Comprehensive Examination

Instructions

July 3, 2015 **09:00 - 13:00**

• *Please use separate notebooks for Classical Mechanics and Electromagnetism.*

• *In each notebook, at the beginning, please write your name and roll number clearly.*

• *You may use loose sheets, available in the exam hall, for rough work.*

• *All problems carry equal marks.* In each secion, you have to do *any three* out of the five problems.

• Passing criterion: **minimum of 10/30** marks in each section *and* a **total of 27/60** in both sections together.

• Duration of examintaion for both the parts together: **09:00 hours to 13:00 hours**

Comprehensive Examination: Paper-I

Classical Mechanics

July 3, 2015 **09:00 - 13:00**

All problems carry equal marks. Solve any *three* problems.

1(a): Obtain the equation of motion for a particle falling vertically under the influence of gravity in the presence of frictional forces obtainable from a dissipative force $(1/2)kv^2$. **[2 marks]**

1(b): Integrate the equation to obtain the velocity as a function of time and show that the maximum possible (terminal) velocity for a particle falling from rest is $v = mg/k$. **[4 marks]**

1(c): A nucleus, originally at rest, beta-decays radioactively by emitting an electron of momentum 1.732 MeV/c and a neutrino with momentum 1.00 MeV/c at right angles to the direction of the electron. In what direction does the nucleus recoil? What is its momentum in MeV/c? If the mass of the residual nucleus is 3.90×10^{-25} kg, what is its kinetic energy in MeV? (Use 1 MeV $= 1.6 \times 10^{-13}$ J.) Are you justified in using non-relativistic approximation for the kinetic energy? **[4 marks]**

2(a): A heavy particle is placed at rest at the top of a vertical hoop. Write down the lagrangian for the particle in terms of the angle θ subtended at the centre of the hoop as it starts to roll down the hoop. What is the equation of constraint?

[2 marks]

2(b): Write down the equations of motion using the method of lagrange multipliers.

[2 marks]

2(c): Find the value of θ when the particle "falls off" the hoop and hence the height at which the particle rolls off the hoop. [Hint: Use $\ddot{\theta} = \dot{\theta}(d\dot{\theta}/d\theta)$.] **[6 marks]**

3(a): Show that the following transformation is canonical for α a fixed parameter:

$$x = \frac{1}{\alpha}\left(\sqrt{2P_1}\sin Q_1 + P_2\right) \ ; \qquad p_x = \frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1 - Q_2\right)$$
$$y = \frac{1}{\alpha}\left(\sqrt{2P_1}\cos Q_1 + Q_2\right) \ ; \qquad p_y = -\frac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right) \ .$$

[4 marks]

3(b): Apply this transformation to the problem of a charged particle with charge q moving in a plane that is perpendicular to a constant magnetic field (assumed in the z direction). Express the hamiltonian for this system in the (Q_i, P_i) coordinates and setting $\alpha^2 = qB/c$. **[3 marks]**

3(c): From this hamiltonian, obtain the motion of the particle as a function of time. Interpret your results. **[3 marks]**

4(a): Two particles move about each other in circular orbits under the influence of mutual gravitational force, with a period τ . At some time $t = 0$, they are suddenly stopped and then they are released and allowed to fall into each other. Write the expression for the total energy of the system. **[5 marks]**

4(b): Find the time T after which they collide. **[5 marks]**

5(a): A rocket of length l_0 in its rest frame is moving with constant speed along the z axis of an inertial system. An observer standing at the origin of this system observes the apparent length of the rocket at any time by noting the z coordinates of the head and tail of the rocket. How does this apparent length vary as the rocket moves past the observer from the extreme left to the extreme right? **[8 marks]**

5(b): How do these results compare with measurements in the rest frame of the observer? (Note: observe, not measure). **[2 marks]**

Comprehensive Examination: Paper-I

Electromagnetism

July 3, 2015 **09:00 - 13:00**

All problems carry equal marks. Solve any *three* problems.

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(1): A charged particle of mass m and charge q is placed at rest at the origin in a magnetic field B that points towards the $\hat{x}$ direction and an electric field E that points along the $\hat{z}$ direction. Write down the equations of motion of the particle and solve for its trajectory. After solving the trajectory draw the path of the particle as a function of time. How does the trajectory change if the particle had an initial velocity $\vec{v}_0$ along $\hat{z}$. **[10 marks]**

(2): Prove that if two charged conducting concentric shells are connected by a wire, the inner one is wholly discharged. If the force law were $r^{-(2+\epsilon)}$, prove that there would be (approximately) a charge q, given by

$$2(R-r)q = -Q\epsilon\Big[2r\ln R - (R+r)\ln(R+r) + (R-r)\ln(R-r)\Big]$$

on the innershell, where Q is the charge on the outer shell and r and R were the radii of the inner and outer shells. **[10 marks]**

(3): The time averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}(1+\frac{\alpha r}{2})$$

where q is the magnitude of the electron charge and $\alpha^{-1} = a_0/2$ is the Bohr radius.Find the distribution of charge both continuous and discrete that will give the potential and interpret your result physically. **[10 marks]**

(4): Write Maxwell's equation generalised with magnetic charge in terms of $\vec{\mathbf{F}} = \vec{\mathbf{E}} + i\vec{\mathbf{B}}$. Show that these equations retain their form under the transformation $\vec{\mathbf{F}} \to e^{-i\phi}\vec{\mathbf{F}}$, where ϕ is an arbitrary constant. Can you give a geometric interpretation?

Defining $\vec{\mathbf{F}}^* = \vec{\mathbf{E}} - i\vec{\mathbf{B}}$, identify the following:

Scalar $\frac{1}{8\pi}\vec{\mathbf{F}}^* \cdot \vec{\mathbf{F}}$

Vector $\frac{1}{8\pi i}\vec{\mathbf{F}}^* \times \vec{\mathbf{F}}$

Dyadic $\frac{1}{8\pi}(\vec{\mathbf{F}}\vec{\mathbf{F}}^* + \vec{\mathbf{F}}^*\vec{\mathbf{F}})$

in terms of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$. What happens to these quantities under the transformation $\vec{\mathbf{F}} \to e^{-i\phi}\vec{\mathbf{F}}$, ϕ being a constant. **[10 marks]**

(5): In a linear crystal the relation between $\vec{D}$ and $\vec{E}$ is of the form

$$D_i = \sum_{k=1}^{3} \varepsilon_{ik} E_k$$

where ε_{ik} is a symmetric tensor.

(a) Give Maxwell's equation in a non-conducting crystal and show that there exist solutions of the form

$$\left.\begin{array}{c} \vec{E} \\ \vec{B} \end{array}\right\} \propto \exp\left[i(\vec{k}\cdot\vec{r} - \omega t)\right]$$

for any direction of $\vec{k}$. Which of the vectors $\vec{E}$, $\vec{D}$, $\vec{H}$, $\vec{B}$ are perpendicular to $\vec{k}$?

(b) Find an expression for the Poynting vector $\vec{S}$ in the crystal. Is $\vec{S}$ in general parallel to $\vec{k}$? If not for how many directions is $\vec{S}$ parallel to $\vec{k}$?

[10 marks]

Comprehensive Examination

Instructions

July 6, 2015 **09:00 - 13:00**

• *Please use separate notebooks for Quantum Mechanics and Statistical Mechanics.*

• *In each notebook, at the beginning, please write your name and roll number clearly.*

• *You may use loose sheets, available in the exam hall, for rough work.*

• *All problems carry equal marks.* In each secion, you have to do *any three* out of the five problems.

• Passing criterion: **minimum of 10/30** marks in each section *and* a **total of 27/60** in both sections together.

• Duration of examintaion for both the parts together: **09:00 hours to 13:00 hours**

Comprehensive Examination: Paper-II

Quantum Mechanics

July 6, 2015 **09:00 - 13:00**

All problems carry equal marks. Solve any *three* problems.

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(1): For convenience we reproduce the following general formula for combining angular momentum l with angular momentum $1/2$. The sum angular momentum is denoted by J and its z component by M:

$$|l{+}1/2, M\rangle = \frac{1}{\sqrt{2l+1}}[\sqrt{l+1/2+M}\,|l,1/2;M{-}1/2,+\rangle + \sqrt{l+1/2-M}\,|l,1/2;M{+}1/2,-\rangle]$$

$$|l{-}1/2, M\rangle = \frac{1}{\sqrt{2l+1}}[\sqrt{l+1/2+M}\,|l,1/2;M{+}1/2,-\rangle - \sqrt{l+1/2-M}\,|l,1/2;M{-}1/2,+\rangle]$$

Consider a particle (a) of spin 3/2 decaying into two particles - one has spin 1/2 and one has spin 0. Total angular momentum is conserved in this process. We place ourself in the rest frame of (a).

(1a): What values can be taken by the relative orbital angular momentum, l, of the two final particles? If the parity of the orbital momentum is known, can we fix l? Would this continue to be true for other values of spin of the decaying particle?

[5 marks]

(1b): Suppose the initial particle has some definite value of z component of spin, say $m_a\hbar$. Suppose we know that the final orbital state has a definite parity. Can we determine this parity by measuring the probabilities of the spin 1/2 particle having its z component of spin equal to +1/2 or -/12. **[5 marks]**

(2): Consider a (non relativistic) particle of mass m interacting with a potential $V(x) = -\alpha\delta(x)$. This has a single (normalizable) bound state $\phi_0(x) = \sqrt{\frac{m\alpha}{\hbar^2}}e^{\frac{-m\alpha|x|}{2\hbar^2}}$. There are also stationary non normalizable (more precisely delta function normalizable) states corresponding to a plane wave coming in from the left and scattering and similarly from the right.

$$\begin{aligned}\chi_k(x) &= \frac{1}{\sqrt{2\pi}}[e^{ikx} - \frac{1}{1+\frac{i\hbar^2 k}{m\alpha}}e^{-ikx}] \quad x<0 \\ &= \frac{1}{\sqrt{2\pi}}\frac{\frac{i\hbar^2 k}{m\alpha}}{1+\frac{i\hbar^2 k}{m\alpha}}e^{ikx} \quad x>0\end{aligned}$$

We would like to calculate the transition probability from the bound state to the scattering state due to a sinusoid perturbation

$$W(t) = -qEX sin\ \omega t$$

We need the ingredients for using the Fermi Golden Rule:

(2a): Calculate the matrix element of this operator, i.e. X between the initial and final states. **[3 marks]**

(2b): Calculate the density of states $\rho(E)$ **[3 marks]**

(2c): Calculate the transition probability. How does it depend on ω? **[4 marks]**

(2d) [Extra credit]: Show that $\langle \chi_k^* | \chi_{k'} \rangle = \delta(k - k')$. This is required for applicability of the Golden rule. Use

$$\int_{-\infty}^{0} e^{iqx} = \int_{0}^{\infty} e^{-iqx} dx = \lim_{\epsilon \to 0} \frac{1}{\epsilon + iq} = \pi\delta(q) - iP(\frac{1}{q})$$

[2 marks]

(Requires a careful calculation. No credit unless all the details are given. So don't waste time on this unless you have plenty of time!)

(3): Note: In this problem remember that the states you write down should have the right symmetry for fermions.

Let $|\phi\rangle$ and $|\chi\rangle$ be two normalized orthogonal states belonging to the orbital states of an electron and let $|+\rangle$ and $|-\rangle$ be the spin state space. Consider a system of two electrons, one in the state $|\phi, +\rangle$ and the other in the state $|\chi, -\rangle$.

(3a): Write down the appropriately symmetrized ket $|\psi\rangle$ that represents this state.

[2 marks]

Let $\rho_{II}(r, r')d^3rd^3r'$ be the probability of finding one electron (in a volume d^3r) at r and the other at r'. This is the two particle density function. The one particle probability similarly is $\rho_I(r)d^3r$.

(3b): What is the physical state describing the outcome of the above measurement? Calculate its overlap with $|\psi\rangle$ and thus determine ρ_{II}. **[2 marks]**

Now consider the case where both electrons are in the spin + state i.e. one is in $|\phi, +\rangle$ and the other is in $|\chi, +\rangle$.

(3c): Repeat (3a) for this situation. **[1 marks]**

(3d): Repeat (3b) for this situation. **[2 marks]**

(3e): What happens in case (3d) if the states ϕ, χ **[2 marks]**

(3e(i)): are not orthogonal?

(3e(ii)): are identical?

(4): Consider a Hamiltonian for a spin 1/2 in a magnetic field: $H = -\vec{M}.\vec{B}$ with $\vec{M} = \gamma\vec{S}$. We define $\omega_i = -\gamma B_i$, $i = 1, 2, 3$ and also $\omega_0 = -\gamma|\vec{B}|$.

(4a): Write an expression for the evolution operator $U(t, 0) = e^{-iHt}$. **[2 marks]**

(4b): Show that it can be written as $U(t, 0) = cos(\omega t/2) - Msin(\omega t/2)$. What is M? **[3 marks]**

(4c): Let the state at time $t = 0$ be $|\psi(0)\rangle = |+\rangle$. Find the probabilities: i) $P_{++}(t)$ (probability of the state being $|+\rangle$ at time t. and ii) $P_{-+}(t)$ **[3 marks]**

(4d): If $B_z = 0 = B_y$ show (by calculating probabilities, or by invoking the rotation group or...) that the spin precesses around the x-axis. How much time does it take for one rotation? **[2 marks]**

(5): Given a harmonic oscillator with

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2}X^2$$

in thermal equilibrium at temperature T.

(5a): We would like to rescale $P = \alpha\bar{P}$ and X to $\beta\bar{X}$, (maintaining the commutation relation when these are quantum operators), so that the Hamiltonian takes the form

$$H = C\frac{(\bar{P}^2 + \bar{X}^2)}{2}$$

Do the rescaling. What are C, α, β? **[1 mark]**

(5b): Write down the partition function and thereby $\langle H\rangle$ for this harmonic oscillator.

[4 marks]

Let $\bar{x}$ stand for eigenvalues of $\bar{X}$ and $\bar{p}$ stand for eigenvalues of $\bar{P}$.

(5c): Write the Schroedinger equation in the $\bar{p}$ representation and find a relation between the nth state in the $\bar{x}$ representation and the $\bar{p}$ representation. **[2 marks]**

(5d): Find $\langle\bar{X}^2\rangle$ and $\langle\bar{P}^2\rangle$ as a function of temperature. Thereby obtain $\langle X^2\rangle$ and $\langle P^2\rangle$. **[3 marks]**

Comprehensive Examination: Paper-II

Statistical Mechanics

July 6, 2015 **09:00 - 13:00**

All problems carry equal marks. Solve any *three* problems.

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1. Consider the quantum mechanical states of N non-interacting distinguishable particles in a box of size $[0, L]^3$.

   (a) Determine the single particle energy levels? [denote the quantum numbers by n_x, n_y, and n_z where $n_i = 1, 2, \ldots$]. [1]

   (b) List the number of microstates Ω when $\sum_{i=x,y,x} \sum_{j=1}^{3} (n_i^j)^2 = \tilde{E}$, for each value of $\tilde{E} \leq 20$. [3]

   (c) Let $\Omega(E, V, N)$ denote the number of states corresponding to energy E when N particles are in volume V. Let $\Sigma(E, V, N) = \sum'_E \Omega(E', V, N)$. Determine $\Sigma(E, V, N)$ for large E, L, N. Volume of a d-dimensional sphere of radius R is

   $$V_d(R) = \frac{2\pi^{d/2} R^d}{d\Gamma(d/2)}$$

   [3]

   (d) Determine entropy $S(E, V, N)$. [You may use $\ln \Gamma(n) \approx n \ln n - n$ for $n \gg 1$.] [1]

   (e) Determine the dependence of energy E on temperature T. [2]

2. Consider a random walk with memory in one dimension. Let the displacements be $x_1, x_2, \ldots, x_t$, where $x_i = \pm 1$. Consider $p > 0$, and

   $$\begin{aligned} P(+1|+1) &= \tfrac{1}{2} + p, \\ P(+1|-1) &= \tfrac{1}{2} - p, \\ P(-1|+1) &= \tfrac{1}{2} - p, \\ P(-1|-1) &= \tfrac{1}{2} + p. \end{aligned}$$

   (a) If $P(x_{i+1}|x_i) = A + Bx_i + Cx_{i+1} + Dx_i x_{i+1}$, determine A, B, C, D. [1]

   (b) Determine $P(x_{i+2}|x_i)$ [Hint: $P(x_{i+2}|x_i) = \sum_{x_{i+1}} P(x_{i+2}|x_{i+1}) P(x_{i+i}|x_{i+1})$] [2]

   (c) Generalizing above, determine $P(x_j|x_i)$, $j > i$. [3]

(d) Show that $\langle x_i x_j \rangle \propto \exp(-|i-j|/\xi)$, and obtain an expression for ξ. [2.5]
(b) Consider $x = x_1 + x_2 + \ldots + x_t$. Obtain $\langle x^2 \rangle$ when $t \gg 1$. [1.5]

3. Consider a system with the Landau free energy functional

$$\mathcal{L}[m] = \frac{b}{2}m^2 + \frac{u}{4}m^4 + \frac{d}{6}m^6,$$

where $d > 0$ for stability. b and u can take both positive and negative values.

(a) Calculate the phase diagram in the b–u plane (b along the y-axis) [5]

(b) What is the order of transition across the phase boundaries? [1]

(c) A first order line meets a second order line at a tricritical point. Give the coordinates of the tricritical point. [1]

(d) Calculate the critical exponents β, γ, and δ at the tricritical point. [3]

4. Let $Z_1(m)$ denote the partition function for a single quantum mechanical particle of mass m in a volume V.

   (a) Calculate the partition function of two such identical non-interacting particles, if they are bosons, and also if they are spinless fermions and also classical particles. You may use $E = \hbar^2 k^2/(2m)$. [5]

   (b) Use the classical approximation $Z_1(m) = V/\lambda^3$ with $\lambda = h/\sqrt{2\pi mkT}$. Calculate the corrections to the energy E and the heat capacity C due to Bose or Fermi statistics, treating the corrections to the classical answer as small. [4]

   (c) At what temperature does the approximation used above break down? [1]

5. Consider the one-dimensional Ising model consisting of N spins S_i, $i = 1, \ldots, N$, where each S_i may take values ± 1. The energy of a configuration is

$$E = -J \sum_{i=1}^{N'} S_i S_{i+1},$$

where $J > 0$, and $N' = N - 1$ for open boundary conditions and $N' = N$ for periodic boundary conditions with $S_{N+1} = S_1$.

   (a) Let $t = \tanh(K)$, where $K = \beta J$. By expanding the exponential, express $e^{K S_i S_{i+1}}$ in terms of $\cosh K$, t and $S_i S_{i+1}$. [3]

   (b) Using (a), determine the partition function for the Ising model with periodic boundary conditions [3]

   (c) Using (a), determine the partition function for the Ising model with open boundary conditions [3]

   (d) Are the results of (b) and (c) the same? If not, does the difference matter for measurable bulk quantities? [1]