Comprehensive Examination

Instructions

July 7, 2017 09:30 - 13:30

- *Please use separate notebooks for Classical Mechanics and Electromagnetism.*

- *In each notebook, at the beginning, please write your name and roll number clearly.*

- *You may use loose sheets, available in the exam hall, for rough work.*

- *All problems carry equal marks.* In each section, you have to do *any three* out of the five problems.

- Passing criterion: **minimum of 10/30** marks in each section *and* a **total of 27/60** in both sections together.

- Duration of examination for both parts together: **09:30 hours to 13:30 hours**

Comprehensive Examination: Paper-I

Part A: Classical Mechanics

All problems carry equal marks. Solve any *three* problems.

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1. A missile is fired from a point A on the surface of earth. It returns and lands at a point B. The angle between the two radial vectors, $\vec{r}_A, \vec{r}_B$, is θ. The trajectory of the missile is such that the launch velocity vector and the landing velocity vector are anti-parallel. The flight is entirely governed by earth's gravity.

   Assuming earth to be spherical with radius R, determine the height reached by the missile. Discuss the limiting cases of $\theta = 0$ and π. .... [**10 marks**]

   [*Hint: Recall the general nature of trajectories in a* $\frac{1}{r}$ *potential.*]

2. (a) A small probe (or planet) of mass $m \ll M_{sun}$ begins a free fall towards a solar mass black hole, from a distance R_0, with initial velocity zero.

   Determine how long will it take to reach the 100 times the Schwarzschild radius, $R_S := 2GM/c^2 (\sim 3$ km for the sun.) ;

   You may assume Newtonian dynamics throughout and $R_0 \gg R_S$.

   Give numerical estimates for the earth-sun combination: $m = 10^{-6} \times M_{sun}$ and $R_0 = 10^8$ km. .... [**5 marks**]

   (b) According to general relativity, the probe also looses energy through gravitational radiation at a rate given by,

   $$Power = \frac{dE_{radiated}}{dt} = \frac{G}{5c^5}\left(\frac{d^3Q}{dt^3}\right)^2 \quad , \quad Q(t) := \frac{2}{3}mr^2(t) \ .$$

   This energy loss may be viewed as exerting an additional force on the probe with a magnitude given by $F_{rad}|\dot{r}| := Power$. Obtain the ratio of the F_{rad} to the Newtonian force $|F_{grav}|$. Estimate the maximum possible value of the ratio when the probe is at $100 \times R_S$. What is the direction of the radiative force?

   In MKS units: $G = 6.7 \times 10^{-11}$, $M_{sun} = 10^{30}$, $m_{probe} = 10^4$. .... [**5 marks**]

3. A free, relativistic particle moving in one dimension has energy $T = -m\sqrt{1 - \dot{x}^2}$ ($c = 1$ units). It is subjected to a harmonic potential $\frac{1}{2}\omega^2 x^2$.

   (i) Write down its Lagrangian and its Hamiltonian. Is the Hamiltonian conserved along the dynamical trajectories? ; .... [**5 marks**]

(ii) Show that the period of oscillation can be expressed as (E is the conserved energy),

$$T = \frac{4}{\omega}\sqrt{\frac{2}{m}}\int_0^{\pi/2} d\theta \frac{E - (E - m)sin^2\theta}{\sqrt{[(E + m) - (E - m)sin^2\theta]}}.$$

Show that in the non-relativistic limit, it goes over to the usual expression. [**5 marks**]

4. A particle of mass m, with initial 4-momentum p^μ scatters-off a target with a final momentum p'^μ. If the exchanged 4-momentum, $q^\mu := p^\mu - p'^\mu$ is taken up by a particle, show that such a particle *cannot* be a real particle i.e. with $q^2 := \eta_{\mu\nu}q^\mu q^\nu \geq 0$ *except* when $m = 0$. Here $\eta_{\mu\nu} = diag(1, -1, -1, -1)$. . . [**10 marks**]

5. Different parts of an extended body such as a planet, experience different forces towards an attracting body such as a star. It causes tidal distortions -squeezing along the transverse direction and stretching along the longitudinal direction- as shown by the dotted ellipse in the figure. Roche limit is the distance D, at which the tidal forces due to a heavier body of mass M and radius R, overcome the forces holding together a lighter body of mass m and radius l.

   Assuming that the lighter body is held together only by the Newtonian force of gravity and that the massive body is almost point-like, eg a black hole of mass M having $R =$ its Schwarzschild (horizon) radius $R_S := 2GM/c^2$, derive the expression for the Roche limit.

   Give numerical estimates for D when

   (i) $M = M_{sun} = 2 \times 10^{33}$ gm, $\ell = 6000$ km and density $\rho = 5$ gm/cc and

   (ii) $M = 10^9\ M_{sun},\ l = 10^5$ km and $\rho = 1.5$ gm/cc.

   Compare D with the horizon radius. What happens if

   (a) a planet size body gets near the horizon of a solar mass black hole and

   (b) it gets near the horizon of a billion solar mass black hole?

You may use $R_S = 3$ km for $M = M_{sun}$. .... [**10 marks**]

Comprehensive Examination: Paper-I

Part B: Electromagnetism

July 7, 2017 **09:30 - 13:30**

All problems carry equal marks. Solve any *three* problems.

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1. A point charge q is a distance d above (in the z direction) an infinite plane grounded conductor. Using the method of images, find

   (a) the electric field at any point $\vec{x}$ and plot it on a y-z plot. [2]

   (b) the surface-charge density induced on the plane. [2]

   (c) the force between the plane and the charge. [2]
   Hint: Use $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$.

   (d) the work necessary to remove the charge q from its position to infinity. [2]

   (e) the potential energy between the charge, q, and its image (compare the answer to part (d) and discuss). [2]

2. (1a) A cylindrical conductor of radius a has a uniform current density J_0 flowing through it. Find the magnitude and direction of the magnetic flux density inside the conductor. [2]

   (1b) A hole of radius b, $b < a$, is now bored parallel to, and centered a distance d from, the cylinder axis along the x direction ($d + b < a$). The current density is still uniform throughout the remaining metal of the cylinder and is parallel to the axis. Use the principle of linear superposition to find the magnitude and direction of the magnetic-flux density in the hole.
   Hint: It may be simpler to convert to rectilinear coordinates. [2]

   (2a) Consider a long, straight wire, parallel to the z axis, centred at $(x = y = 0)$ and carrying current I in the $+z$ direction. Find the direction and magnitude of the magnetic field. [2]

   (2b) Now, consider the field due to two parallel wires centred at $(x = \pm d/2, y = 0)$, distance d apart and carrying currents I in opposite directions and express the field in terms of a scalar potential $\vec{H} = -\nabla\Phi$. Solve for the potential in terms of cylindrical coordinates, (ρ, ϕ, z). [3]

   (2c) In the limit $d \ll \rho$, where $\rho = \sqrt{x^2 + y^2}$, show that the leading term in the potential is a dipole, [1]

$$\Phi \approx -\frac{I\,d\,\sin\phi}{2\pi\,\rho}\ .$$

3. A uniformly magnetised conducting sphere of radius R rotates about its magnetisation axis z with angular velocity ω. In the steady state no current flows in the conductor. Define the magnetic moment density (magnetisation) of the sphere in terms of the magnetic moment m:

$$\vec{m} = \int \vec{M} d^3x \ ,$$

along the z direction. The magnetic field inside the uniformly magnetised sphere is

$$\vec{B} = \frac{2\mu_0}{3} M\hat{z} \ .$$

Let the sphere rotate slowly with angular velocity ω so that the electric field that is generated does not vary with time (to a good approximation).

(a) Solve for the magnetic field H inside the sphere and use Ampere's law to show that $\nabla \times \vec{H} = 0$. [2]

(b) Use the non-relativistic transformation relating the electric field E' in the rotating frame to those in the lab frame:

$$\vec{E'} = \vec{E} + \vec{v} \times \vec{B} \ ,$$

to show that an electric field $\vec{E}$ is induced and calculate its magnitude and direction. [4]

(c) Use Gauss' law to find the volume charge density inside the conductor and show that it is uniform. [4]

4. (1a) Consider a long straight wire of electrical conductivity σ and cross-sectional radius a carrying a uniform current with axial density J. Calculate the magnitude and direction of the Poynting vector at the surface of the wire. [4]
Hint: Assume the conductor to be ohmic.

(2a) A localized electric charge distribution produces an electrostatic field, $\vec{E} = -\nabla\Phi$. Into this field is placed a small localized time-independent current density $J(x)$, which generates a magnetic field, $\vec{H}$. Show that the momentum of these electromagnetic fields can be transformed to [4]

$$P_{field} = \frac{1}{c^2} \int \Phi \, \vec{J} d^3x \ ,$$

provided $\Phi \, H$ falls rapidly enough at large distances.
(2b) Quantify this. [2]
Hint: $P_{field} = (1/c^2) \int (\vec{E} \times \vec{H}) d^3x$. Start by computing the z-component over a cube of side L and then let $L \to \infty$.

5. (a) Use Gauss' theorem to find the electric field inside and outside a charged conducting sphere of radius a. [2]

(b) What is the electric field if the sphere has a uniform charge density within its volume? [2]

(c) What if the spherically symmetric charge density varies radially as r^n, $n > -3$? [3]

(d) Sketch the behavior of the fields as a function of radius for the first two spheres, and for the third with $n = 2$. [3]

Comprehensive Examination

Instructions

July 10, 2017 09:30 - 13:30

• *Please use separate notebooks for Quantum Mechanics and Statistical Mechanics.*

• *In each notebook, at the beginning, please write your name and roll number clearly.*

• *You may use loose sheets, available in the exam hall, for rough work.*

• *All problems carry equal marks.* In each section, you have to do *any three* out of the five problems.

• Passing criterion: **minimum of 10/30** marks in each section *and* a **total of 27/60** in both sections together.

• Duration of examination for both the parts together: **09:30 hours to 13:30 hours**

Comprehensive Examination: Paper-II

Part C: Quantum Mechanics

All problems carry equal marks. Solve any *three* problems.

1. A spin 1/2 particle and its anti-particle are placed a fixed distance a apart. They have magnetic-moments $\vec{\mu}_1$ and $\vec{\mu}_2$ with interaction energy given by
$$V = \frac{1}{r^3}\left(\vec{\mu}_1 \cdot \vec{\mu}_2 - 3\,\frac{\vec{\mu}_1 \cdot \vec{r}\,\vec{\mu}_2 \cdot \vec{r}}{r^2}\right) ,$$
where $\vec{r}$ is the separation-vector between the pair, and $r \equiv |\vec{r}|$. Find the energy eigenvalues of eigenstates of the total-spin operator. **(10 marks)**

2. A 1-dimensional double-well potential of the form
$$V(x) = V_0 - \frac{\kappa}{2}x^2 + \frac{\lambda}{4}x^4 ,$$
has in it a quantum particle of mass m. Assume $V_0 \gg E_0$ where E_0 is the ground-state energy in one well, that the potential is approximately harmonic near the minimum, and that the potential energy at the minimum is zero. If the particle starts out in the right-well ground-state at time $t = 0$, find the probability as a function of time for the particle to be found in the left-well at a later time, and compute the time-taken for the particle to be found in the left-well with unit probability. **(10 marks)**

3. A hydrogen atom is in a region of constant electric field $\vec{E}_f$, which is a small perturbation. To leading non-zero order in the perturbation, compute the splitting of energy levels for (a) the $n = 1$ level (ground-state); (b) the $n = 2$ level. The radial integrals can be left in terms of the radial wave-function $R_{nl}(r)$ (i.e. no need to perform the radial integrals), but the selection rule for the angular quantum numbers is to be fixed and the angular integrals evaluated. Recall that
$$P_l^m(x) = (1 - x^2)^{m/2}\frac{d^m}{dx^m}P_l(x) \text{ ,and } P_0(x) = 1 \text{ , } P_1(x) = x .$$
(10 marks)

4. A hydrogen atom initially in its ground-state is in a spatially uniform time-dependent electric field $\vec{E}_f(t) = \vec{E}_f e^{-t/\tau}$ that is turned on at $t = 0$, and is a small perturbation. Find the probability of finding the atom in the $(n, l, m) = (2, 1, 0)$ excited-state after time t to first-order in the perturbation, for large $t \gg \tau$.
Note: The following radial wavefunctions $R_{nl}(r)$ may be useful $\mathrm{R}_{10}(r) = \left(\frac{1}{\pi a^3}\right)^{1/2} e^{-r/a}$; $R_{21}(r) = \frac{1}{\sqrt{4\pi}}\left(\frac{1}{2a}\right)^{3/2}\left(\frac{r}{a}\right) e^{-r/(2a)}$ $, where \mathrm{a} = \hbar^2/(\mu e^2)$ is the Bohr radius. **(10 marks)**

5. Two relativistic particle species a and b can be transformed into each other by interactions, with $H_{int}\psi_a\psi_b \equiv H_{ab}$. Recall that the free-particle Hamiltonian leads to the relativistic relation $E_i^2 = p^2 + m_i^2$ (in $c = 1$ units), with $i = \{1, 2\}$ denoting the energy eigenstates, the m_i being the masses. Assume $H_{ab} \ll (E_1 - E_2)$. If at time $t = 0$, a is produced with momentum $p = |\vec{p}|$, with $p \gg m_i$, find the probability that the particle is detected as b after time T in which it travels a distance L. **(10 marks)**

Comprehensive Examination: Paper-II

Part D: Statistical Mechanics

July 10, 2017 **09:30 - 13:30**

All problems carry equal marks. Solve any *three* problems.

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1. Consider a classical, relativistic ideal gas confined in a two-dimensional area ($A = L_x \times L_y$). Consider also that there is no external field applied to the system, so the gas has zero potential energy. For such a relativistic ideal gas system, the kinetic energy is given by $E = c|\vec{p}|$, where $\vec{p}$ is the particle momentum and c, the velocity of light. Calculate

   (a) Partition function Z

   (b) Free energy F

   (c) Entropy S

   (d) two-dimensional pressure of the gas P_2 **(10 marks)**

2. Consider a region (of volume $V = L^3$) within a fluid (with N particles) described by the van der Waals equation $\beta p = [\rho/(1 - bp)] - \beta a\rho^2$, where $\rho = \langle N \rangle / V$. Due to the spontaneous fluctutations in the system, the instantaneous value of the density can differ from its average by an amount $\delta\rho$. Use the following identities:

$$\frac{\langle(\delta\rho)^2\rangle^{1/2}}{\rho} = \frac{\langle(\delta N)^2\rangle^{1/2}}{\langle N\rangle},$$
$$(\delta N)^2 = \frac{\partial\langle N\rangle}{\partial(\beta\mu)},$$
$$\left(\frac{\partial\beta p}{\partial\rho}\right) = \rho\left(\frac{\partial\beta\mu}{\partial\rho}\right)$$

   (a) Evaluate $\frac{\langle(\delta\rho)^2\rangle^{1/2}}{\rho}$ as a function of β, a, b, ρ, L^3

   (b) Show that when the size of the region becomes macroscopic ($L^3 \to \infty$), the relative fluctuations become negligible.

   (c) A fluid is said to be at its "critical point" when

$$(\partial\beta p/\partial\rho)_\beta = (\partial^2\beta p/\partial\rho^2)_\beta = 0$$

   Determine the critical point density (ρ_c) and temperature (β_c) for the fluid obeying the van der Waal's equation. **(10 marks)**

3. Consider an isomerization process

   A¡=¿B ,

   where A and B refer to the different isomer states of a molecule and N_A and N_B denote the populations in each isomer state. Also consider that g_A and g_B are the degeneracies of states A and B respectively. The particles in each state are non-interacting and identical and single-particle partition function in states A and B are q_A and q_B respectively and the total partition function is represented by Q. Starting from the equilbrium condition of chemical potential $\mu_A = \mu_B$, show that

$\langle N_A \rangle / \langle N_B \rangle = (g_A/g_B)e^{-\Delta\epsilon}$, where $\Delta\epsilon = E_A - E_B$ **(10 marks)**

4. Consider a system of N distinguishable non-interacting spins in a magnetic field H. Each spin has a magnetic moment of size μ, and each can point either parallel or antiparallel to the field. The energy of a particular state is

$$\sum_{i=1}^{N} -n_i \mu H, \; n_i = \pm 1$$

where $n_i \mu$ is the magnetic moment in the direction of the field.

(a) Determine the partition function of the system Z

(b) Determine the internal energy of this system as a function of β, H, and N.

(c) Determine the entropy of this system from free energy ($F = \langle E \rangle - TS$) and its relation to partition function Z, as a function of β, H, and N

(d) Determine the behavior of the energy and entropy for this system as $T \to 0$. **(10 marks)**

5. The Gibbs entropy formula is given by

$$S = -k_B \sum_{\nu} P_\nu ln P_\nu$$

Consider a system contained in two boxes, A and B. Denote the total entropy of the system by S_{AB} and consider that the subsystems are uncoupled ($P_{AB}(\nu_A, \nu_B) = P_A(\nu_A)P_B(\nu_B)$) and that the individual probabilities are normalised for each system.

(a) Using the Gibbs entropy formula, show

$$S_{AB} = S_A + S_B$$

(b) Show that if one assumes a general functional formula for entropy as

$$S = \sum_{\nu} P_\nu f(P_\nu)$$

where $f(x)$ is some function of x, then the requirement that S is extensive implies that $f(x) = c \ln x$, where c is an arbitrary constant. **(10 marks)**