Sample Problems in Classical Mechanics

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1. Two particles move about each other in circular orbits under the influence of mutual gravitational force, with a period τ . At some time $t = 0$, they are suddenly stopped and then they are released and allowed to fall into each other. Find the time T after which they collide, in terms of τ.

2. Solve either (a) below or assuming (a) solve (b).

   (a) Show that for any repulsive central force, a formal solution for the scattering angle Θ is given by,

   $$\Theta = \pi + 2\int_0^{u_0} \frac{s du}{\sqrt{1 - V(u)/E - s^2u^2}} \quad \text{where}$$

   $u = 1/r$, V is the potential energy, E is the total energy, s is the impact parameter and u_0 corresponds to the turning point of the orbit.

   (b) Now consider a repulsive central force $f = kr^{-3}$. Show that

   $$\sigma(\Theta)d\Theta = \frac{k}{2E}\frac{(1-x)dx}{x^2(2-x)^2 sin(\pi x)} \quad \text{where} \quad x \equiv \Theta/\pi$$

   and $E > 0$ is the energy.

3. Consider central force motion with the force given by,

   $$f(r) = -\frac{K}{r^2} + \frac{C}{r^3}$$

   Obtain the orbit equation and show that the orbit is a precessing ellipse. What is the approximate rate of precession to first order in C?

4. On a 2N dimensional phase space, consider the set of functions which are purely quadratic in the generalised coordinates and momenta.

   (a) Show that the Poisson bracket of any two members of the set is again in the set.

   (b) Let the Hamiltonian be given by,

   $$H = \frac{1}{2}\sum_{i=1}^{N}\{(p_i)^2 + (q^i)^2\}$$

   Which functions in the above set constitute infinitesimal symmetries of the system? How many independent symmetries are there?

5. Consider a one dimensional system with Hamiltonian given by,

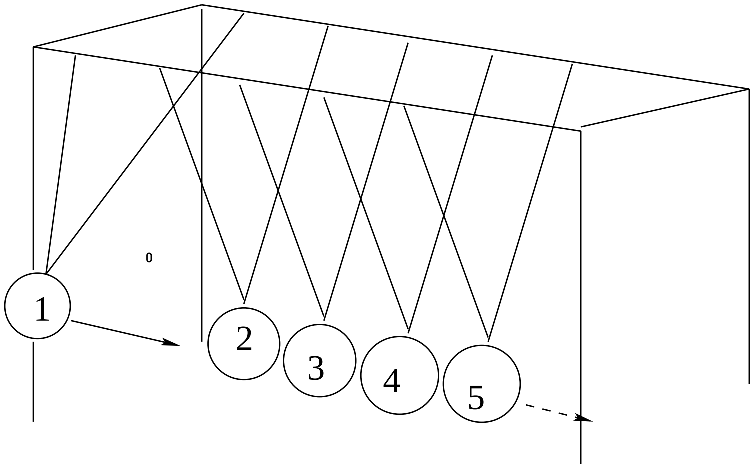
   $$H(q,p) = \frac{1}{2}\left[ap^2 + bq^2 + 2cpq\right] \qquad \text{where } a, b, c \text{ are constants.}$$

   Find a canonical transformation $(q,p) \to (Q,P)$ such that

   $$H(Q,P) = AP^2 + BQ^2.$$

   What is the frequency of the harmonic oscillator in the new variables? Is it the same as in the old variables?

6. Consider Newton's cradle in the figure below. It is well known that if k balls are lifted together and released, after collision, exactly k balls move out at the other end. The task is to explain why. To simplify the problem, first consider a cradle with two balls of *un-equal masses*. The 1st ball with mass m_1 is used to strike the stationary one of mass m_2. Is the state (i.e. the two momenta) after the collision unique? If so, give the final momenta of the two balls as fractions of the initial momentum. What happens when the two masses are equal?

   Now consider the case of three balls of *equal* masses. As before, one ball at one extreme is used to strike. Is the final state unique? If not, why does one see only one ball exit the group, at the other extreme?

   Comment on generalization to N (not too large) balls in the cradle.

Electromagnetism :

Sample questions for Comprehensive Exams

Comments

* The questions below (except **4**) are from Jackson, with a few changes here and there.

* I did not write down the questions for many topics, notably radiating systems.

* There are many internet sites with solutions to (most of) Jackson's exercises. Anyway, I'll attach the solutions later – either downloading from some site, or my own. But there are often more than one way of solving a given problem, including the ones below.

Sample Questions

(1) Consider an electrostatic potential Φ given by

$$\Phi = \frac{q}{4\pi\epsilon_0} \, \frac{e^{-ar}}{r} \, (1 + br)$$

where a and b are constants. Find the distribution of charge (both continuous and discrete) that will give this potential.

(a) What, if anything, is special when $a = 2b$?

(a) Interpret your results physically.

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(2) A point charge q is at a distance d away from an infinite plane conductor held at zero potential. Using the method of images (or otherwise), find:

(a) surface charge density induced on the plane, and plot it.

(b) the force between the plane and the charge

(c) the work necessary to remove the charge q from its position to infinity

(d) express the answer in part **(c)** in electron volts when $d = 1$ Angstrom.

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(3) A straight-line charge with linear charge density λ is located perpendicular to the $x - y$ plane at (x_0, y_0) in the first quadrant. The intersecting planes $x = 0,\ y \geq 0$ and $y = 0,\ x \geq 0$ are conducting boundary surfaces held at zero potential. Consider the potential and fields in the first quadrant.

(a) The electrostatic potential for an isolated line charge at (x_0, y_0) is given by

$$\Phi(x, y) = \frac{\lambda}{4\pi\epsilon_0}\, ln\, \frac{R^2}{r^2}$$

where R is a constant and $r^2 = (x - x_0)^2 + (y - y_0)^2$. Find the potential Φ for the line charge in the presence of the intersecting planes. Verify explicitly that the potential and the tangential electric field vanish on the bundary surfaces.

(b) Show that far from the origin, the potential asymptotes to

$$\Phi \ \rightarrow\ \frac{4\lambda}{\pi\epsilon_0}\ \frac{(x_0 y_0)\ (xy)}{\rho^4}$$

where $\rho^2 = x^2 + y^2 \ \gg\ (x_0)^2 + (y_0)^2$.

**** If you are unable to solve this problem, then solve the simpler one by assuming ONLY ONE conducting plane. The expression in part **(b)** will now be different. The correct solution will be given 60 percent marks. ****

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(4) Consider three charges: a charge $(-2q)$ is placed at $(0, 0, 0)$; a charge

$(+q)$ is placed at $(0, 0, a)$; and, the third charge $(+q)$ is placed at $(0, 0, -a)$. Find the point(s), if any, at finite distance(s) where the net force vanishes.

Such points are stationary points. Infinitesimal motion away from such points in any of the three independent directions may be stable or unstable in some or all directions. The possibilities are: (1) stable in all three directions; (2) stable in two and unstable in the other one direction; (3) stable in one and unstable in the other two directions; (4) unstable in all three directions.

(a) Which of the possibilities are realised by the stationary point(s) of the three charge distribution given above.

(b) Note that, for static charge distributions, not all the above four possibilities may be realised. If you assert that some of the possibilities cannot be realised by any static charge distributions then: which possibities or they? Prove your assertion.

(c) By giving atleast one example of charge distribution (besides the example given above), show that each of the remaining possibilities is realised. (That is, in your examples, you must find all the stationary points and analyse explicitly the infinitesimal motions near them.)

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(5) A transverse plane wave is incident normally in vacuum on a perfectly absorbing flat screen.

(a) From the law of conservation of linear momentum, show that the pressure – radiation pressure – exerted on the screen is equal to the field energy per unit volume in the wave.

(b) Let the incident radiation have a flux of 1.4 kW/m^2 . The absorbing screen has a mass of 1 gm/m^2 . What is the screen's acceleration due to radiation pressure?

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(6) Consider a layered slab of thickness d, with its bottom at $z = 0$ and its top at $z = d$. The refractive index of the slab is n . The refractive index of the medium below it, namely for $z < 0$, is n_1, and of the medium above, namely for $z > 0$, is n_2 .

A plane electromagnetic wave is incident normally on the slab from below, namely from the $z < 0$ side.

(a) Calculate the reflection and transmission coefficients (ratios of reflected and transmitted Poynting's flux to the incident flux).

(b) Take $n_1 = 3,\ n = 2$, and $n_2 = 1$. For these values, plot the reflection coefficient as a function of frequency.

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(7) Consider a conductor whose conductivity $\sigma(\omega)$ is frequency dependent and is given by

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}$$

where $\sigma_0 = \epsilon_0 \omega_p^2 \tau$ and τ is damping time. Consider electric fields in a conductor, Ohm's law, continuity equation, and the differential form of Coulomb's law.

(a) Show that the time-Fourier transformed charge density satisfies the equation

$$(\sigma(\omega) - i\omega\epsilon_0)\ \rho(\vec{x}, \omega)\ =\ 0$$

(b) Assume that $\omega_p \tau \gg 1$. Show that, in this approximation, an initial disturbance will oscillate with a frequency ω_p and its amplitude will decay with a decay constant $\frac{1}{2\tau}$.

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(8) A particle with mass m and charge q moves in a uniform static electric field $\vec{E}_0$. Its initial velicity is $\vec{v}_0$ and is perpendicular to the electric field. Note that the particle motion may be relativistic.

(a) Solve for the velocity and position and obtain them explicitly as functions of time. Plot these functions.

(b) Eliminate time and obtain velocity as a function of the position. Plot this function.

(c) Discuss the motion for 'short' and 'long' times.

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(9) Static uniform electric and magnetic fields, $\vec{E}$ and $\vec{B}$, make an angle θ with respect to each other.

(a) By a suitable choice of axes, solve the force equation for the motion of a particle of charge q and mass m in rectangular coordinates.

(b) For $\vec{E}$ and $\vec{B}$ parallel, show that with appropriate constants of integration et cetera, the parametric solution can be written as

$$x = AR\ sin\ \phi\ \ ,\quad y = AR\ cos\ \phi\ \ ,$$

$$z = \frac{R}{\rho}\ \sqrt{1+A^2}\ Cosh\ (\rho\phi)\ \ ,\quad ct = \frac{R}{\rho}\ \sqrt{1+A^2}\ Sinh\ (\rho\phi)$$

where $R = \frac{mc^2}{qB}$ $\rho = \frac{E}{B}$, A is an arbitrary constant, and ϕ is a parameter (actually, $\frac{c}{R}$ times the proper time).

Sample Questions in Quantum Mechanics

1. Let H be the Hamiltonian of a physical system. Denote by $|\phi_n\rangle$ the eigenvectors of H with eigenvalues E_n:

   $$H|\phi_n\rangle = E_n|\phi_n\rangle$$

   (a) For an arbitrary operator A prove the relation:

   $$\langle\phi_n|[A, H]|\phi_n\rangle = 0$$

   (b) Consider a one-dimensional problem, where the physical system is a particle of mass m and of potential energy $V(X)$. In this case H is written:

   $$H = \frac{P^2}{2m} + V(X)$$

      i. In terms of P, X and $V(X)$, find the commutators: $[H, P], [H, X]$.

      ii. Show that the mean value of the matrix element $\langle\phi_n|P|\phi_n\rangle$ is zero.

      iii. Establish a relation between $E_k = \langle\phi_n|\frac{P^2}{2m}|\phi_n\rangle$ (the mean value of kinetic energy in the state $|\phi_n\rangle$) and $\langle\phi_n|X\frac{dV}{dX}|\phi_n\rangle$:

      $$\langle\phi_n|X\frac{dV}{dX}|\phi_n\rangle = 2E_k$$

      Since the mean value of the potential energy is $\langle\phi_n|V(X)|\phi_n\rangle$, how is it related to the mean value of the kinetic energy when:

      $$V(X) = V_0X^\lambda$$

      ($\lambda = 2, 4, 6...; V_0 > 0$)

2. Consider spin half particles, which can be in states $|+\rangle, |-\rangle$. Let ρ be the density matrix, correctly normalized so that $Tr\rho = 1$.

   (a) Consider a system with one spin. Assume the system is in a *pure* state. Take a general pure state, construct ρ in the $|\pm\rangle$ basis. Using this or otherwise and show that $\rho^2 = \rho$.

   (b) Consider a system with two spins and use $|\pm, \pm\rangle$ as the four basis states for the combined system. Let the system be in a pure state. Argue that $\rho^2 = \rho$ again.

   (c) For a state $\frac{1}{\sqrt{2}}|+-\rangle + \frac{1}{\sqrt{2}}|-+\rangle$ evaluate the "partial trace" - where the states of the second particle have been traced over. This gives $\rho(1) = Tr_2\rho$ - the density matrix for the first spin. Evaluate $\rho(1)$. Is $\rho(1)^2 = \rho(1)$?

   (d) Consider a special pure state that is in a product state, i.e. a state that can be described as $|\phi_1\rangle \otimes |\phi_2\rangle$ where the subscripts denote the two spins. Show, by taking an example, that the partial traced density matrix $\rho(1)$ satisfies $\rho(1)^2 = \rho(1)$.

3. Take the Hamiltonian for one spin half in a magnetic field to be $H = -\mu\vec{S}.\vec{B}$. Let us take $\vec{B} = B_z\hat{k}$.
   (a) If the spin was in the state $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$ at time $t = 0$, what is the density matrix after a time t_1?
   (b) Is it a pure state or a mixed state at $t = t_1$? (Evaluate ρ^2 and compare with ρ).
   (c) Evaluate $Tr\rho S_x$ as a function of time and thus find the precession frequency.
   (d) This frequency is the same for *any* ρ. Establish this. (Hint: Use the fact that $\rho(t) = e^{-iHt}\rho(0)e^{iHt}$ and the various properties of Pauli matrices such as $\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k$, $Tr\sigma_i\sigma_j = 2\delta_{ij}$ and $e^{i\theta\sigma_z} = cos\theta + i\sigma_z sin\theta$)

4. Consider a charged particle (of mass m and charge e) moving in a uniform magnetic field $\vec{B} = B_z\hat{k}$.
   (a) Hamiltonian $H = \frac{(\vec{p}-e\vec{A}).(\vec{p}-e\vec{A})}{2m} \equiv \frac{\vec{\Pi}.\vec{\Pi}}{2m}$. Choose a convenient gauge for $\vec{A}$ and evaluate the commutators $[\Pi_i, \Pi_j]$ for $i, j = x, y, z$.
   (b) Show that the Hamiltonian splits naturally into two parts: $H = H_z + H_{xy}$ where H_z is a free particle Hamiltonian with eigenfunctions as plane waves $e^{ik_z z}$ and H_{xy} is a harmonic oscillator Hamiltonian. Hence the eigenvalues of this Hamiltonian are known exactly. What are they in terms of the original parameters?

5. Consider a particle P of mass m constrained to move in a circle in the $x - y$ plane, centered at O and of radius r. The position is characterized by an angle θ and the wave function is $\psi(\theta)$. $\psi(\theta) = \psi(\theta + 2\pi)$ because it has to have a unique value at a space time point. Assume it is normalized $\int_0^{2\pi} d\theta|\psi(\theta)|^2 = 1$.
   (a) Consider the operator $M = -i\hbar\frac{d}{d\theta}$. Is it Hermitian? What are the eigenvalues and normalized eigenfunctions?
   (b) The kinetic energy of the particle is
   $$H_0 = \frac{M^2}{2mr^2}$$
   What are the eigenvalues and eigenfunctions? Is there any degeneracy?
   (c) Assume the particle has charge q and that it interacts with a small uniform electric field $\mathcal{E}$ in the x direction. This adds a potential energy perturbation
   $$W = -q\mathcal{E}x = -q\mathcal{E}rcos\theta$$
   to the Hamiltonian. Calculate the new ground state wave function to first order in $\mathcal{E}$.

(d) Determine the susceptibility χ which is the proportionality coefficient between the electric dipole along x and the electric field $\mathcal{E}$.

6. Consider two spin 1/2's $\vec{S}_1$ and $\vec{S}_2$ coupled by an interaction of the form $a(t)\vec{S}_1.\vec{S}_2$. Assume that $a(t)$ is a pulse of some arbitrary shape extending over a finite duration.

   (a) Show that $\vec{S}_1+\vec{S}_2$ is conserved by this interaction. This means that if we start in the state $|+-\rangle$ then the systems remains in the subspace $|+-\rangle, |-+\rangle$ for all time.

   (b) Show that the Hamiltonian as a 2×2 matrix is

   $$H = \frac{a(t)}{4}[-I + 2\sigma_x]$$

   Diagonalize it. Find the eigenfunctions and eigenvalues.

   (c) Solve the time dependent Schroedinger equation. Show that the probability for transition from $|+-\rangle$ at $t = -\infty$ to $|-+\rangle$ at $t = +\infty$ involves only the integral $\int_{-\infty}^{\infty} a(t)dt$ and calculate it.

Sample questions for Statistical Mechanics part of Comprehensive exam

(Dated: November 8, 2013)

1) An equation of state for a rubber band is either

$$S = L_0\gamma\left[\sqrt{\frac{\theta E}{L_0}} - \frac{1}{2}\left(\frac{L}{L_0}\right)^2 - \frac{L_0}{L} + \frac{3}{2}\right]$$

or

$$S = L_0\gamma\left[e^{\theta nE/L_0} - \frac{1}{2}\left(\frac{L}{L_0}\right)^2 - \frac{L_0}{L} + \frac{3}{2}\right]$$

where $L_0 = nl_0$, and γ, l_0 and θ are constants, and L the length of the rubber band. Which of the two possibilities are acceptable? Why? For the acceptable choice, deduce the dependence of the tension f upon T and L/n; that is determine $f(T, L/n)$.

2) The probability of observing a closed equilibrated system with energy E is $P(E) \propto \Omega(E)\exp(-\beta E) = \exp[\ln(\Omega(E) - \beta E]$. Both $\ln(\Omega(E)$ and $-\beta E]$ are of the order of N which suggests that $P(E)$ is a very narrow distribution centred about E. Verify this suggestion by performing a steepest descent calculation with $P(E)$. That is, expand $\ln P(E)$ is powers of $\delta E = E - \langle E\rangle$, and truncate the expansion after the quadratic term. Use this expansion to estimate for 0.001 moles of gas the probability for observing a spontaneous fluctuation of E of the size $10^{-6}\langle E\rangle$.

3) For the photon gas, derive the formula for the correlation function $\langle\delta n_i\delta n_j\rangle$ where $\delta n_i = n_i - \langle n_i\rangle$, and n_i is the occupation number of the i^{th} oscillator state.

4) (a) Calculate the value of n_x, n_y and n_z for the case $n_x = n_y = n_z$ for a hydrogen atom (atomic weight 1.00) in a box of dimension 1 cc if the particle has a kinetic energy $3kT/2$ for $T = 300K$. What significance does this calculation illustrate?

(b) Calculate the De Broglie wavelength of an argon atom at 300 K and compare this with the average inter atomic spacing at 1 atm.

5) Consider the infinite range Ising model where the coupling constant $J_{ij} = J_0$ for all i, j

(no restriction to nearest neighbour).

$$H(\{S\}) = -h\sum_i S_i - \frac{J_0}{2}\sum_{i\neq j} S_i S_j,$$

with $J_0 > 0$ and $S_i = \pm 1$.

(a) Explain why this model only makes sense if $J_0 = J/N$, where N is the number of spins in system.

(b) Show that

$$\exp\left(\frac{ax^2}{2N}\right) = \int_{-\infty}^{\infty} dy \sqrt{\frac{Na}{2\pi}} \exp\left(-Nay^2/2 + axy\right), \quad Re(a) > 0.$$

c) Hence, show that

$$\exp\left(\frac{\beta J}{2}\right) Z = \int_{-\infty}^{\infty} dy \sqrt{\frac{N\beta J}{2\pi}} e^{-N\beta L},$$

where

$$L = \frac{Jy^2}{2} - \frac{1}{\beta}\ln\left[2\cosh(\beta H + \beta J y).\right]$$

When can this expression become non-analytic?

(d) In the thermodynamic limit, this integral can be evaluated exactly by the method of steepest descent. Show that

$$Z(\beta, h, J) = \sum_i e^{-\beta N L(h,J,\beta,y_i)},$$

and find the equation satisfied by y_i. What is the probability of the system being in the state specified by y_i? Hence show that the magnetisation is given by

$$m = y_0,$$

where y_0 is the position of the global minimum of L.

(e) Consider the case $h = 0$. By considering how to solve the equation for y_i graphically, show that there is a phase transition and find the transition temperature T_c. Discuss the acceptability of all the solutions of the equation for y_i both above and below T_c.