An Introduction to Single Conservation Law

First Chapter of P. Prasad, 2001, Nonlinear Hyperbolic Waves in Multi-dimensions, Chapman and Hall/CRC, London, 2001.

1 An Introduction to Single Conservation Law*

1.1 A wave equation with genuine nonlinearity

We are concerned in this article with hyperbolic waves having nonlinearity of a special type i.e. waves governed by quasilinear hyperbolic partial differential equations. We attempt to introduce the terms and concepts gradually in this chapter with the help of a model equation and examples of its solutions.

Simplest example of a linear hyperbolic partial differential equation in two independent variables x and t is

$$u_t + cu_x = 0, \quad c = \text{ constant} \tag{1.1.1}$$

Its solution $u = f(x - ct)$, where f is an arbitrary function with continuous first derivatives, represents a wave every point of whose profile propagates with the same constant velocity c.
An extension of (1.1.1) is another linear equation

$$u_t + cu_x = \alpha u \tag{1.1.2}$$

which also represents a wave $e^{\alpha t} f(x - ct)$ in which the amplitude either decays to zero ($\alpha < 0$) or tends to infinity ($\alpha > 0$) but these limiting values are reached asymptotically in infinite time.

Consider now a nonlinear equation

$$u_t + cu_x = u^2 \tag{1.1.3}$$

The solution of this equation with initial condition

$$u(x, 0) = f(x) \ , \ \ f \in C^1(I\!R) \tag{1.1.4}$$

is

$$u(x,t) = \frac{f(x - ct)}{1 - tf(x - ct)} \tag{1.1.5}$$

*A new and simpler treatment of the topic, using the language of physics, is available in a popular article by Prasad (1997).

which also represents a wave propagating with the constant velocity c. However, the wave profile now deforms in such a way that a negative amplitude at a point on the initial pulse decays to zero as t tends to infinity but a positive amplitude of a point propagating with the velocity c tends to infinity in finite time. Topic of discussion in this article are nonlinear waves which appear as solutions of equations having a different type of nonlinearity. The simplest and yet the most beautiful example of an equation having this type of nonlinearity, called *genuine nonlinearity*, is

$$u_t + uu_x = 0 \tag{1.1.6}$$

whose solutions represent waves in which the velocity of propagation of a point on the pulse is equal to the amplitude at that point. This equation is called Burgers' equation.

The term genuine nonlinearity was first defined by PD Lax in 1957. It represents a property of wave propagation, namely dependence of the propagation velocity on amplitude, which may be present in certain modes but not in others of the system. In a small amplitude unimodel * genuinely nonlinear wave in a homogeneous system, the propagation velocity of a point on the pulse exceeds a constant velocity c by a quantity whose leading term is proportional to the amplitude u of the wave. Denoting the distance in the frame of reference moving with velocity c also by x and rescaling the amplitude and assuming that the wave is free from dispersion and diffusion we get the approximate partial differential equation (1.1.6) for the unimodel waves (Bhatnagar 1979). This equation also appears as a natural approximate model for a large class of physical processes governed by a single conservation law in which the flux function depends on the density alone (Whitham, 1974). Therefore, we note that Burgers' equation models a large class of important physical problems. The solution set of this equation has a very rich variety of properties which form the subject matter of discussion in this chapter.

Consider the solution of the equation (1.1.6) satisfying the initial condition

$$u(x, 0) = e^{-x^2} \tag{1.1.7}$$

Properties of the solution will be discussed in subsequent sections. We take up here a simple geometrical construction of the successive shapes of the initially single humped pulse given by (1.1.7). The graph of the solution at any time t (i.e. the pulse at time t) is obtained by translating a point P on pulse (1.1.7) by a distance in positive $x-$ direction, the magnitude of the translation being equal to t times the amplitude of the pulse at the point P. Fig.1.1.1 shows the pulse at times $t = 0$, 1.166, 2,. We note that

(a) since different points of the pulse move with different velocity, the pulse now deforms;

(b) at a critical time t_c (for the initial pulse (1.1.7), $t_c = \sqrt{e/2} \cong 1.166$ see section 1.2) the pulse has a vertical tangent for the first time at some point on it;

(c) after $t > t_c$, the pulse ceases to represent the graph of a function (for example, at $x = a$

*By unimodel wave in a general system, we mean a wave in which the amplitudes of waves in all other modes is small compared to that in one mode.

it has three values u_1, u_2 and u_3) and the physical interpretation fails (for example, if u represents pressure in a fluid there cannot exist three values of pressure at $x = a$).

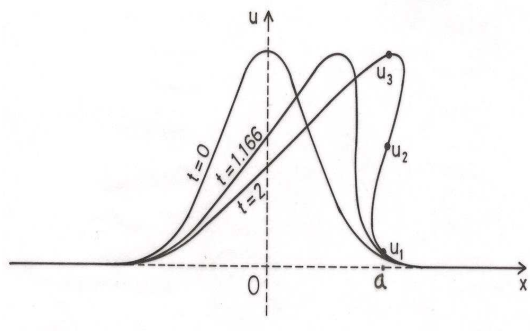

Fig. 1.1.1: As t increases, the pulse of the nonlinear wave deforms.

It has been observed in nature that a discontinuity appears in the quantity u immediately after the time t_c, this moving discontinuity at a point $x = X(t)$ is called *shock*, which we shall define formally in section 1.4. A shock, when appears in the solution, fits into the multi-valued part of the solution in such a way that it cuts off lobes of areas on two sides of it in certain ratio from the graph of the solution at any time $t > t_c$ and makes the solution single valued. The ratio in which the lobes on the two sides are

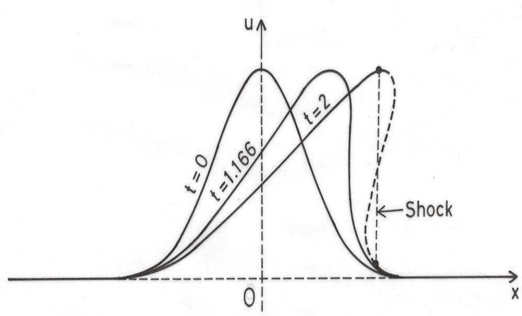

Fig. 1.1.2 : Shock (shown by broken vertical line) fitted into the multi-valued part of the curve at $t = 2$ assuming that the shock cuts off lobes of equal areas on two sides of it.

cut off depends on a more primitive property (conservation of an appropriate density) of the physical phenomena represented by the equation (1.1.6). When the primitive property is conservation of the density $\rho(u) = u$, the shock cuts off lobes of equal area on the two sides of it.

1.2 Breakdown of a genuine solution

A large number of physical phenomena are modelled by partial differential equations on the assumption that the variables which describe the state of a phenomenon are sufficiently smooth. In many realistic situations the state variables of these phenomena are smooth. However, there exist other situations when they are not. This is reflected by the fact that for certain initial-boundary value problems associated with the equations either smooth solutions (also called *genuine solutions*) do not exist even locally or the solutions cease to be smooth after some critical time t_c even if the initial and boundary values are smooth. Burgers' equation (1.1.6) is an example for which a solution for a certain type of initial data, however smooth, always develops a singularity at a finite time.

An initial value problem or a Cauchy problem for the Burgers' equation consists in finding a solution of

$$u_t + uu_x = 0, \ (x,t) \in \mathbf{R} \times \mathbf{R}_+ \tag{1.2.1}$$

satisfying initial data

$$u(x,0) = \varphi(x), \ x \in \mathbf{R} \tag{1.2.2}$$

where $I\!R$ is the set of real numbers and $I\!R_+ = (0,\infty)$.

Definition A *genuine* (or *classical*) solution of the partial differential equation (1.2.1) in a domain D in (x,t)-plane is a function $u(x,t) \in C^1(D)$ which satisfies (1.2.1).

A sufficient condition for the existence of a local genuine solution (i.e. a solution valid for $0 < t < t_c$ with some $t_c < \infty$) of the initial value problem (1.2.1) and (1.2.2) is that $\varphi(x) \in C^1(\mathbf{R})$ (Prasad and Ravindran, 1985). A genuine solution can be obtained by solving the compatibility condition

$$\frac{du}{dt} = 0 \tag{1.2.3}$$

along the characteristic curves

$$\frac{dx}{dt} = u \tag{1.2.4}$$

These equations imply that u is constant along the characteristics $\xi \equiv x - ut =$ constant. Hence the solution of (1.2.1) and (1.2.2) is given by

$$u = \varphi(x - ut) \tag{1.2.5}$$

From implicit function theorem, it follows that the relation (1.2.5) defines a C^1 function $u(x,t)$ as long as

$$1 + t\varphi'(\xi) \neq 0 \tag{1.2.6}$$

which is satisfied for $|\,t\,|$ small. The x-derivative of the solution is given by

$$u_x = \phi'(\xi)/\{1 + t\phi'(\xi)\}, \ \xi = x - ut \tag{1.2.7}$$

If the initial data is such that $\varphi' < 0$ on some interval of the x-axis, there exists a time $t_c > 0$ such that as $t \to t_c - 0$, the derivative $u_x(x,t)$ of the solution tends to $-\infty$ for some value of x and thus the genuine solution can not be continued beyond at $t = t_c$. The critical time t_c is given by

$$t_c = -\frac{1}{\min\limits_{\xi \in \mathbf{R}}\{\varphi'(\xi)\}} > 0 \tag{1.2.8}$$

If $\varphi'(x) > 0$ for all $x \in \mathbf{R}$; the relation (1.2.5) gives a genuine solution of (1.2.1) and (1.2.2) for all $t > 0$. Breakdown of the genuine solution at $t = t_c$ can be explained not only graphically as in the previous section but also from the geometry of the characteristic curves of the equation (1.2.1). Let I be an interval on the x-axis such that $\varphi'(x) < 0$ for $x \in I$. The characteristics in the (x, t)-plane starting from the various points of the interval I converge and in general envelop a cusp starting from the time t_c. Consider the domain in the (x, t)-plane which is bounded by the two branches of the cusp. Three characteristics starting from some three points of I pass through any point of this domain. Since characteristics carry different constant values of the solution, u is not defined uniquely at interior points of this domain. Difficulty in continuation of a genuine solution beyond a finite time t_c is quite common for a hyperbolic system of *quasilinear* partial differential equations. A smooth solution of an initial value problem for the semilinear equation (1.1.3) also breaks down after a finite time due to an unbounded increase in the amplitude of the solution. However, the breakdown of the solution of (1.1.6) is due to a different reason: due to an unbounded increase in the absolute value of the first derivatives, the solution itself remains finite. Here, we are concerned with the breakdown of the type exhibited by a solution of (1.1.6). For a hyperbolic system of quasilinear equations, such a breakdown is due to a very special property of a characteristic velocity namely the velocity of propagation (which is the same as the characteristic velocity) of the waves depends essentially on the amplitude of the waves that is the characteristic field is “genuinely nonlinear” (see Prasad and Ravindran, 1985).

1.3 Conservation law and jump condition

We have seen in the last section that a genuine solution of the initial value problem (1.2.1) and (1.2.2) ceases to be valid after a critical time t_c. However, equation (1.2.1) models quite a few physical phenomena, where the function u becomes discontinuous after the time t_c and the discontinuous states of the phenomena persist for all time $t > t_c$. Hence we must generalise the notion of a solution to permit u, which are not necessarily C^1. In order to do that we write (1.2.1) in a divergence form

$$\frac{\partial}{\partial t}(u) + \frac{\partial}{\partial x}(\frac{1}{2}u^2) = 0, (x, t) \in \mathbf{R} \times \mathbf{R}_+. \tag{1.3.1}$$

Definition A conservation law is an equation in a divergence form.

(1.3.1) is just one of an infinity of conservation laws,

$$\frac{\partial}{\partial t}(u^n) + \frac{\partial}{\partial x}(\frac{n}{n+1}u^{n+1}) = 0, \; n = \text{constant} \tag{1.3.2}$$

which can be derived from (1.2.1). Both these conservation forms are particular cases of a general form

$$\frac{\partial H(u)}{\partial t} + \frac{\partial F(u)}{\partial x} = 0 \tag{1.3.3a}$$

where the *density* H and the *flux* F are smooth functions of the state variable u. Equation (1.2.1) can be derived from (1.3.3a) if

$$H'(u) \equiv \frac{dH}{du} \neq 0 \text{ and } F'(u)/H'(u) = u \tag{1.3.3b}$$

In physics, a balance equation representing the conservation of a quantity such as mass, momentum or energy of a physical system is not expressed in the form of a differential equation (1.3.3a). The original balance equation is stated in terms of integrals, rather than in the form of (1.3.3a), as

$$\int_{x_1}^{x_2} H(u(\xi,t_2))d\xi - \int_{x_1}^{x_2} H(u(\xi,t_1))d\xi \;=\; \int_{t_1}^{t_2} \{F(u(x_1,t)) - F(u(x_2,t))\}dt \tag{1.3.4}$$

which holds for every fixed space interval (x_1, x_2) and for every time interval (t_1, t_2). This equation is meaningful even for a discontinuous function $u(x,t)$. We can now define a *weak solution* of the conservation law (1.3.3) to be a bounded measurable function $u(x,t)$ which satisfies the integral form (1.3.4). We shall not use the balance equation in the general form such as (1.3.4) but in a more restricted form

$$\frac{d}{dt}\int_{x_1}^{x_2} H(u(\xi,t))d\xi = F(u(x_1,t)) - F(u(x_2,t)),\; x_1, x_2 \text{ fixed} \tag{1.3.5}$$

which we shall assume to be valid almost everywhere for $(x_1, x_2, t) \in I\!R^2 \times I\!R_+$. For smooth solutions, the equations (1.3.3), (1.3.4) and (1.3.5) are equivalent. We state this result as theorem.

THEOREM 1.1 *Every weak solution of (1.3.3a-b) which is C^1 is a genuine solution of the partial differential equation (1.1.6).*

Consider now a solution $u(x,t)$ of (1.3.5) such that $u(x,t)$ and its partial derivatives suffer discontinuities across a smooth isolated curve $\Omega : x = X(t)$ in the (x,t)-plane and is continuously differentiable elsewhere. It is further assumed that the limiting values of u and its derivatives as we approach Ω from either side exist. The function $u(x,t)$ is a genuine solution of (1.2.1) in the left and right subdomains of the curve of discontinuity Ω. Let the fixed points x_1 and x_2 be so chosen that $x_1 < X(t) < x_2$ for $t \in$ an open interval. Writing $\int_{x_1}^{x_2} H(u(\xi,t))d\xi = \int_{x_1}^{X(t)} H(u(\xi,t))d\xi + \int_{X(t)}^{x_2} H(u(\xi,t))d\xi$ in (1.3.5) and then taking the time derivative, we get

$$\int_{x_1}^{X(t)} H'u_t(\xi,t)d\xi + \int_{X(t)}^{x_2} H'u_t(\xi,t)d\xi + \dot{X}(t)\{H(u(X(t)-0,t)) - H(u(X(t)+0,t))\}$$

$$= \{F(u(x_1,t)) - F(u(x_2,t))\}$$

The first two terms tend to zero as $x_1 \to X(t)-$ and $x_2 \to X(t)+$. Hence taking the point x_1 on the left of $X(t)$ very close to it and the point x_2 on the right of $X(t)$ also very close to it, we get in the limit

$$\dot{X}(t)(H(u_\ell(t)) - H(u_r(t))) \ = \ F(u_l) - F(u_r) \tag{1.3.6}$$

where

$$u_\ell(t) = \lim_{x\to X(t)-0} u(x,t) \quad \text{and} \quad u_r(t) = \lim_{x\to X(t)+0} u(x,t) \tag{1.3.7}$$

(1.3.6) gives the following expression for the velocity of propagation $\dot{X}$ of the discontinuity

$$\dot{X}(t) \ = \ [F]/[H] \tag{1.3.8}$$

where the symbol [ ] is defined by

$$[f] = f(u_r) - f(u_l) \tag{1.3.9}$$

The relation (1.3.8), connecting the speed of propagation $\dot{X}(t)$ of a discontinuity and the limiting values u_ℓ and u_r on the two sides of the discontinuity, is called the *jump relation*. Such jump relations derived from the conservation laws of gas dynamics are called Rankine–Hugoniot conditions. The usual initial and boundary conditions of the gas dynamics equations combined with the three Rankine–Hugoniot conditions and an entropy condition (to be discussed in the next section) are sufficient to calculate many problems of great practical importance (Courant and Friedrichs,1948) and are also sufficient to prove many theorems on the existence and uniqueness of solutions (see Smoller, 1983, Chapters 17 and 18). This mathematical completeness in the theory of discontinuous solutions was probably the main reason that mathematicians did not take up the question of further jump relations on the derivatives of the function $u(x,t)$.

For a discontinuity with nonzero jump, $u_l \neq u_r$. Hence for the conservation laws (1.3.1) and (1.3.2), the jump relation (1.3.8) become

$$\dot{X}(t) = \frac{1}{2}(u_r + u_l) \tag{1.3.10}$$

and

$$\dot{X}(t) = \frac{n}{n+1}\Big(\sum_{i=0}^{n} u_r^{n-i}\, u_l^i\Big)\Big/\Big(\sum_{i=0}^{n-1} u_r^{n-1-i}\, u_l^i\Big) \tag{1.3.11}$$

1.4 Stability consideration, entropy condition and shocks

Let us discuss solutions of a number of initial value problems of the balance equation (1.3.4) with $H(u) = u, F(u) = \frac{1}{2}u^2$ i.e. weak solutions of (1.3.1). The solutions need not be smooth now. First we consider the smooth initial data

$$u(x,0) \ = \ 0 \quad (\equiv \varphi_1(x) \text{ say}) \tag{1.4.1}$$

One obvious solution with this initial data is

$$u(x,t) \ = 0, \quad (x,t) \in \mathbf{R} \times \mathbf{R}_+ \tag{1.4.2}$$

which is a genuine solution of (1.2.1). Consider now the function

$$u(x,t) = \begin{cases} 0\,, & x \le -\frac{1}{2}t, \\ -1\,, & -\frac{1}{2}t < x \le 0, \\ 1\,, & 0 < x \le \frac{1}{2}t, \\ 0\,, & \frac{1}{2}t < x \end{cases} \tag{1.4.3}$$

which has been shown in Fig 1.4.1

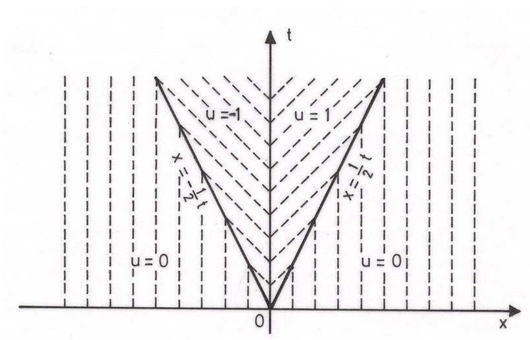

Fig. 1.4.1 : A discontinuous solution of (1.3.1) with initial data given by $u(x,0) = 0$. The characteristic curves of (1.2.1) are shown by broken lines.

The characteristic curves of (1.2.1) are shown by broken lines. The function (1.4.3) is constant and hence a genuine solution in each of the subdomains of the upper half of the (x,t)-plane in which it is divided by the straight lines $x = 0, x = -\frac{1}{2}t$ and $x = \frac{1}{2}t$. The function satisfies the jump relation (1.3.10) along these lines. Hence (1.4.3) is another weak solution (now a discontinuous solution) of the equation (1.3.1) with initial condition (1.4.1). This is a spectacular result of a nonzero solution coming of a zero initial condition and is due to the fact that the discontinuity along the line $x = 0$ is not admissible which we shall explain later.

Note that even if the value of a function which satisfies (1.3.4), be changed at a set of moving points $x_i(t)$ finite in number (or of measure zero) in (x_1, x_2) the new function will still satisfy (1.3.4). Therefore, it is immaterial whether we take the weak solution of (1.3.1) to be continuous to the left (as in the case (1.4.3)) or to the right of the point of discontinuity.

In the class of discontinuous solutions, we can consider not only smooth initial data but initial data which could be discontinuous. Let us consider a discontinuous initial data of the form

$$u(x,0) \ = \varphi_2(x) \ \equiv \ \begin{cases} 0\,, & x \le 0 \\ 1\,, & 0 < x. \end{cases} \tag{1.4.4}$$

The equation (1.3.1) has an infinity of discontinuous weak solutions for this initial data:

$$u(x,t) \;=\; \begin{cases} 0 & , \quad x \le 0 \\ x/t & , \quad 0 < x \le \alpha t \\ \alpha & , \quad \alpha t < x \le \frac{1}{2}(1+\alpha)t \\ 1 & , \quad \frac{1}{2}(1+\alpha)t < x \end{cases} \tag{1.4.5}$$

depending on a parameter α satisfying $0 \le \alpha \le 1$. This solution has been shown in Fig 1.4.2

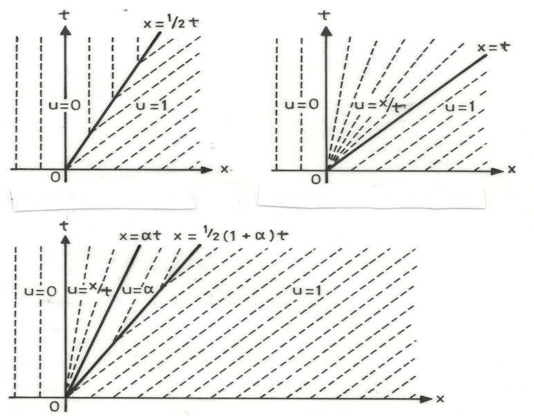

Fig. 1.4.2 : Solution (1.4.5) for $\alpha = 0$
Solution (1.4.5) for $\alpha = 1$
Solution (1.4.5) for $0 < \alpha < 1$.
The characteristic curves have been shown by broken lines.

The characteristic curves have been shown by broken lines. For $\alpha = 1$, the solution (1.4.5) becomes continuous but it is not a genuine solution of (1.2.1) since it is not continuously differentiable. We note that the non-uniqueness in the solution arises because of the possibility of fitting a line of discontinuity of an arbitrary slope $\frac{1}{2}(1+\alpha)$ joining a constant state continuation ($u = \alpha$) of the *centered fan* $u = x/t$ on the left and the constant state $u = 1$ on the right. Another simple solution with initial condition (1.4.4) consists of three constant states $0, 1/2$ and 1 separated by lines of discontinuity $x = (1/4)t$ and $x - (3/4)t$.

Instead of the initial data $\varphi_2(x)$, if we take

$$u(x,0) \;=\; \varphi_3(x) \equiv \begin{cases} 1 & , \quad x \le 0 \\ 0 & , \quad x > 0 \end{cases} \tag{1.4.6}$$

we get a discontinuous solution of (1.3.4) :

$$u(x,t) \;=\; \begin{cases} 1 & , \quad x - \frac{1}{2}t \le 0 \\ \\ 0 & , \quad x - \frac{1}{2}t > 0 \end{cases} \tag{1.4.7}$$

In this case, it is not possible to have a centered fan with centre at the origin and hence (1.4.7) is the only weak solution which has a curve of discontinuity through $(0,0)$.

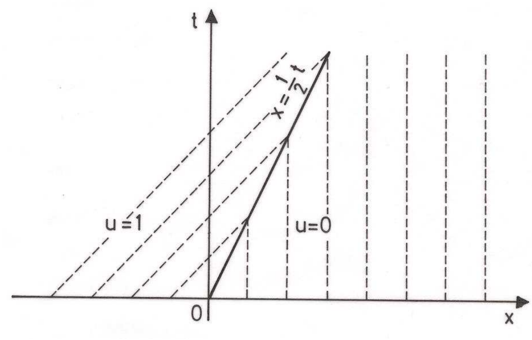

Fig. 1.4.3 : Solution (1.4.7) with characteristic curves shown by broken lines.

The above example shows that, in general, a discontinuous solution of an initial value problem for (1.3.4), i.e. a weak solution of the conservation law (1.3.1) is not unique. What is needed now is a mathematical principle characterising a class of permissible solutions in which every initial value problem for the conservation law has a unique solution. We can deduce such a principle from the following consideration. A genuine solution satisfying smooth initial data is unique and this is true even for a solution with piecewise continuous derivatives such as (1.4.5) with $\alpha = 1$. In this case a unique characteristic originating from a point on the initial line passes through any point (x,t) in the domain where solution is determined. However, for a discontinuous solution satisfying (1.4.4) the initial data is unable to control the solution in the domain $\alpha < \frac{x}{t} < 1$. The failure of the initial data to control the solution in this domain leads to nonuniqueness on the curve of discontinuity of the solution. When a discontinuity appears in the solution (1.4.5) for $0 \le \alpha < 1$, the characteristics starting from a point $(\frac{1}{2}(1+\alpha)t, t)$ diverge (into the domains on the two sides of it) as t increases, so that discontinuity could have been replaced by a continuous centered wave from this point onward. The situation is different when the initial data is (1.4.6), which gives rise to a situation in which characteristics starting from the points on the two sides of the point of discontinuity converge and start intersecting as t increases. In this and all other such situations a discontinuity must necessarily appear to prevent multivaluedness in the solution. Therefore, a discontinuity is permissible only if it prevents the intersection of the characteristics coming from the points of the initial line on the two sides of it, i.e.

$$u_r(t) < \dot{X}(t) < u_\ell(t). \tag{1.4.8}$$

If we accept this as a principle, Fig. 1.4.1 and Fig. 1.4.2 show that the only admissible solution for initial data φ_1 is the zero solution, and that for the initial data φ_2 is the continuous solution obtained for $\alpha = 1$. The solution (1.4.7) is also the only admissible solution for the initial data φ_3.

The mathematical criterion (1.4.8), known as *Lax's entropy condition*, for admissible discontinuities can be derived from the following stability consideration (first shown by Gel'fand in 1962).

"A discontinuity is admissible if when small amplitude waves are incident upon the discontinuity, the resulting perturbations in the velocity of the discontinuity and the resulting waves moving away from the discontinuity are uniquely determined and remain small."

The derivation is trivial for the single conservation law.

Definition An admissible discontinuity satisfying the stability criterion is called a *shock*.

Using (1.3.10), we find that for the conservation law (1.3.1) the stability condition (1.4.8) is equivalent to an easily verifiable condition

$$u_r(t) < u_\ell(t) \tag{1.4.9}$$

The term "shock" was first used for a compression discontinuity in gasdynamics, where an expansion discontinuity is ruled out by the second law of thermodynamics which implies that the specific entropy of the fluid particles must increase after crossing the discontinuity. Hence, the stability condition (1.4.8) is also called the *entropy condition*. We note two important results regarding solutions with shocks of nonlinear problems :

(1) In contrast to the results for linear equations, not only a discontinuity may appear in the solution of nonlinear equations with continuous data, but also a discontinuity in the initial data may be immediately resolved in the solution. This is shown by the continuous solution (1.4.5) with $\alpha = 1$ for the discontinuous initial data $\varphi_2(x)$.

(2) Physical processes described by genuine solutions of a hyperbolic system of quasilinear equations are reversible in time, i.e. if we know the solution at some time, we can use the differential equation to get the solution uniquely in the past as well as in future. However, if a process is described by a discontinuous solution (where discontinuities are shocks) of a system of balance equations, then it is irreversible. We shall illustrate this mathematically by means of an example. The weak solution of the conservation law (1.3.1) satisfying

$$u(x,0) = \begin{cases} 2 & , \quad x \le \frac{1}{4} \\ 0 & , \quad x > \frac{1}{4} \end{cases} \tag{1.4.10}$$

is

$$u(x,t) = \begin{cases} 2 & , \quad x \le t + \frac{1}{4} \\ 0 & , \quad t + \frac{1}{4} < x \end{cases} \qquad \text{for } t > 0 \tag{1.4.11}$$

and that satisfying

$$u(x,0) = \begin{cases} 2 & , \quad x \le 0 \\ 1 & , \quad 0 < x \le \frac{1}{2} \\ 0 & , \quad \frac{1}{2} < x \end{cases} \tag{1.4.12}$$

is

$$u(x,t) = \begin{cases} 2 \quad , & x \le \frac{3}{2}t \\ 1 \quad , & \frac{3}{2}t < x \le \frac{1}{2}t + \frac{1}{2} \quad , \quad \text{for } 0 < t < \dfrac{1}{2} \\ 0 \quad , & \frac{1}{2}t + \frac{1}{2} < x \end{cases} \tag{1.4.13}$$

$$u(x,t) = \begin{cases} 2 \quad , & x \le t + \frac{1}{4} \\ 0 \quad , & t + \frac{1}{4} < x. \end{cases} \quad , \quad \text{for } t > \frac{1}{2}$$

The solution of (1.4.12), for $0 < t < \frac{1}{2}$, has two shocks and it is interesting to draw them in the (x,t)-plane.

(1.4.11) and (1.4.13) are admissible and unique solutions with two different initial values. However, both represent the same function for $t \ge \frac{1}{2}$. Thus, the same state given by the two solutions at time $t \ge \frac{1}{2}$ corresponds to two initial states. This shows irreversibility-past can not be uniquely determined by the future.

1.5 Some examples

Even though the conservation form (1.3.1) of the Burgers' equation (1.2.1) looks innocently simple, explicit solution (see section 1.9) of an initial value problem with arbitrary initial data (1.2.2) is so involved that it requires a lot of mathematical analysis to deduce even some simple properties of the solution. In this section we present a number of exact solutions and asymptotic form of some other solutions which shows that genuine nonlinearity significantly modifies the linear solution.

Example 1.5.1 Solution of an initial value problem for (1.3.1), with continuous initial data

$$\phi(x) = \begin{cases} 1 \quad , & x \le -\frac{1}{2} \\ \frac{1}{2} - x \quad , & -\frac{1}{2} < x \le \frac{1}{2} \\ 0 \quad , & x > \frac{1}{2} \end{cases} \tag{1.5.1}$$

remains continuous for all t in the interval $0 \le t < 1$ and is given by

$$u(x,t) = \begin{cases} 1, \quad , & x \le -\frac{1}{2} + t \\ \frac{(1/2)-x}{1-t} \quad , & -\frac{1}{2} + t < x \le \frac{1}{2} \\ 0 \quad , & x > \frac{1}{2} \end{cases} \tag{1.5.2}$$

For $t \ge 1$ the solution has a shock, which separates two constant states 1and 0 and which moves along the path $x = X(t) \equiv \frac{1}{2}t$ as in the case of the solution satisfying the initial condition (1.4.6). The characteristic curves starting from the various points of the x-axis have been shown in Fig 1.5.1

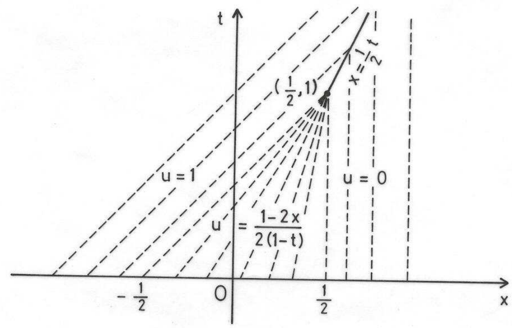

Fig. 1.5.1 : All characteristic curves starting from the points of the initial data between $-\frac{1}{2} \leq x \leq \frac{1}{2}$ meet at the point $\left(\frac{1}{2}, 1\right)$.

Example 1.5.2 Consider an initial data

$$\phi(x) = \begin{cases} \frac{1}{2}A \quad , & -1 < x \leq 1 \\ 0 \quad , & x \leq -1 \text{ and } x > 1 \end{cases} \tag{1.5.3}$$

where we take $A > 0$. If the evolution of the initial data is described according to the conservation law (1.3.1) then the solution has two distinct representations in two different time intervals :

(i) $0 < t \leq \frac{8}{A}$.

$$u(x,t) = \begin{cases} 0 \quad , & x \leq -1 \\ \frac{x+1}{t} \quad , & -1 < x \leq -1 + \frac{A}{2}t \\ \frac{1}{2}A \quad , & -1 + \frac{A}{2}t < x \leq 1 + \frac{A}{4}t \\ 0 \quad , & 1 < x \end{cases} \tag{1.5.4}$$

which has been shown in the Fig. 1.5.2.

Note that there is a centered wave in the wedged shape region $-1 < x \leq -1 + \frac{A}{2}t$ and a shock along the curve $x = \frac{1}{2}At$ in the (x,t)-plane. At the time $t = \frac{8}{A}$ the leading front of the centered wave overtakes the shock at $x = 3$. After this time the shock interacts with the centered wave.

(ii) $t \geq \frac{8}{A}$.

The shock path $x = X(t)$ is obtained by solving

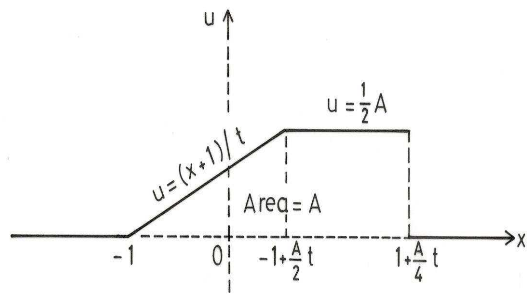

Fig. 1.5.2 : Graph of the solution with initial value (1.5.3) valid in the time interval $0 < t < \frac{8}{A}$.

$$\frac{dX}{dt} = \frac{1}{2}(u_l(X(t)) + u_r(X(t)) = \frac{X+1}{2t}\ , \ \ X\left(t = \frac{8}{A}\right) = 3$$

which gives

$$X(t) = -1 + \sqrt{2At} \tag{1.5.5}$$

At $x = X(t)$, the amplitude of the pulse i.e. the shock strength $u_l - u_r$ is given by

$$u = \frac{x+1}{t}|_{x=X(t)} = \sqrt{\frac{2A}{t}} \tag{1.5.6}$$

The pulse now takes a triangular shape whose base is spread over a distance $\sqrt{2At}$ and whose height is $\sqrt{\frac{2A}{t}}$ (Fig.1.5.3). The total area of the pulse remains constant equal to A which is also the area of the initial pulse (1.5.3). This agrees with a general property of the conservation law (1.3.1) *"when the solution u vanishes outside a closed bounded interval of x-axis, $\int_{-\infty}^{\infty} u(\xi, t)d\xi$ is independent of t."* Fig.1.5.3 is the limiting form of the shape of the graph of any solution for which the initial data $\phi(x)$ is positive everywhere and is of compact support.

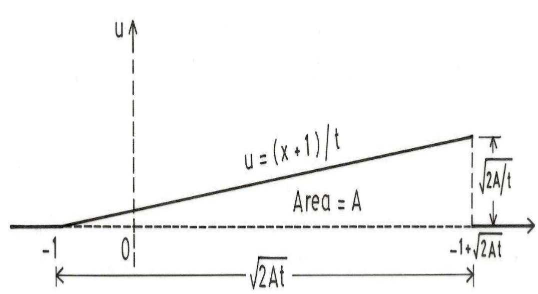

Fig. 1.5.3 : Graph of the solution with initial condition (1.5.3) valid from $t > \frac{8}{A}$.

Example 1.5.3 Consider an initial data

$$\phi(x) = \begin{cases} 0 \quad , & -\infty < x < -1 \\ -x-1 \quad , & -1 < x \leq -\frac{1}{2} \\ x \quad , & -\frac{1}{2} < x \leq 1 \\ -x+2 \quad , & 1 < x \leq 2 \\ 0 \quad , & 2 < x < \infty \end{cases} \tag{1.5.7}$$

which has been shown in Fig.1.5.4.

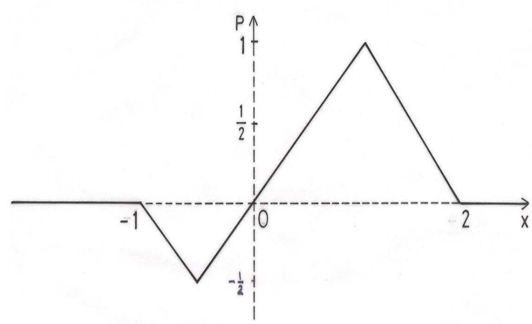

Fig. 1.5.4 : Graph of the initial data (1.5.7).

The solution of the conservation law (1.3.1) with initial condition (1.5.7) remains continuous for $t < 1$ and its expression can be easily written. During this initial stage, the interval in which the solution is nonzero remains fixed i.e. $(-1, 2)$. At $t = 1$, a pair of shocks appear at -1 and 2 and for $t > 1$ the solution is given by

$$u(x,t) = \begin{cases} 0 \quad , & -\infty < x \leq -\sqrt{\frac{1}{2}(1+t))} \\ \frac{x}{1+t} \quad , & -\sqrt{\frac{1}{2}(1+t)} < x < \sqrt{2(1+t)} \\ 0 \quad , & \sqrt{2(1+t)} < x < \infty \end{cases} \tag{1.5.8}$$

with shocks at the leading and the trailing ends. The values of u at these two ends are $\sqrt{2/(1+t)}$ and $-\sqrt{1/\{2(1+t)\}}$ respectively.

The area of the positive pulse on the right side of $x = 0$ is

$$\frac{1}{2}\sqrt{2(1+t)}\sqrt{\frac{2}{1+t}} = 1$$

and that of the negative pulse on the left side of $x = 0$ is

$$\frac{1}{2}\sqrt{\frac{1+t}{2}}\;\frac{1}{\sqrt{2(1+t)}} = \frac{1}{4}$$

Thus the areas on the two sides of the origin are conserved. This is because the flux function $\frac{1}{2}u^2$ in (1.3.1) vanishes at the origin.

The ultimate shape of the solution is depicted in Fig.1.5.5 which is called a N wave.

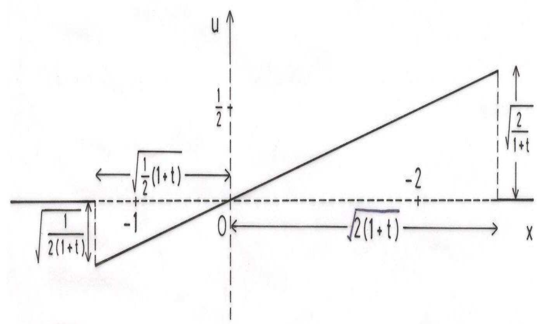

Fig. 1.5.5 : A N-wave solution (1.5.8) at $t = 3$.

In this figure, the leading and trailing shocks are not of equal strength.

Example 1.5.4 Consider an initial data

$$\phi(x) = \begin{cases} 0 \quad , & -\infty < x \leq -\lambda \\ 2a(x+\lambda)/\lambda \quad , & -\lambda < x \leq -\frac{1}{2}\lambda \\ -2ax/\lambda \quad , & -\frac{1}{2}\lambda < x \leq \frac{1}{2}\lambda \\ 2a(x-\lambda)/\lambda \quad , & \frac{1}{2}\lambda < x \leq \lambda \\ 0 \quad , & \lambda < x < \infty \end{cases} \tag{1.5.9}$$

where $\lambda > 0$. This initial data has been shown graphically in Fig.1.5.6

Example 1.5.5 The solution of the conservation law (1.3.1) with initial data (1.5.9) remains continuous for $0 < t < \frac{\lambda}{2a}$. At $t = \frac{\lambda}{2a}$, a shock appears at the origin with $u_l = a$ and $u_r = -a$. According to the jump relation, this shock does not move away from the origin but its amplitude decays. The solution in the interval $-\lambda < x \leq -\frac{1}{2}\lambda + at$ for $t \leq \frac{\lambda}{2a}$ and in $-\lambda < x < 0$ for $t > \frac{\lambda}{2a}$ are given by

$$u(x,t) = \frac{x+\lambda}{t+\frac{\lambda}{2a}} \tag{1.5.10}$$

Since $u_r(t) = -u_l(t)$, (1.5.10) shows that the shock strength at the origin is

$$u_l - u_r = 2\lambda/(t + \frac{\lambda}{2a}) \tag{1.5.11}$$

showing that, unlike all previous examples where the shock strength decays as $0\left(\frac{1}{\sqrt{t}}\right)$, in this case it decays as $0\left(\frac{1}{t}\right)$. The asymptotic solution as $t \to \infty$, retaining only the first term is

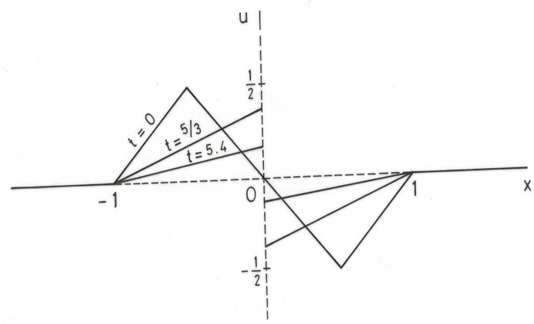

Fig. 1.5.6 : The initial pulse is given by (1.5.9) with $\lambda = 1, a = \frac{1}{2}$. The solution develops a shock at the origin which decays to zero with time as $0(\frac{1}{t})$.

$$u(x,t) = \begin{cases} 0 \quad , & -\infty < x \le -\lambda \\ (x+\lambda)/t \quad , & -\lambda < x \le 0 \\ (x-\lambda)/t \quad , & 0 < x \le \lambda \\ 0 \quad , & \lambda < x < \infty \end{cases} \tag{1.5.12}$$

which has a shock of strength $2\lambda/t$. It is interesting to note that the asymptotic solution is the same whatever may be the initial amplitude but remembers (apart from the initial total area of the pulse) the length 2λ where it is non-zero.

As an another interesting example, the reader is asked to find the solution of (1.3.1) with initial data

$$\phi(x) = \begin{cases} 0 \quad , & -\infty < x \le -2 \\ x+2 \quad , & -2 < x \le -1 \\ -x \quad , & -1 < x \le \frac{1}{2} \\ x-1 \quad , & \frac{1}{2} < x \le 1 \\ 0 \quad , & 1 < x < \infty \end{cases} \tag{1.5.13}$$

Example 1.5.6 We now consider an important example of a solution of (1.3.1) with periodic initial condition:

$$\phi(x) = -a \sin \frac{\pi x}{\lambda}\ ,\ \lambda > 0 \tag{1.5.14}$$

The asymptotic form of the solution as $t \to \infty$ can be deduced with the help of Lax-Oleinik formula. Here we deduce the form by noting that at any time t, the solution must be periodic of the period 2λ and assuming that in the period $-\lambda < x < \lambda$, the solution is given by a pair of centered waves

$$u(x,t) = \begin{cases} (x+\lambda)/t\ ,\ -\lambda < x \le 0 \\ (x-\lambda)/t\ ,\ 0 < x \le \lambda \end{cases} \tag{1.5.15}$$

According to the jump condition, the shocks do not move away from the points $0, \pm 2\lambda$, $\pm 4\lambda, \ldots$ and their strengths decay as $\frac{2\lambda}{t}$ as was the case in the previous example. The asymptotic periodic solution "forgets" the amplitude a of the initial data but "remembers" its period 2λ. The asymptotic solution in the form of a saw-tooth has been shown in Fig.1.5.7.

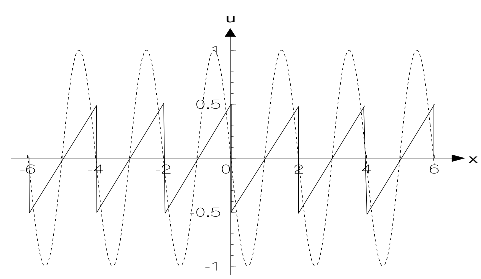

Fig. 1.5.7 : The saw-tooth solution arising from a periodic initial data shown by a dotted line.

Example 1.5.7 An equation, governing the propagation of small perturbations trapped at a point on the sonic surface of a steady gas flow, is given by (Prasad, 1973)

$$u_t + (u - Kx)u_x = Ku \tag{1.5.16}$$

The dependent and independent variables have been properly scaled. The constant K is proportional to the deceleration of the fluid elements at the sonic point in the steady flow. When the fluid is passing from a supersonic state to a subsonic state, $K > 0$.

Had the genuine nonlinearity not been present, the approximate equation would have been

$$u_t - Kxu_x = Ku \tag{1.5.17}$$

The solution of the initial value problem

$$u(x,0) = \phi(x) \tag{1.5.18}$$

for (1.5.17) is

$$u(x,t) = e^{Kt}\phi(xe^{Kt}) \tag{1.5.19}$$

This solution shows that for $K > 0$ the amplitude u would tend to infinity as $t \to \infty$. However, for an initial data ϕ which is nonzero only on an bounded interval on the x- axis, the solution will get concentrated near the point $x = 0$

A conservation form of the genuinely nonlinear equation (1.5.16) is

$$u_t + (\frac{1}{2}u^2 - Kxu)_x = 0 \tag{1.5.20}$$

It is simple to show that the jump relation for a shock appearing in a weak solution of (1.5.20) is

$$\frac{dX(t)}{dt} = \frac{1}{2}(u_l + u_r) - KX(t) \tag{1.5.21}$$

When a shock appears in a solution of an initial value problem for (1.5.20), it cuts off the growing part of pulse as shown in Fig. 1.5.8 for a special initial data.

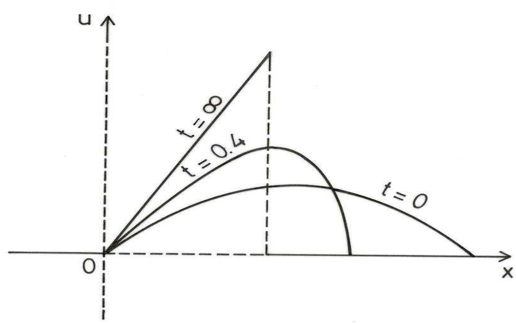

Fig. 1.5.8 : Solution of (1.5.20), with an initial data which is positive and nonzero only on a closed bounded interval, gets trapped on the right hand side of the origin and ultimately attains a triangular shape.

1.6 Shock structure, dissipation and entropy condition

Shock front, a surface of discontinuity, is a mathematical idealization of a physical phenomenon in which a physical variable, say pressure, density or particle velocity vary continuously but rapidly in a narrow zone containing the surface. This idealisation, in which the model conservation law (1.3.3) is valid, breaks down across a shock. In (1.3.3) the *flux* F has been taken to be a function of the *density* u alone whereas in a physical phenomenon F depends not only on u but also on its gradient u_x:

$$F = Q(u) - \nu u_x \;, \;\; \nu > 0 \tag{1.6.1}$$

where ν is a small constant. In an interval where u varies slowly, the term νu_x is negligible compared to $Q(u)$. But in an interval where the gradient u_x is large and of the order of $1/\nu$, the two terms are comparable the model equation (1.3.3) needs to be modified. Choosing $H = u, Q(u) = \frac{1}{2}u^2$ as in (1.3.1) and assuming that the function u is smooth, we get a modification of the equation (1.1.6) in the form

$$u_t + uu_x = \nu u_{xx},\ \nu > 0 \tag{1.6.2}$$

In order to study how a shock front in a solution of (1.3.1) is replaced by a continuous solution of (1.6.2) with a narrow zone of rapid variation in u, we consider the solution

$$u(x,t) \ = \ \begin{cases} u_l & , \quad x \le St \\ \\ u_r & , \quad x > St \end{cases} \tag{1.6.3}$$

of (1.3.1) where the shock velocity $\dot{X}(t) = S$ satisfies

$$S = \frac{1}{2}(u_l + u_r)\ ,\ u_l > u_r \tag{1.6.4}$$

This solution becomes steady in a frame of reference moving with the velocity S and hence we look for a solution of (1.6.2) in the form

$$u(x,t) = u(\xi),\ \ \xi = x - St \tag{1.6.5}$$

i.e. we solve the two point boundary value problem for the equation

$$-Su_\xi + uu_\xi - \nu u_{\xi\xi} = 0 \tag{1.6.6}$$

satisfying

$$\lim_{\xi\to-\infty} u(\xi) = u_l,\ \ \lim_{\xi\to\infty} u(\xi) = u_r \tag{1.6.7}$$

We further assume that this solution tends to the two limiting values u_l and u_r smoothly which requires.

$$\lim_{\xi\to\pm\infty} u_\xi(\xi) = 0 \tag{1.6.8}$$

Integrating (1.6.6) we get $-Su + \frac{1}{2}u^2 - \nu u_\xi =$ a constant $= A$, say. In order that the solution tends to u_l and u_r as x tends infinity on the two sides, the constant A should be such that

we can write this equation in the form

$$u_\xi = -\frac{1}{2\nu}(u_l - u)(u - u_r) \tag{1.6.9}$$

with

$$u_l = S + \sqrt{S^2 + 2A}\ \ ,\ \ u_r = S - \sqrt{S^2 + 2A} \tag{1.6.10}$$

Since u_l and u_r are real we must have $S^2 + 2A > 0$. Integrating (1.6.9) we get

$$u(x - St) \equiv u(\xi) = \frac{1}{2}\left[(u_l + u_r) - (u_l - u_r)\tanh\left\{\frac{u_l - u_r}{4\nu}(x - St)\right\}\right] \tag{1.6.11}$$

where we have chosen the constant of integration such that $u = \frac{1}{2}(u_l + u_r)$ at $\xi = 0$. We note that $u \to u_r$ as $x \to \infty$ and $u \to u_l$ as $x \to -\infty$.

(1.6.11) is a continuous solution of (1.6.2) joining the two *states*: u_l at $-\infty$ and u_r at $+\infty$ and thus represents the structure of a shock wave separating u_l on its left and u_r on its right. The shock speed $S = \frac{1}{2}(u_l + u_r)$ is the speed of translation of the whole continuous profile in the structure of the shock.

Since the transition from u_l to u_r takes place over an infinite distance, we measure the shock thickness by a distance over which a given fraction $1 - 2\alpha$, where α is a small positive number, of the shock strength is observed. Let us denote by ξ_- the value of ξ where $u(\xi_-) = u_l - \alpha(u_l - u_r)$ and by ξ_+ that where $u(\xi_+) = u_r + \alpha(u_l - u_r)$. Here $0 < \alpha < 1$. Then from (1.6.11), we obtain $\xi_+ + \xi_- = 0$ and the following expression for the shock thickness

$$\xi_+ - \xi_- = \frac{2\nu}{u_l - u_r} ln \frac{1-\alpha}{\alpha} \tag{1.6.12}$$

The shock thickness is inversely proportional to the shock strength $u_l - u_r$, showing that the transition through the shock from u_l to u_r takes place over a large distance for a weak shock and over a small distance for a strong shock. For a shock of moderate strength, the shock thickness is of the order of the *diffusion* coefficient ν. Since ν is small the shock thickness is generally very small compared to the length scales normally we consider in day to day life and for all practical purposes it can be treated as a surface of discontinuity as dealt in the previous sections.

In the limit as $\nu \to 0+$, the travelling wave solution (1.6.11) satisfying (1.6.7) becomes a solution

$$u(x,t) = \begin{cases} u_l \quad , \quad x - \frac{1}{2}(u_l + u_r)t \le 0 \\ \\ u_r \quad , \quad x - \frac{1}{2}(u_l + u_r)t > 0 \end{cases} \tag{1.6.13}$$

of (1.3.1) with a single shock. We also note that an expansion shock solution (1.6.13) with $u_l < u_r$ can not be obtained as a limit of a solution of the *viscosity equation* (1.6.2) because the expression on the right hand side of (1.6.9) is negative for $u_l < u < u_r$ and hence in any solution of (1.6.9) u can not increase from u_l to u_r as ξ increases. This is a general property of the conservation law (1.3.1), every weak solution of this equation with only discontinuities which are shocks can be obtained as a limit almost everywhere of the viscosity equation (1.6.2) as $\nu \to 0+$. The converse is also true. All solutions of (1.6.2) are smooth. The limit of any solution of this equation is a weak solution of (1.3.1) with discontinuities which, if any, are shocks.

Problems Solve $u_t + \left(\frac{1}{2}u^2\right)_x = 0$ with following initial conditions

1.

$$u(x,0) = \begin{cases} 1 & , \quad -\infty < x \leq 0 \\ 2 & , \quad 0 < x \leq 1 \\ 0 & , \quad x > 0 \end{cases}$$

2.

$$u(x,0) = \begin{cases} 0 & , \quad |x| \geq 1 \\ -1 & , \quad -1 < x < 0 \\ 1 & , \quad 0 < x < 1 \end{cases}$$

3.

$$u(x,0) = \begin{cases} 0 & , \quad -\infty < x \leq -1 \\ 1 & , \quad -1 < x \leq 0 \\ 2 & , \quad 0 < x \leq 1 \\ 0 & , \quad x > 1 \end{cases}$$

4.

$$u(x,0) = \begin{cases} 2 & , \quad x \leq 0 \\ 0 & , \quad 0 < x \leq 1 \\ 1 & , \quad x > 1 \end{cases}$$

References

[1] Bhatnagar, P. L. (1979) *Nonlinear waves in one-dimensional dispersive systems*, Oxford Mathematical Monograph Series, Clarendon Press, Oxford.

[2] Courant, R. and Freidrichs, K O (1948) *Supersonic flow and shock waves*, Interscience (reprinted by Springer Verlag).

[3] Prasad, P. (1973) *Nonlinear wave propagation on an arbitrary steady transonic flow*, Journal of Fluid Mechanics, **57**, 721-737.

[4] Prasad, P. and Ravindran, R (1985) *Partial differential equations*, Wiley Eastern.

[5] Prasad, P. (1997) *Nonlinearity, Conservation Law and Shocks, Part I: Genuine Nonlinearity and Discontinuous Solutions*, RESONANCE- Journal of Science Education by Indian Academy of Sciences, Bangalore, Vol-2, No.2, 8-18.

(1997) *Nonlinearity, Conservation Law and Shocks, Part II: Stability Consideration and Examples*, RESONANCE- Journal of Science Education by Indian Academy of Sciences, Bangalore, Vol-2, No.7, 8-19.

[6] Smoller, J. (1983) *Shock waves and reaction diffusion equation*, Springer-Verlag.

[7] Whitham, G.B. (1974) *Linear and nonlinear waves*, John Wiley.