Comprehensive Examination

July 4, 2014 | **Instructions** | **09:30 - 12:30**

- *Please use separate notebooks for Classical Mechanics and Electromagnetism.*
- *In each notebook, at the beginning, please write your name and roll number clearly.*
- *You may use loose sheets, available in the exam hall, for rough work.*
- *All questions carry equal marks.* In each section, you have to do *any three* out of the four problems.

- Passing criterion: **minimum of 11/30** marks in each section *and* a **total of 30/60** in both sections together.

Comprehensive Examination: Paper-I

July 4, 2014 **Classical Mechanics** 09:30 - 12:30

All problems carry equal marks. Solve any *three* problems.

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1. Write down the Lagrangian, and Lagrange's equations of motion, for a disk of radius r and uniform mass density ρ, pivoted at one point on its circumference and free to swing in two dimensions (in its own plane) under gravity (acceleration due to gravity = g) with no other forces or constraints, in coordinates of your choice. What are the conserved quantities?

2. Write down a first-order differential equation for x such that, depending on initial conditions (value of x at $t = 0$), $x(t)$ tends to either 0 or 1. Specifically, if $x(0) < 0.5$ then $x(t) \to 0$ as $t \to \infty$, and if $x(0) > 0.5$ then $x(t) \to 1$ as $t \to \infty$. What are the fixed points of this system, and which points are stable and which are unstable?

3. Consider a Hamiltonian that can be written

$$H = H_0 + \epsilon H_1$$

where H_0 is a function only of the momenta (and hence exactly solvable). Consider the Lie perturbation treatment of this Hamiltonian,

$$H' = e^{\epsilon L_W} H$$

where $L_W A \equiv \{A, W\}$, and W is chosen to satisfy

$$L_W H_0 + H_1 \equiv \{H_0, W\} + H_1 = 0.$$

Show that

$$H' = H_0 + \sum_{n \geq 2} \frac{\epsilon^n}{n!}(n-1) L_W^{n-1} H_1$$

4. Write down

   (a) four linearly-independent timelike 4-vectors;

   (b) four linearly-independent spacelike 4-vectors;

   (c) four linearly-independent null 4-vectors.

   Choose $c = 1$, and use the zero component for time, i.e. (t, x, y, z). Are linear combinations of these also timelike, spacelike and null, respectively?

Comprehensive Examination: Paper-I

July 4, 2014 **Electromagnetism** 09:30 - 12:30

All problems carry equal marks. Solve any *three* problems.

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1. Consider a rectangular box of length l, width w, and height h . Its six faces are at $x = 0$, $x = l$; $y = 0$, $y = w$; and $z = 0$, $z = h$. See the figure.

   All the faces, except the face at $z = h$, are maintained at zero electrostatic potential. The electrostatic potential on the face at $z = h$ is given by

   $$\Phi(x, y, h) = V_0 \, sin^2 \left(\frac{3\pi x}{l} \right) \quad .$$

   (a) Find the electrostatic potential $\Phi(x, y, z)$ inside the box, *i.e.* for $0 < x < l$, $0 < y < w$, $0 < z < h$ 7/10

   (b) Find the electrostatic energy inside the box. 3/10
   (HINT: Use the technique of separation of variables in Cartesian coordinates.)

2. Consider an infinite slab of a medium located occupying the region between $0 \leq z \leq a$. There is vacuum for $z < 0$ and $z > a$. In this medium, the refractive indices for left and right circular polarised electromagnetic (EM) waves are n_L and n_R respectively.

   (a) Give an example of such a medium where $n_L \neq n_R$ 1/10

   (b) Consider a plane EM wave in the vacuum $z < 0$, which propagates in the $\hat{z}$ direction and is incident on the medium from below. Let the electric field of this incident wave be given, in the standard complex notation, by

   $$\vec{E}(x, y, z) = (\hat{x} \; E_1 + \hat{y} \; E_2) \; e^{i(kz - \omega t)}$$

   where E_1 and E_2 are real.

   (c) Find the electric and magnetic field in the regions $0 \leq z \leq a$ and $z > a$. For the purposes of this problem, you may neglect reflection at the interfaces between the media. 9/10

3. Let the electric displacement $\vec{D}$ and the electric field E be related, in the standard notation, by $D_j = \sum_k \epsilon_{jk} E_k$ where ϵ_{jk} are constants and are given by

   $$\epsilon_{jk} = (\delta_{jk} + \chi_{jk}) \; \epsilon_0 \quad .$$

   Consider a monochromatic plane EM wave with frequency ω propagating along $\hat{n}$ direction. Its wave vector is then $\vec{k} = k \; \hat{n}$ with $\hat{n} \cdot \hat{n} = 1$.

(a) Show that the dispersion relation between k and ω is given, in the usual notation, by

$$(1-\xi)\ E_j + \xi n_j\ (\hat{n} \cdot \vec{E}) + \sum_k \chi_{jk}\ E_k\ =\ 0$$

where $\xi = \frac{c^2 k^2}{\omega^2}$ 7.5/10

(b) For non trivial constants χ_{ij}, describe qualitatively

i. how to use this dispersion relation, and

ii. the effects of χ_{ij} 2.5/10

4. Let there be constant electric field $\vec{E}$ and magnetic field $\vec{B}$ with $\hat{E} \cdot \hat{B} = cos\ \theta$. Consider the motion of a particle with charge q and mass m. Assume the motion to be non relativistic. Let $\vec{x}(t)$ and $\vec{v}(t)$ be the location and velocity of the particle at time t . Let $\vec{x}(0) = \vec{v}(0) = 0$ be the initial conditions.

(a) Take $\theta = \frac{\pi}{2}$. Find $\vec{x}(t)$ and $\vec{v}(t)$. Draw the trajectory of the particle and describe its salient features. What happens in the limit $\vec{B} \to 0$? 6/10

(b) Find $\vec{x}(t)$ and $\vec{v}(t)$ for $\theta \neq \frac{\pi}{2}$ 4/10

Comprehensive Examination

July 7, 2014 **Instructions** **09:30 - 12:30**

- *Please use separate notebooks for Quantum Mechanics and Statistical Mechanics.*
- *In each notebook, at the beginning, please write your name and roll number clearly.*
- *You may use loose sheets, available in the exam hall, for rough work.*
- *All questions carry equal marks.* In each section, you have to do *any three* out of the four problems.

- Passing criterion: **minimum of 11/30** marks in each section *and* a **total of 30/60** in both sections together.

Comprehensive Examination: Paper-II

July 7, 2014 **Quantum Mechanics** 09:30 - 12:30

All problems carry equal marks. Solve any *three* problems.

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1. Consider the Hermitian operators A, B and C satisfying, $[A, B] = iC$.
   (a) Derive an inequality for the product of uncertainties, $\Delta A \cdot \Delta B$. For $C = \hbar \mathbb{1}$ deduce the usual uncertainty relation.
   (b) For $C = \hbar \mathbb{1}$, Show that $e^{i\alpha B} A e^{-i\alpha B} = A + \alpha\hbar\mathbb{1}$.
   (c) Given that the eigenvalues of a Hamiltonian operator are bounded below, can one have a Hermitian operator T such that $[T, H] = i\hbar\mathbb{1}$?

2. Consider a two level atom with energies $E > E_0$. It is coupled to an oscillator of frequency $\omega = (E - E_0)/\hbar$, with an interaction Hamiltonian $V_{atom}(a + a^\dagger)$. Consider the two processes: (1) atom makes a transition from $E \to E_0$ and the oscillator state changes by addition of a quantum, and (2) the atom absorbs a quantum from the oscillator and makes a transition from $E_0 \to E$.
   (a) Obtain the transition probabilities for the two processes when the initial oscillator state has exactly k quanta.
   (b) Assume that if the oscillator is in a coherent state $|z\rangle$, defined by $a|z\rangle = z|z\rangle$, $z \in \mathbb{C}$, $\langle z|z\rangle = 1$, its state is *unchanged* when the atom make a transition either way. In this case, what are the transition probabilities for the two atomic transitions.

3. Consider particle in a 1-dimensional box occupying the interval $[-L, L]$. The particle is in the ground state when the box walls are suddenly changed to $[-L', L']$. Obtain the transition probability for transition to the n^{th} state of the new box.

   Distinguish the cases $L < L'$ and $L > L'$ as well as when L'/L, is a rational number.

4. Consider a spin 1/2 system. Obtain the eigenstates of $\hat{n} \cdot \vec{\sigma}$ where $\hat{n}$ is a unit vector and σ's are the usual Pauli matrices. Obtain the transition probability for the transition to a state pointing in the direction $\hat{n}'$.

   Consider an ensemble with 30% spins pointing in the direction $\hat{n}_1$ and 70% spins pointing in the direction $\hat{n}_2$. What is the average of the observable $\hat{n}_3 \cdot \vec{\sigma}$.

Comprehensive Examination: Paper-II

July 7, 2014 **Statistical Mechanics** 09:30 - 12:30

All problems carry equal marks. Solve any *three* problems.

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1. An ideal Bose-Einstein gas of bosons with mass m having an internal degree of freedom η which can take two values ($\eta = 1, 2$). The energy spectrum is given by $E_\eta(p) = p^2/2m + (\eta - 1) \times \Delta$. Assume that $\Delta >> kT_c = 1/\beta_c$ and compute the Bose-Einstein condensation temperature (T_c). Compare this condensation temperature to the case with no internal degree of freedom.

2. Consider a tetrahedron of 4 Ising spins. Each spin can take values ± 1 and coupled to every other spin. The Hamiltonian of the tetrahedran is given by

$$H = -J \sum_{i,j} s_i s_j$$

Compute (and draw) thermal average of the absolute value of the magnetisation and the internal energy as function of temperature. Note that because of symmetry of the Hamiltonian under $s_i \to -s_i$, the thermal average of the magnetisation will be zero unlike thermal average of its absolute value.

3. Find the efficiency of the ideal gas engine whose cycle in the $PV-$diagram is given by $(P_1, V_1) \to (P_2, V_2)(V_2 > V_1) \to (P_3, V_2)(P_3 < P_2) \to (P_4, V_1) \to (P_1, V_1)$.

The processes $1 \to 2$ and $3 \to 4$ are adiabatic while $4 \to 1$ and $2 \to 3$ are at constant volume.

4. Consider a $1-$dimensional Ising chain(ring) of $N-$spins, the Hamiltonian is given by

$$H = -J \sum_{i=1}^{N} S_i S_{i+1}, \quad S_{N+1} = S_1, \quad S_i = \pm 1,$$

Consider the system in the canonical ensemble and Calculate the leading order temperature dependence of the specific heat in the $T \to 0$ limit.

(Hint: No need to evaluate the partition function exactly.)