the Cauchy data as

\[ x = x_0(\eta), \quad y = y_0(\eta) \quad (A); \quad u = u_0(\eta). \quad (B) \]

2. Solve Characteristic equations and compatibility condition

\[
\frac{dx}{d\sigma} = a(x, y), \quad \frac{dy}{d\sigma} = b(x, y); \quad \frac{du}{d\sigma} = c(x, y)
\]

with initial data at \( \sigma = 0 \) as in (A) and (B), i.e.

\[ (x, y, u)|_{\sigma=0} = (x_0(\eta), y_0(\eta), u_0(\eta)). \]

3. We get

\[
\begin{align*}
x &= x(\sigma, x_0(\eta), y_0(\eta), u_0(\eta)) \equiv X(\sigma, \eta) \\
y &= y(\sigma, x_0(\eta), y_0(\eta), u_0(\eta)) \equiv Y(\sigma, \eta) \\
u &= u(\sigma, x_0(\eta), y_0(\eta), u_0(\eta)) \equiv U(\sigma, \eta)
\end{align*}
\]

4. Solving the first two for \( \sigma = \sigma(x, y), \eta = \eta(x, y) \) we get the solution

\[ u = U(\sigma(x, y), \eta(x, y)) \equiv u(x, y). \]
Example 7

Cauchy problem: Solve

\[ 2u_x + 3u_y = 1 \]

with data \( u|_\gamma = u(\alpha \eta, \beta \eta) = u_0(\eta) \) on

\[ \gamma : \beta x - \alpha y = 0; \ \alpha, \beta = \text{constant} \]

A parametric representation of Cauchy data is

\[ x|_\gamma = \alpha \eta = x_0(\eta), \ \text{say}, \ y|_\gamma = \beta \eta = y_0(\eta), \ \text{say}; \ \ u|_\gamma = u_0(\eta) \]

where \( u_0(\eta) \) is a given function.
Example 7 contd..

- **Solution of characteristic equations**

  \[
  \frac{dx}{d\sigma} = 2, \quad \frac{dy}{d\sigma} = 3
  \]

  satisfying \( x(\sigma = 0) = \alpha \eta, \quad y(\sigma = 0) = \beta \eta \) are

  \[x = \alpha \eta + 2\sigma, \quad y = \beta \eta + 3\sigma, \quad \eta = \text{const}, \quad \sigma \text{ varies.} \quad (40)\]

  These are characteristic curves staring from the points \( x(\sigma = 0) = \alpha \eta, \quad y(\sigma = 0) = \beta \eta \) of \( \gamma \).

- **Solution of the compatibility condition**

  \[
  \frac{du}{d\sigma} = 1
  \]

  satisfying \( u(\sigma = 0) = u_0(\eta) \) is

  \[u = u_0(\eta) + \sigma \quad (41)\]
To get the solution of the Cauchy problem, we

- first solve $x = \alpha \eta + 2\sigma$, $y = \beta \eta + 3\sigma$ for $\sigma$ and $\eta$

$$
\sigma = \frac{\beta x - \alpha y}{2\beta - 3\alpha}, \quad \eta = \frac{2y - 3x}{2\beta - 3\alpha} \quad (42)
$$

- and then substitute in expression $u_0(\eta) + \sigma$ for $u$

$$
u(x, y) = \frac{\beta x - \alpha y}{2\beta - 3\alpha} + u_0 \left( \frac{2y - 3x}{2\beta - 3\alpha} \right) \quad (43)$$
Example 7 contd.. Existence and Uniqueness

The solution exists as long as $2\beta - 3\alpha \neq 0$ i.e., the datum curve is not a characteristic curve.

**Uniqueness:**
Compatibility condition carries information on the variation of $u$ along a characteristic in unique way. This leads to uniqueness.

What happens when $2\beta - 3\alpha = 0$?
**Example 8: Characteristic Cauchy problem**

$2\beta - 3\alpha = 0 \Rightarrow$ datum curve is a characteristic curve.

Choose $\alpha = 2, \beta = 3 \Rightarrow x = 2\eta, y = 3\eta$.
Check with (40) with $\sigma = 0$. 

\[ \text{Diagram: Lines representing the characteristic curves.} \]
Example 8: Characteristic Cauchy problem contd...

- The characteristic Cauchy problem: Solve

\[ 2u_x + 3u_y = 1 \]

with data

\[ u(2\eta, 3\eta) = u_0(\eta) \]

- Since

\[
\frac{du_0(\eta)}{d\eta} = \frac{d}{d\eta} u(2\eta, 3\eta) = 2u_x + 3u_y = 1, \text{ using PDE,} \quad (44)
\]

- the Cauchy data \( u_0 \) cannot be prescribed arbitrarily on \( \gamma \).

\[ u_0(\eta) = \eta = \frac{1}{2} x, \text{ ignoring constant of integration.} \]
Example 8 contd..

- $u = \frac{1}{2}x$ is a particular solution satisfying the Cauchy data and $g(3x - 2y)$ is solution of the homogeneous equation.

- Hence

$$u = \frac{1}{2}x + g(3x - 2y), \ g \in C^1 \text{ and } g(0) = 0$$

is a solution of the Cauchy problem.

- Since $g$ is any $C^1$ function with $g(0) = 0$, solution of the Characteristic Cauchy problem is not unique.

- We verify an important theorem “in general, solution of a characteristic Cauchy problem does not exist and if exists, it is not unique”. 
Quasilinear equation

Consider the equation

\[ a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \quad (46) \]

- Since \( a \) and \( b \) depend on \( u \), it is not possible to interpret

\[ a(x, y, u) \frac{\partial}{\partial x} + b(x, y, u) \frac{\partial}{\partial y} \quad (47) \]

as a directional derivative in \((x, y)\)-plane.

- We substitute a known solution \( u(x, y) \) for \( u \) in \( a \) and \( b \), then at any point \((x, y)\), it represents directional derivative \( \frac{\partial}{\partial \sigma} \) in the direction given by

\[ \frac{dx}{d\sigma} = a(x, y, u(x, y)), \quad \frac{dy}{d\sigma} = b(x, y, u(x, y)) \quad (48) \]

- Along characteristic curves, given by (48), we get compatibility condition

\[ \frac{du}{d\sigma} = c(x, y, u(x, y)) \quad (49) \]
(49) is true for every solution $u(x, y)$. The Characteristic equations

$$\frac{dx}{d\sigma} = a(x, y, u), \quad \frac{dy}{d\sigma} = b(x, y, u)$$  \hspace{1cm} (50)

along with the Compatibility condition

$$\frac{du}{d\sigma} = c(x, y, u)$$

forms closed system.
Method of solution of a Cauchy problem

Solve

\[ a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \]  \hspace{1cm} (51)

in a domain \( D \) containing

\[ \gamma: x = x_0(\eta), \ y = y_0(\eta) \]

with Cauchy data

\[ u(x_0(\eta), y_0(\eta)) = u_0(\eta) \]  \hspace{1cm} (52)
Method of solution of a Cauchy problem contd..

Solve

\[
\frac{dx}{d\sigma} = a(x, y, u), \quad \frac{dy}{d\sigma} = b(x, y, u), \quad \frac{du}{d\sigma} = c(x, y, u) \tag{53}
\]

\[
(x, y, u)|_{\sigma=0} = (x_0(\eta), y_0(\eta), u_0(\eta)) \tag{54}
\]

\[
\Rightarrow x = x(\sigma, x_0(\eta), y_0(\eta), u_0(\eta)) \equiv X(\sigma, \eta) \\
y = y(\sigma, x_0(\eta), y_0(\eta), u_0(\eta)) \equiv Y(\sigma, \eta) \\
u = u(\sigma, x_0(\eta), y_0(\eta), u_0(\eta)) \equiv U(\sigma, \eta) \tag{55}
\]

Solving the first two for \(\sigma = \sigma(x, y), \eta = \eta(x, y)\) we get the solution

\[
u = U(\sigma(x, y), \eta(x, y)) \equiv u(x, y) \tag{56}
\]
Method of solution of a Cauchy problem contd..

\[\text{Fig. 1.1. Solution of a Cauchy problem with the help of characteristic curves } C_c\]
Quasilinear equations conti...

Do not worry about the complex statement in the theorem below. As long as the datum curve $\gamma$ is not tangential to a characteristic curve and the functions involved are smooth, the solution exist locally and is unique (see previous slide).

**Theorem:**

1. $x_0(\eta), y_0(\eta), u_0(\eta) \in C^1(I)$, say $I = (0, 1)$
2. $a(x, y, u), b(x, y, u), c(x, y, u) \in C^1(D_2)$, where $D_2$ is a domain in $(x, y, u)$ – space
3. $D_2$ contains curve $\Gamma$ in $(x, y, u)$-space
   $$\Gamma : x = x_0(\eta), y = y_0(\eta), u = u_0(\eta), \eta \in I$$
4. $$\frac{dy_0}{d\eta} a(x_0(\eta), y_0(\eta), u_0(\eta)) - \frac{dx_0}{d\eta} b(x_0(\eta), y_0(\eta), u_0(\eta)) \neq 0, \eta \in I$$

There exists a unique solution of the Cauchy problem in a domain $D$ containing $I$.

**Note 1:** Condition 4 rules out that datum curve $\gamma : (x_0(\eta), y_0(\eta))$ is a characteristic curve.
Example 9

Cauchy problem

\[ u_x + u_y = u \]
\[ u(x, 0) = 1 \Rightarrow x_0 = \eta, \ y_0 = 0, \ u_0 = 1 \quad (57) \]

Step 1. Characteristic curves

\[ \frac{dx}{d\sigma} = 1 \Rightarrow x = \sigma + \eta \]
\[ \frac{dy}{d\sigma} = 1 \Rightarrow y = \sigma \quad (58) \]

Step 2. Therefore \( \sigma = y, \ \eta = x - y \)

Step 3. Compatibility condition

\[ \frac{du}{d\sigma} = u \Rightarrow u = u_0(\eta)e^\sigma = e^\sigma \]

Step 4. Solution \( u = e^y \)

exists on \( D = \mathbb{R}^2 \)
Example 10

Cauchy problem (same as problem 9 with a small change on the RHS of the PDE)

\[ u_x + u_y = u^2 \]
\[ u(x, 0) = 1 \Rightarrow \]
\[ x_0 = \eta, \ y_0 = 0, \ u_0 = 1 \] (59)

Step 1. Characteristic equations give

\[ x = \sigma + \eta, \ y = \sigma \]

Step 2. Compatibility condition gives

\[ \frac{du}{d\sigma} = u^2 \Rightarrow u = \frac{1}{u_0(\eta) - \sigma} \]

Step 3. Solution \( u = \frac{1}{1-y} \) exists locally on the domain \( D = y < 1 \) and \( u \to +\infty \) as \( y \to 1^- \).
Example 11

Cauchy problem

\[ uu_x + u_y = 0 \]
\[ u(x, 0) = x, \; 0 \leq x \leq 1 \]  \hspace{1cm} (60)

\[ \Rightarrow x = \eta, \; y = 0, \; u = \eta, \; 0 \leq \eta \leq 1 \text{ at } \sigma = 0 \]

Step 1. Characteristic equations and compatibility condition

\[ \frac{dx}{d\sigma} = u, \quad \frac{dy}{d\sigma} = 1, \quad \frac{du}{d\sigma} = 0 \]  \hspace{1cm} (61)

Step 2. Quasilinear equations, characteristics depend on the solution

\[ u = \eta \]
Example 11 contd..

Step 3. Substituting $u = \eta$ in (61) we get

$$x = \eta(\sigma + 1), \ y = \sigma$$

Step 4. From solution of characteristic equations $\sigma = y$ and $\eta = \frac{x}{y+1}$

Step 5. Solution is $u = \frac{x}{y+1}$, but what is domain $D$ of the solution?

Step 6. Characteristic curves are straight lines

$$\frac{x}{y + 1} = \eta, \ 0 \leq \eta \leq |$$

which meet at the point $(-1, 0)$.

Step 7. $u$ is constant on these characteristics (see next slide).
Example 11 contd..

Figure: Solution is determined in a wedged shaped region in \((x, y)\)-plane including the lines \(y = 0\) and \(y = x + 1\).

We note

\[ u(0, -1) \]  \hspace{1cm} (62) 

is not defined.
Example 12
Cauchy problem

\[ uu_x + u_y = 0 \]
\[ u(x, 0) = \frac{1}{2}, \ 0 \leq x \leq 1 \]  \hspace{1cm} (63)

Step 1. Parametrization of Cauchy data

\[ \Rightarrow x_0 = \eta, \ y_0 = 0, \ u_0 = \frac{1}{2}, \ 0 \leq \eta \leq 1. \]

Step 2. The compatibility condition along characteristic curves gives

\[ u = \text{constant} = \frac{1}{2}. \]

Step 3. The characteristic curves are

\[ y - 2x = -2\eta, \ 0 \leq \eta \leq 1. \]  \hspace{1cm} (64)

on which solution has the same value \( u = \frac{1}{2}. \)
Example 12 contd..

Step 4. The solution \( u = \frac{1}{2} \) of the Cauchy problem is determined in an infinite strip \( 2x - 2 \leq y \leq 2x \) in \( (x, y) \)-plane.

![Diagram](image)

**Fig. 1.3. The domain \( D \) when the Cauchy data is \( u(x, 0) = 1/2 \) for \( 0 \leq x \leq 1 \)**

**Important:** From examples 11 and 12, we notice that the domain, where solution of a Cauchy problem for a quasilinear equation is determined, depends on the Cauchy data.
Example 13
Consider initial data for $uu_x + u_y = 0$:

$$u(x, t) = \begin{cases} 
1 &, x \leq 0 \\
1 - x &, 0 < x \leq 1 \\
0 &, x > 1.
\end{cases} \quad (65)$$

Solution remains continuous for $0 \leq y < 1$

$$u(x, y) = \begin{cases} 
1 &, x \leq y \\
\frac{1-x}{1-y} &, y < x \leq 1 \\
0 &, x > 1
\end{cases} \quad (66)$$

Solution is not valid at $y = 1$ but data at $y = 1$

$$u(x, 1) = \begin{cases} 
1 &, -\infty < x \leq \frac{1}{2} \\
0 &, 1 < x < \infty
\end{cases} \quad (67)$$

Draw the figure in $(x, y)$-plane.
Example 14

Cauchy problem

\[ uu_x + u_y = 0 \]

\[ u(x, 0) = 0, x < 0, \]
\[ u(x, 0) = x, 0 \leq x \leq 1, \]
\[ u(x, 0) = 1, x > 1. \]

Initial data is continuous but solution (given below) is not a genuine solution - why?

\[ u(x, y) = 0, \ x < 0; \ u(x, y) = 1, \ x > 1 + y; \]

\[ u(x, y) = \frac{x}{y + 1}, \ 0 \leq \frac{x}{y + 1} \leq 1, \ y > -1. \]  (68)

Solution as \( y \to (-1)^+ \) is

\[ u(x, -1) = 0, x < 0; \quad u(x, -1) = 1, x > 0. \]  (69)

Solution is shown graphically on next slide.
Example 14 \textellipsis \text{conti.}
Example 15

For the Cauchy problem

$$uu_x + u_y = 0$$

$$u(x, 0) = x, \ 0 \leq x \leq 1/2, \quad u(x, 0) = \frac{1}{2}, \ 1/2 \leq x \leq 1. \quad (70)$$

- Find the solution,
- find the domain of the solution,
- draw characteristic curves and
- note that the solution is continuous but not a genuine solution.
- Why is it not a genuine solution?
General solution

General solution of a first order PDE contains an arbitrary function.

**Theorem**: If \( \phi(x, y, u) = C_1 \) and \( \psi(x, y, u) = C_2 \) be two independent first integrals of the ODEs

\[
\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{c(x, y, u)} \tag{71}
\]

and \( \phi^2_u + \psi^2_u \neq 0 \), the general solution of the PDE \( au_x + bu_y = c \) is given by

\[
h(\phi(x, y, u), \psi(x, y, u)) = 0 \tag{72}
\]

where \( h \) is an arbitrary function.

For proof see PP-RR PDE.
Example 16

\[ uu_x + u_y = 0 \quad (73) \]

\[ \frac{dx}{u} = \frac{dy}{1} = \frac{du}{0} \quad (74) \]

Note 0 appearing in a denominator to be properly interpreted

\[ \Rightarrow u = C_1 \]

\[ x - C_1 y = C_2 \]

\[ \Rightarrow x - uy = C_2 \Rightarrow \quad (75) \]

General solution is given by

\[ \phi(u, x - uy) = 0 \]

\[ \text{or } u = f(x - uy) \quad (76) \]

where \( h \) and \( f \) arbitrary functions.

**Note**: Solution of this nonlinear equation may be very difficult.
Numerical method is generally used.
Example 17
Consider the differential equation

\[(y + 2ux)u_x - (x + 2uy)u_y = \frac{1}{2}(x^2 - y^2)\]  \hspace{1cm} (77)

The characteristic equations and the compatibility condition are

\[\frac{dx}{y + 2ux} = \frac{dy}{-(x + 2uy)} = \frac{du}{\frac{1}{2}(x^2 - y^2)}\]  \hspace{1cm} (78)

To get one first integral we derive from these

\[\frac{xdx + ydy}{2u(x^2 - y^2)} = \frac{2du}{x^2 - y^2}\]  \hspace{1cm} (79)

which immediately leads to

\[\varphi(x, y, u) \equiv x^2 + y^2 - 4u^2 = C_1\]  \hspace{1cm} (80)
Example 17 contd..

For another independent first integral we derive a second combination

\[
\frac{y\,dx + x\,dy}{y^2 - x^2} = \frac{2\,du}{x^2 - y^2} \quad (81)
\]

which leads to

\[
\psi(x, y, u) \equiv xy + 2u = C_2 \quad (82)
\]

The general integral of the equation (55) is given by

\[
h(x^2 + y^2 - 4u^2, \, xy + 2u) = 0
\]

\[
x^2 + y^2 - 4u^2 = f(xy + 2u) \quad (83)
\]

where \( h \) or \( f \) are arbitrary functions of their arguments.

We can use a general solution to solve a Cauchy problem. See next slide.
Consider a Cauchy problem for equation (77) with Cauchy data \( u = 0 \) on \( x - y = 0 \)

\[
\Rightarrow \quad x = \eta, y = \eta, u = 0
\]

- From (58) and (60) we get \( 2\eta^2 = C_1 \) and \( \eta^2 = C^2 \) which gives a relation between constants in (80) and (82): \( C_1 = 2C_2 \).

- Therefore, the solution of the Cauchy problem is obtained, when we take \( h(\varphi, \psi) = \varphi - 2\psi \).

- This gives, taking only the suitable one,

\[
 u = \frac{1}{2} \left\{ \sqrt{(x - y)^2 + 1} - 1 \right\}. \tag{84}
\]

We note that the solution of the Cauchy problem is determined uniquely at all points in the \((x, y)\)-plane.
Two Important References

- In 1992 I gave a lecture at The Larmor Society, which is the Natural Sciences Society, St Johns College at Cambridge.

- The lecture was meant for undergraduate students and hence I used the language of physics without any mathematical equations.

- Based on the idea in this lecture, I wrote a popular article Nonlinearity, Conservation Laws and Shocks in two parts and it was published in 1997 in Resonance. See reference [4].

- But a reader has to pause and think a lot to understand the mathematical concepts.
Exercise

1. Show that all the characteristic curves of the partial differential equation

\[(2x + u)u_x + (2y + u)u_y = u\]

through the point (1,1) are given by the same straight line \(x - y = 0\).

2. Discuss the solution of the differential equation

\[uu_x + u_y = 0, \ y > 0, \ -\infty < x < \infty\]

with Cauchy data

\[u(x, 0) = \begin{cases} 
\alpha^2 - x^2 & \text{for } |x| \leq \alpha \\
0 & \text{for } |x| > \alpha.
\end{cases}\]  

\[\text{(85)}\]
Exercise contd..

3. Find the solution of the differential equation

\[(1 - \frac{m}{r}u)u_x - mM u_y = 0\]

satisfying

\[u(0, y) = \frac{My}{\rho - y}\]

where \(m, r, \rho, M\) are constants, in a neighbourhood of the point \(x = 0, y = 0\).

4. Find the general integral of the equation

\[(2x - y)y^2u_x + 8(y - 2x)x^2u_y = 2(4x^2 + y^2)u\]

and deduce the solution of the Cauchy problem when the \(u(x, 0) = \frac{1}{2x}\) on a portion of the \(x\)-axis.


2. P. Prasad. A theory of first order PDE through propagation of discontinuities. Ramanujan Mathematical Society News Letter, 2000, 10, 89-103; see also the webpage:


Thank You!