Comprehensive Examination - Paper I

Instructions

July 1, 2016 **09:30 - 13:30**

• *Please use separate notebooks for Classical Mechanics and Electrodynamics.*

• *In each notebook, at the beginning, please write your name and roll number clearly.*

• *You may use loose sheets, available in the exam hall, for rough work.*

• *All problems carry equal marks.* In each section, you have to do *any three* out of the five problems.

• Passing criterion: **minimum of 10/30** marks in each section *and* a **total of 27/60** in both sections together.

• Duration of examination for both the parts together: **09:00 hours to 13:00 hours**

Paper-I: Part A

Classical Mechanics

All problems carry equal marks. Solve any *three* problems.

1].

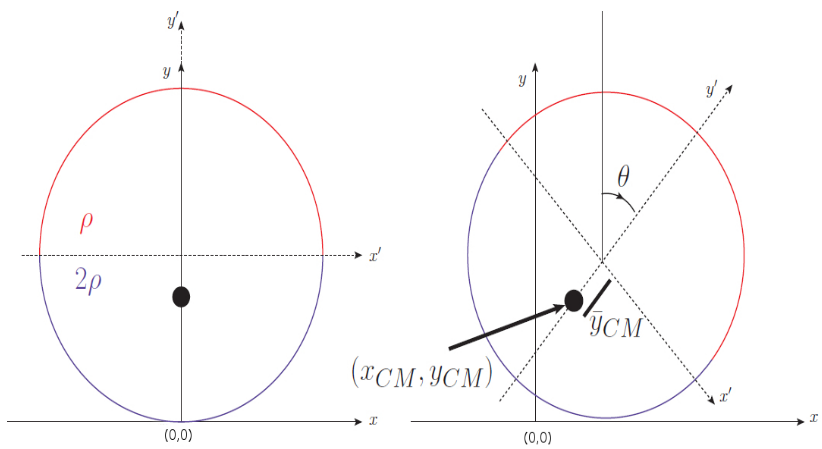


Figure 1: Figure for Q1A.

(A) Consider a thin disk composed of two homogeneous halves connected along a diameter of the disk. One half has density ρ and the other half density 2ρ. The disk rolls without slipping along a horizontal surface as shown in Fig 1 and the rotation takes place in the plane of the disk. Show that the Lagrangian is

$$L = \frac{1}{2}MR^2\dot{\theta}^2\left[\frac{3}{2} - \frac{8}{9\pi}\cos\theta\right] - MgR\left[1 - \frac{4}{9\pi}\cos\theta\right]$$

.

[full marks of (A) = 7]

(B) Write down the Equation of Motion (EoM) for the above Lagrangian.

[full marks of (B) = 3;
*****You will not get the any mark for (B) unless you do (A).]***

2].

(A) A particle moves along X-axis. It quadruples its momentum when its speed doubles. What was the initial speed of the particle in units of c?

[full marks of (A) = 2].

(B) An electron of kinetic energy 1 MeV is travelling along X-axis. It makes a head-on collision with a positron at rest. The collision of two particles annihilate each other and are replaced by two photons of equal energy travelling in the $X-Y$ plane at angles $+\theta$ and $-\theta$ with the X-axis. Determine the energy E, momentum $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$, and angle θ of each photon. Rest mass of electron (as well as of positron) is $m_0 = 0.511\ \mathrm{MeV\,c^{-2}}$.

[full marks of (B) = 4]. Write the expressions in units of c whenever applicable.

(C) In frame F' with coordinates (t', x', y', z'), a straight thin rod rotates in the $x' - y'$ plane with angular velocity ω' about its centre. The centre of the rod is fixed at the spatial origin of F'. At time $t' = 0$, the rod lies along the x'-axis. F is another frame with coordinates (t, x, y, z). There is *no* relative rotation between the two frames. F' moves with a constant speed $v\hat{x}$ with respect to F. Find an equation which gives the shape of the rod in the frame F at $t = 0$. Does it represent the equation of a straight rod?

[full marks of (C) = 3+1].

3].

(A) Two particles (both of mass m) move in one dimension at the junction of three springs, as shown in the figure. The springs all have unstretched lengths equal to a and the spring constants k, $3k$, and k as shown in the figure. Find the frequencies of normal modes.

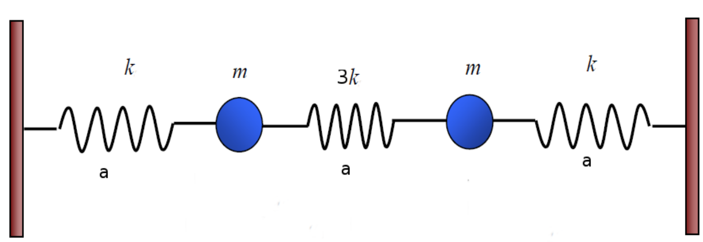


Figure 2: Figure for Q3A.

[full marks of (A) = 4].

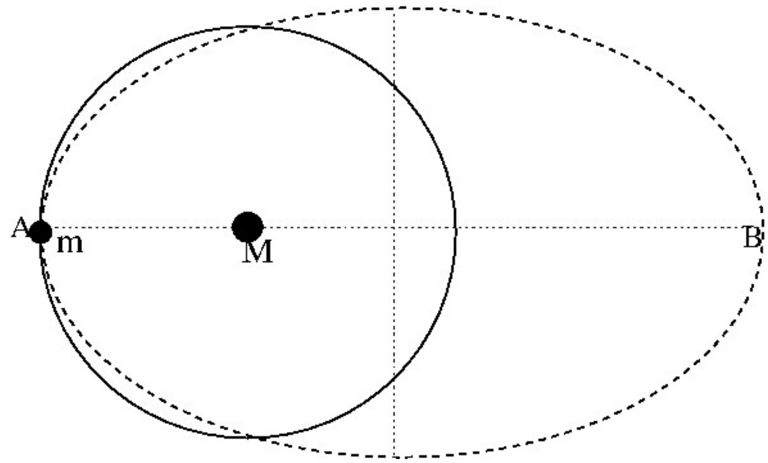


Figure 3: Figure for Q3B.

(B) There are two gravitationally bound objects of masses M and m, and $M >> m$ so that the reduced mass of the system is m and one can consider the object of mass m is orbiting around the object of mass M. For the sake of simplicity, let us consider that the orbit of m is circular with a radius of r_E and the orbital velocity is v_E. When m is at the position A (on its circular orbit) as shown in the figure, an instantaneous external cause changed its orbit into an elliptical one with turning points as A and B. If the velocity of m at A and B are v_A and v_B respectively, prove that (i) $v_B = \beta v_A$ and (ii) $v_A^2 = \frac{2v_E^2}{(1+\beta)}$ where $\beta = r_E/r_B$, r_A is the distance of the point A from M and r_B is the distance of the point B from M.

[full marks of (B) = 4].

(C) A particle of mass m and angular momentum l moves in a central force $f(r)\hat{r}$. Find the expression of $f(r)$ which gives the shape of the orbit as $r(\theta) = \frac{2b}{\theta^2}$.

[full marks of (C) = 2].

4].

(A) The Lagrangian of a simple pendulum within small angle approximation is known as:

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}mgl\theta^2$$

where the symbols bear their conventional meanings. Using Hamilton's equations of motion, show that the motion of the pendulum in the phase-space is an ellipse.

[full marks of (A) = 2]

(B) Show that the transformation:

$$Q = \ln(q^{-1}\sin p)$$
$$P = q\cot p$$

satisfies the symplectic condition for canonical transformation.

[full marks of (B) = 3]

(C) Show that

$$F_1(q,Q) = q\,\sin^{-1}(q\,e^Q) + e^{-Q}\,\left(1 - q^2 e^{2Q}\right)^{1/2}$$

is a generating function of the canonical transformation given in (B).

[full marks of (C) = 3]

[full marks of (C) = 3]

(D) Find another generating function $F_2(q,P)$ for the transformation given in (B) and F_1 given in (C).

[full marks of (D) = 2]

[full marks of (D) = 2]

5].

Let us consider the following Lagrangian,

$$\tilde{L} \;=\; L - aL' \;,$$

where a is an arbitrary real constant,

$$L \;=\; \dot{x}^2 - x^2 \;,$$

is the harmonic oscillator Lagrangian and,

$$L' \;=\; (3x\cos t)L - (\dot{x}\sin t)(\dot{x}^2 + 3x^2) \;.$$

(A) Show that the equation of motion (EoM) for $\tilde{L}$ is given by:

$$\{3aQ(x,\dot{x},t) - 1\}\,(\ddot{x} + x) \;=\; 0 \;, \qquad (1)$$

where,

$$Q(x,\dot{x},t) \;=\; x\cos t - \dot{x}\sin t \;.$$

[full marks of (A) = 5].

(B) Argue that the set of solutions of Eqn. (1) is precisely the same as that of harmonic oscillator.

[full marks of (B) = 2].

(C) Show that $Q(x, \dot{x}, t)$ is a constant of motion for the harmonic oscillator. Find the general solution for which,

$$Q(x, \dot{x}, t) \quad = \quad A \ , \quad \text{(constant)} \ .$$

[full marks of (C) = 3].

***Electrodynamics question paper starts from the next page.*

Paper-I: Part B

Electrodynamics

All problems carry equal marks. Solve any *three* problems.

1.
   - A particle of charge Q is moved from infinity to the center of a hollow conducting spherical shell of radius R and thickness t, through a very tiny hole in the shell. How much work is required?

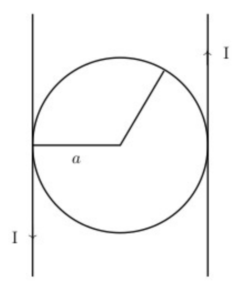
     [5]

   - The electric field inside a dielectric sphere, placed in between a large parallel-plate capacitor is uniform, denoted by $\vec{E}_0$. Let the radius of the sphere be R and relative dielectric constant $K = \epsilon/\epsilon_0$. Find $\vec{E}$ at a point on the outer surface of the sphere. Determine the bound surface charge density at that point.

     [2+3]

2.
   - In the following figure two parallel conductors of infinite length, separated by a distance $2a$ carry a current I. An insulated circular conducting ring of radius a is placed in between them. Find the coefficient of mutual inductance between the circular conductor and the two straight conductors.

     [5]



   - Consider a cylinder of radius R and length L carrying a uniform current I along its axis.
     (a) Find the direction and magnitude of the magnetic field inside the cylinder.
     (b) A beam of positively charged particle each with charge q and momentum p parallel to the axis of the cylinder, enters on its end along the direction of the current. Find out the magnetic force acting on a single particle. Show that after passing through the cylinder, the particles will focus to a point (Neglect the slowing down and scatterings).

     [2+3]

3. A conducting spherical shell of radius R is cut into two halves. The two pieces are electrically separated from each other with an infinitesimally small distance. Let $\phi = \phi_0$ and $\phi = 0$ are the potentials for the upper and lower halves, respectively. Compute the electrostatic potential ϕ in space outside the surface of the conductors (Neglect terms falling faster than $\frac{1}{r^4}$ where r is the distance from the center).

   Use
$$\int_0^1 P_n(x)dx = C_n \qquad C_0 = 1, C_1 = \frac{1}{2}, C_2 = 0, C_3 = -\frac{1}{8}$$

[10]

4. A long coaxial cable consists of a solid inner cylindrical conductor of radius R_1 and a thin outer cylindrical conducting shell of radius R_2. At one end the two conductors are connected together by a resistor of resistance R and at the other end they are connected to a battery.

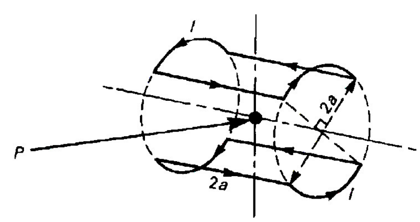
   (a) Find the magnetic field $\vec{B}$ and the electric field $\vec{E}$ between the conductors.

   (b) Find the magnetic and electric energy per unit length in the region between the conductors.

   (c) Assuming that the magnetic energy in the inner conductor is negligible, find the inductance and the capacitance per unit length.

[4+4+2]

5. Find an expression for the magnitude and direction of the magnetic flux density at the point P for the arrangement of a set of saddle coils as shown in the following figure .



[5+5]

Comprehensive Examination - Paper II

Instructions

July 4, 2016 **09:00 - 13:00**

• *Please use separate notebooks for Quantum Mechanics and Statistical Mechanics.*

• *In each notebook, at the beginning, please write your name and roll number clearly.*

• *You may use loose sheets, available in the exam hall, for rough work.*

• *All problems carry equal marks.* In each section, you have to do *any three* out of the five problems.

• Passing criterion: **minimum of 10/30** marks in each section *and* a **total of 27/60** in both sections together.

• Duration of examination for both the parts together: **09:00 hours to 13:00 hours**

Paper-II: Part A

Quantum Mechanics

All problems carry equal marks. Solve any *three* problems.

(1): The wave function for a particle at time $t = 0$ for three different cases is given below. Discuss the change in the wave function (spreading of the wave packet) for each case.

(A) free motion

$$\psi(\mathbf{r}, 0) = \frac{1}{(\pi\delta^2)^{3/4}} \exp\left\{\frac{i\mathbf{p_0 r}}{\hbar} - \frac{r^2}{2\delta^2}\right\};$$

[4 marks]

(B) motion in a homogeneous field

$$\psi(\mathbf{r}, 0) = \frac{1}{(\pi\delta^2)^{3/4}} \exp\left\{\frac{i\mathbf{p_0 r}}{\hbar} - \frac{r^2}{2\delta^2}\right\};$$

[3 marks]

(C) motion of a particle in the potential field $V = \frac{1}{2}\mu\omega^2 x^2$

$$\psi(x, 0) = cexp\left\{\frac{ip_0 x}{\hbar} - \frac{\alpha^2(x - x_0)^2}{2}\right\}, \alpha = \left(\frac{\mu\omega}{\hbar}\right)^{1/2}$$

[3 marks]

(2): Two identical plane rotators with coordinates θ_1, θ_2 are coupled according to the Hamiltonian

$$H = A(p_{\theta_1}^2 + p_{\theta_2}^2) - B\cos(\theta_1 - \theta_2),$$

where A and B are positive constants. (Note: $\theta_i + 2\pi$ is equivalent to θ_i.) From Schrodinger equation determine the energy eigenvalues and eigen-functions when the following conditions hold:

(A) In the case $B \ll A\hbar^2$, discussing only terms linear in B. Watch out for degeneracies.

[5 marks]

(B) In the case $B \gg A\hbar^2$, by reducing the problem to an oscillator problem (small oscillations).

[5 marks]

(3): Consider two identical linear oscillators with spring constant k. The interaction potential is given by $H' = cx_1x_2$ where x_1 and x_2 are the oscillator variables.

(A) Find the exact energy levels

[5 marks]

(B) Assume $c \ll k$ and compute the lowest pair of excited states in first-order perturbation theory. (Give energy levels in first order and eigenfunctions in zeroth order.)

[5 marks]

(4): Consider an electron in a uniform magnetic field in the positive z-direction. The result of a measurement has shown that the electron spin is along the positive x-direction at $t = 0$. For $t > 0$ compute quantum-mechanically the probability for finding the electron in the state

(A) $S_x = \frac{1}{2}$ **[4 marks]**

(B) $S_x = -\frac{1}{2}$ **[3 marks]**

(C) $S_z = \frac{1}{2}$. **[3 marks]**

(5): A nucleus of spin I and quadrupole moment Q_0 is in a magnetic field of intensity $\mathcal{H}$ and in a nonhomogeneous electric field. Considering the operator of the quadrupole energy as a small perturbation, find the energy levels of the nucleus in the external field in the second order approximation.

[10 marks]

***Statistical Mechanics question paper starts from the next page.*

Paper-II: Part B

Statistical Mechanics

All problems carry equal marks. Solve any *three* problems.

(1): A gas is initially confined to one half of a thermally isolated container. The other half is initially empty. The gas is suddenly permitted to expand into the the entire container. If the initial temperature of the gas is T_i, what is the final temperature for the following two cases:

(A) Equation of state: $PV = nRT$ **[5 marks]**

(B) Equation of state: $b(P + \frac{a}{V^2}) = nRT$. **[5 marks]**

(2): Consider a system of Heisenberg-like classical spins (vectors with 3 components) that are located on the sites of a 2D square lattice.

(A) Assume that the lattice constant is so large that the spins do not interact with each other. Calculate the magnetisation in the presence of an applied field.

[4 marks]

(B) The magnetic susceptibility is defined by

$$\mathbf{M} = \chi \mathbf{B}_{ext}. \tag{1}$$

In the limit of high temperatures, show that the magnetic susceptibility, obeys $\chi \sim 1/T$. (Curie law). Useful identity: for small x, $\{\coth(x) - 1/x\} \sim x/3$.

[2 marks]

(C) Suppose each spin interacts with its four nearest neighbours on the square lattice via a ferromagnetic coupling J. The effect of neighbouring spins can be thought of as an additional magnetic field ('Weiss' field) – what is the effective magnetic field acting on each site?

[2 marks]

(D) Assume that we can replace the magnetization terms in the Weiss field with their average values. Rewrite Eq. 1 replacing the external magnetic field with the effective magnetic field. Show that this leads to $C \sim 1/(T - \Theta)$ in the limit of high temperature. (Curie-Weiss law)

[2 marks]

(3): Let $Z_1(m, V)$ denote the partition function of a single free particle of mass m in a volume V.

(A) Write an expression for $Z_1(m, V)$.

[2 marks]

(B) Consider two such non-interacting particles which are spinless fermions. Write an arbitrary state for this two particle system.

[2 marks]

(C) Write the partition function for the two particle problem $Z_2(m, V))$. Express the final answer in terms of $Z_1(m', V')$.

[5 marks]

(D) What would $Z_2(m, V))$ be if the two particles were distinguishable?

[1 mark]

(4): Consider a lattice gas model with a lattice of N sites. Each site may be empty – in which case its energy is zero, or occupied by a single particle – in which case its energy is ϵ. In addition, each particle has a magnetic moment of magnitude μ which can orient either along or opposite to an applied magnetic field $\mathbf{B}$. The strength of the coupling to the magnetic field, g, is known.

(A) Write an expression for the energy of a given configuration in terms of local variables n_i and s_i, denoting the occupation number and the moment orientation.

[1 mark]

(B) Working in the grand canonical ensemble, evaluate the partition function of the system.

[4 marks]

(C) Derive expressions for the average energy and the average magnetisation at any given temperature, T.

[5 marks]

(5): Consider a gas of non-interacting relativistic bosons in a 3 dimensional box of volume V. The energy of a particle is given by $\epsilon = c|\vec{p}|$.

(A) Taking the system to be a fixed temperature T and coupled to a particle reservoir with chemical potential μ, write an expression for the grand potential ($\mathcal{G} = -k_B T \ln Z$). Express your answer in terms of

$$f_m^+(z) = \frac{1}{(m-1)!} \int_0^\infty \frac{x^{m-1}}{z^{-1}e^x - 1} dx, \tag{2}$$

where $z = \exp(\mu/k_B T)$.

[6 marks]

(B) What is the average number of particles? *Hint*: $z\partial f_{d+1}^+/\partial z = f_d(z)$.

[2 marks]

(C) Write an expression for the critical temperature at which this system will undergo Bose condensation.

[2 marks]