# Comprehensive Examination

# Instructions

## January 6, 2017 09:00 - 13:00

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• *Please use separate notebooks for Classical Mechanics and Electromagnetism.*

• *In each notebook, at the beginning, please write your name and roll number clearly.*

• *You may use loose sheets, available in the exam hall, for rough work.*

• *All problems carry equal marks.* In each section, you have to do *any three* out of the five problems.

• Passing criterion: **minimum of 10/30** marks in each section *and* a **total of 27/60** in both sections together.

• Duration of examination for both the parts together: **09:00 hours to 13:00 hours**

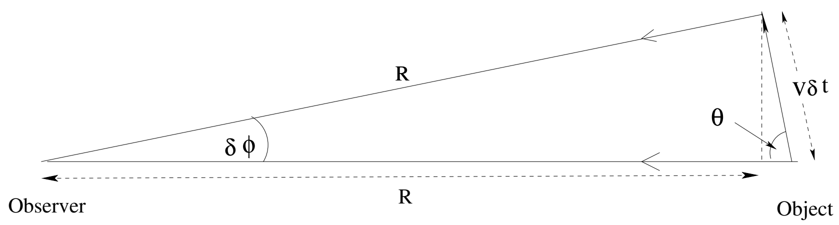
# Comprehensive Examination: Paper-I

## Part A: Classical Mechanics

All problems carry equal marks. Solve any *three* problems.

1. A certain astronomical body is moving with a speed $v$ towards earth, making an angle $\theta$ with the line of sight. It emits two electromagnetic pulses at times $t_1, t_2 = t_1 + \delta t$. These pulses are received on earth at times $t'_1, t'_2 = t'_1 + \delta t'$. The line of sight changes during the interval $\delta t$, by a very small angle $\delta\phi$ as the body is very far from earth at a distance $R$. An observer measures the angular speed, $\omega := \delta\phi/\delta t'$ and infers the *transverse speed* of the object to be $v_T := R\omega$ (see figure).

   The transverse speed for that object was measured to be *twice the speed of light in vacuum*. Explain this apparently contradictory observation by estimating $v$ and $\theta$.

   **Hint:** $v_T$ is dependent on $v, \theta$ and you may assume that the observed transverse speed is the *maximum possible* transverse speed as the angle $\theta$ is varied.

   Incidentally, such objects are observed and are referred to as superluminal sources. Jets coming out of active galactic nuclei are an example of such sources. ......[**10 marks**]

2. A small body of mass $m$ is in a circular orbit of radius $r$ around a larger body of mass $M$. Due to gravitational radiation, energy is lost at a rate of,

   $$P(r) = -\frac{32}{5}\frac{G^4}{c^5}\frac{M^2m^2(M+m)}{r^5} \quad Jsec^{-1}$$

   Assuming the validity of the energy loss formula and Newtonian orbits, obtain (a) the time for merger as a function of the masses and the initial radius, (b) the energy radiated away in *one period* and (c) the rate of change of the period. ..[**5 marks**]

   Give numerical estimates of the merger time for

   (i) $M = 35M_\odot, m = 30M_\odot$ and $R_0 = 350$ km (the GW150924 binary black hole system) , ......................................................... [**2.5 marks**];

   (ii) $M = M_\odot, m = 10^{-6}M_\odot$ and $R_0 = 1.5 \times 10^{11}$ meters (the Sun-earth system). [**2.5 marks**]

   Use, solar mass $M_\odot = 2 \times 10^{30}$kg, $G = 6.6 \times 10^{-11}\ m^3kg^{-1}sec^{-2}$ and $c = 3 \times 10^8\ m\ sec^{-1}$.

3. Earth is being bombarded by cosmic dust at a rate estimated to be between 3 to 300 tons per day which may be taken to be about $10^3$ gm/sec for order of magnitude estimates. Assume the rate has remained roughly constant for the past billion years and that all the dust has been spread uniformly over the surface, at the same average

density as that of the earth (about $\rho \sim 5.5$ gm/cc). This process results in increasing the earth's radius by $\Delta R \ll R_{earth}$ centimeters. ...................... **[10 marks]**

*Estimate* the fractional change in the length of the day over the past billion years. You may use: $M_{earth} \sim 6 \times 10^{27}$ *gms,* $R_{earth} \sim 6300$ *km, and 1 year* $\sim 3 \times 10^7$ *seconds.*

4. Consider motion in a central potential, $V(r) = -\frac{k}{r}$, but now incorporate the special relativistic expressions for momentum ($= m_0\gamma\vec{v}$) and total energy ($= m_0\gamma c^2 + V$, $\gamma = (1 - v^2/c^2)^{-1/2}$).

   Derive the orbit equation and show that generically a bound orbit *precesses*. Find the angle gained per revolution.

   For a nearly circular orbit of radius $R$, and $k = GMm_0$ show that the perihelion advances through an angle $\delta \approx \frac{\pi GM}{Rc^2}$. ................................ **[10 marks]**

5. (a) Given a first order autonomous system in $N$ variables,

$$\dot{x}^i = V^i(x^1, x^2, ..., x^N)$$

   List all the conditions under which the above can be considered as a Hamiltonian system. ........................................................ **[2.5 marks]**

   (b) Is the autonomous system given below, a Hamiltonian system?

$$\dot{x} \; = \; y \quad , \quad \dot{y} \; = \; x - x^3 - \mu y(2y^2 - 2x^2 + x^4) \quad , \; \mu > 0$$

   ............................................................ **[2.5 marks]**

   (c) When is an infinitesimal transformation of generalized coordinates and momenta,
   $(q^i, p_j)_{i,j=1,\ldots,n}$, a canonical transformation?..................... **[2.5 marks]**

   (d) Show that every function on the phase space of a Hamiltonian system generates an infinitesimal canonical transformation........................ **[2.5 marks]**

# Comprehensive Examination: Paper-I

## Part B: Electromagnetism

**January 6, 2017** **09:00 - 13:00**

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All problems carry equal marks. Solve any *three* problems.

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1. Three point charges, $q$, $-2q$ and $q$ lie in a straight line equidistant $a$ apart.

   (a) Find the potential at an arbitrary point $\vec{x}$, $x \equiv |\vec{x}| > a$, assuming the negative charge lies at the origin. Express the potential in spherical polar coordinates. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . [**5 marks**]

   (b) In the limit $a \to 0$ with $qa^2 \to Q =$ constant, show that the potential is finite. What is the dominant pole in the potential? . . . . . . . . . [**2 marks**]

   (c) If the charges are enclosed by a spherical, grounded, conducting sphere of radius $b$ ($b > a$), centred around the origin, how will the potential change for (i) $r < b$, (ii) $r > b$? Just indicate the changes and how you would compute the potential in this case. . . . . . . . . . . . . . . . . . . . . . . . . . . [**3 marks**]

2. (a) A circular disk of radius $a$ rotates uniformly, anticlockwise in the $x$-$y$ plane. A uniform magnetic field exists over the entire region in the positive $z$ direction (normal to disk). Show that there is a voltage generated between the centre (axle) and the rim. . . . . . . . . . . . . . . . . . . . . . . . [**5 marks**]

   If you connect a resistance $R$ between the centre (axle) and rim, in which direction would the current flow? . . . . . . . . . . . . . . . . . . . . . . . . . . . [**1 mark**]

   (b) A current $I$ flows in the $x$ direction through a rectangular bar of conducting material. There is a transverse magnetic field, $B_z$.

   Show that a potential difference develops between the surfaces of the conductor in the $y$ direction. Express this potential difference in terms of the currents and fields in the problem. . . . . . . . . . . . . . . . . . . . [**4 marks**]

3. Consider an electric charge $e$ and a magnetic charge $g$ distance $d$ apart, aligned along the $z$ axis. The magnetic charge at the origin gives rise to a magnetic field of magnitude $B = (\mu_0/(4\pi))\, g/r^2$ at a distance $r$ from it and the electric charge gives rise to the usual electric field.

   (a) Find the direction of the linear momentum density of the fields at an arbitary point $\vec{x}$ from the origin; $x \equiv |\vec{x}| \gg d$. . . . . . . . . . . . . [**3 marks**]

   (b) Express the angular momentum density of the fields in terms of the linear momentum density. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . [**2 marks**]

   (c) Find the total (conserved) angular momentum stored in the fields. . . [**5 marks**]

4. Consider an electromagnetic wave travelling in the positive $x$ direction with speed $c$:

$$
\begin{aligned}
\vec{E}(x,y,z,t) &= E_y(x,t)\hat{j} = E_{y,0}\sin\left((2\pi/\lambda)(x-ct)\right)\hat{j} \;, \\
\vec{B}(x,y,z,t) &= B_y(x,t)\hat{k} = E_{y,0}\sin\left((2\pi/\lambda)(x-ct)\right)\hat{k} \;.
\end{aligned}
$$

   (a) Find the Poynting vector for this wave. . . . . . . . . . . . . . . . . . . . [**3 marks**]

(b) What is the time-averaged Poynting vector field on the fixed plane $x = 0$ over one period? ........................................ [**2 marks**]

(c) What is the time-average (over one period) of the energy density stored in the electric and magnetic fields at the point $(x, y, z, t)$. Recall that $u_{elec} = (1/2)\epsilon_0 E^2$; $u_{mag} = B^2/2\mu_0$ and $c^2 = 1/\mu_0\epsilon_0$. How is it related to the magnitude of the time-averaged Poynting vector? ...... [**3 marks**]

(d) At the upper surface of the Earth's atmosphere, the time- averaged magnitude of the Poynting vector, referred to as the solar constant, is given by,

$$\langle|\vec{S}|\rangle = 1.35 \times 10^3 \text{ W m}^{-2} \ .$$

If you assume that the Sun's electromagnetic radiation is a plane sinusoidal wave, what are the magnitudes of the electric and magnetic fields? Use $\epsilon_0 = 9 \times 10^{-12}$ $\text{C}^2/(\text{Nm}^2)$ . ............................ [**2 marks**]

5. Two long, cylindrical conductors of radii $a_1$ and $a_2$ are parallel and separated by a distance $d$, which is large compared with either radius.

(a) Show that the capacitance per unit length is given approximately by

$$C \sim \pi\epsilon_0 \left(\ln \frac{d}{a}\right)^{-1} ,$$

where $a$ is the geometrical mean of the two radii. ........... [**5 marks**]
Hint: The capacitance of two conductors carrying charge per unit length $Q$, $-Q$ respectively, is equal to the ratio of the charge on one conductor to the potential between them.

(b) What is the approximation used here? ...................... [**2 marks**]

(c) Approximately what gauge wire (state diameter in millimeters) would be necessary to make a two-wire transmission line with a capacitance of $1.2 \times 10^{-11}$ F/m if the separation of the wires was 0.5 cm? . [**3 marks**]

# Comprehensive Examination

## Instructions

**Jan 9 2017** **09:00 - 13:00**

• *Please use separate notebooks for Quantum Mechanics and Statistical Mechanics.*

• *In each notebook, at the beginning, please write your name and roll number clearly.*

• *You may use loose sheets, available in the exam hall, for rough work.*

• *All problems carry equal marks.* In each section, you have to do *any three* out of the five problems.

• Passing criterion: **minimum of 10/30** marks in each section *and* a **total of 27/60** in both sections together.

• Duration of examination for both the parts together: **09:00 hours to 13:00 hours**

# Comprehensive Examination: Paper-II

## Part C: Quantum Mechanics

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All problems carry equal marks. Solve any *three* problems.

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1. A quantum particle constrained to move in 1-dimension is in a potential $V(x) = (1/2)Kx^2$. It also carries a charge $+q$ and an electric field directed in the $+x$ direction is present. Add the interaction term to $V(x)$ due to the electric field.

   (a) Find exact solutions for the energy eigenvalues $E_n$ with the electric field present. **(3 marks)**

   (b) Find the ground state wave function with the electric field present. **(3 marks)**

   (c) If the electric field is a small perturbation over the dominant harmonic oscillator potential, solve for the energy eigenvalues using perturbation theory. Compare this result with your previous exact result. **(4 marks)**

2. Consider the Schrodinger equation in the co-ordinate representation in 3-dimensional spherical co-ordinates $(r, \theta, \phi)$ for a time-independent central potential $V(r)$. This question deals with quantum mechanical tunneling (barrier penetration).

   Note: The Laplacian in spherical coordinates can be written as

   $$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{r^2\hbar^2}\hat{L}^2 \ ,$$

   where $\hat{L}(\theta, \phi)$ is the angular momentum operator.

   (a) Write the spatial wave-function in a separated form $\psi(r, \theta, \phi) = R(r)Y_{lm}(\theta, \phi)$ where the $Y_{lm}$ are the spherical harmonics, and obtain the differential equation for the radial part $R(r)$. **(3 marks)**

   (b) With the classical turning points given by $r_1$ and $r_2$ and for $V(r) > E$ in the region $r_1 < r < r_2$, obtain an *approximate* expression for the *barrier penetration factor* $P_B \equiv R(r_2)^2/R(r_1)^2$, in the WKB approximation for a slowly varying potential $V(r)$. **(4 marks)**

   (c) Consider the $\alpha$-decay process which leaves a heavy nucleus with charge $+Ze$ and radius $R$, and the charge on the $\alpha$ particle being

$+Z'e$. If the relevant interaction is due to the Coulomb potential $V(r) = ZZ'e^2/r$, and if the final $\alpha$ particle comes out with a kinetic energy $E = mv^2/2$ (non-relativistic, with $m$ being the $\alpha$-particle mass), for $l = 0$, write an expression for $P_B$ for this decay. The $\alpha$-decay life-time is given by $\tau_\alpha = \tau_0/P_B$, where $\tau_0 \approx 10^{-21}s$. (Hint: take $r_1 = R$ and ignore the $r < r_1$ region where strong interaction effects are important. It is sufficient to write an integral equation for $P_B$; no need to integrate it.) **(3 marks)**

3. Consider a Hydrogen-like atom, i.e. a bound state of two non-identical spin-half fermions. For use below, we note that the Pauli spin matrices are given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ; \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; .$$

   (a) Write the Hamiltonian for the spin-spin interaction of the two particle quantum system and find the energy eigenvalues and eigenvectors in the two-spin tensor-product space of the system. You can take the common energy of the states as $E_0$; we are interested in how the states are split by the spin-spin interaction. **(5 marks)**

   (b) A magnetic field $B_z$ is applied in the $+z$ direction that couples to the magnetic moments $\mu_1$ and $\mu_2$ of the two particles respectively. Add this interaction along with the above spin-spin interaction Hamiltonian, diagonalize the Hamiltonian and find the energy eigenvalues. **(5 marks)**

4. For a quantum state $|\alpha\rangle$ (in the Schrodinger picture), the projection operator corresponding to this state is defined as $\mathbf{P}_\alpha \equiv |\alpha\rangle\langle\alpha|$.

   (a) For a pure state $|\alpha\rangle$, and a time-independent operator $\mathcal{O}$, the average value is defined as $\langle\mathcal{O}\rangle \equiv \langle\alpha|\mathcal{O}|\alpha\rangle$. Show that $\langle\mathcal{O}\rangle = \mathrm{tr}(\mathbf{P}_\alpha\mathcal{O})$, where the trace is over a complete set of basis states. **(2 marks)**

   Now, let a spin-half system be in a statistical state with the density matrix $\rho \equiv \sum_\alpha p_\alpha \mathbf{P}_\alpha$ where $p_\alpha$ (constant in time) is the probability for it to be in the state $|\alpha\rangle$. Let the average value of the operator $\mathcal{O}$ in the statistical state be $\langle\mathcal{O}\rangle_{\text{avg}} \equiv \sum_\alpha p_\alpha \langle\alpha|\mathcal{O}|\alpha\rangle$.

   (b) Show that the following equation of motion holds:

$$i\hbar\frac{d\rho}{dt} = [H, \rho] \; ,$$

   where $H$ is the Hamiltonian. **(2 marks)**

(c) Show that $\langle \mathcal{O} \rangle_{\text{avg}} = \text{tr}(\rho\mathcal{O})$. Also, for an operator $\mathcal{O}$ with no explicit time dependence, show that

$$i\hbar \frac{d}{dt} \langle \mathcal{O} \rangle_{\text{avg}} = \text{tr}\left(\rho[\mathcal{O}, H]\right) \ .$$

**(2 marks)**

(d) The magnetic moment 3-vector is given by $\mu^i = \frac{\gamma}{2}\hbar\sigma^i$, i={1,2,3}, where $\sigma^i$ are the Pauli matrices and $\gamma$ is a constant (the gyromagnetic ratio). An interaction term in the Hamiltonian between the magnetic moment and an external magnetic field is $H_B = -\mu^i B^i$. Due to this interaction term, find the rate of change of the average value of the spin operator, i.e. find $d\left\langle \sigma^i \right\rangle_{\text{avg}}/dt$, which gives spin precession. **(4 marks)**

5. A particle and its antiparticle, denoted $K^0$ and $\bar{K}^0$ respectively, has the matrix elements of the Hamiltonian in the $(|K^0\rangle, |\bar{K}^0\rangle)$ basis given by both diagonal entries being $E_K$ and both offdiagonal entries being $\Delta m$, in this $2 \times 2$ system.

(a) Find the energy eigenstates $|K_1\rangle$ and $|K_2\rangle$, and the corresponding eigenvalues $E_1$ and $E_2$. **(2 marks)**

(b) Add imaginary terms (constants) to the Hamiltonian matrix elements, with $-i\Gamma$ added to the diagonal terms and $-i\Delta\Gamma$ added to the off-diagonal terms. Show that $\Gamma$ will cause a loss of probability in each state (i.e. $\Gamma$ will account for the decay of the $K^0$ and $\bar{K}^0$). Evolve the energy eigenstates $|K_1\rangle$ and $|K_2\rangle$ in time, i.e. find the states $|K_1(t)\rangle$ and $|K_2(t)\rangle$, including the effect of the $\Gamma$. **(4 marks)**

(c) An *entangled* $K^0$–$\bar{K}^0$ pair is produced at $t = 0$ and travels in opposite directions, one to the right and the other to the left. A detector is available that can detect whether the particle is a $K^0$ or $\bar{K}^0$. Using this detector, a measurement on the particle on the left at time $t_0$ reveals that it is a $\bar{K}^0$. Find an expression for the probability that at a later time $t > t_0$ the particle on the right is also detected as a $\bar{K}^0$? (Hint: The $K^0$–$\bar{K}^0$ pair being entangled means that knowing that at time $t_0$ the state on the left is a $\bar{K}^0$ implies that the state on the right at the same time $t_0$ is a $K^0$.) **(4 marks)**

# Comprehensive Examination: Paper-II

## Part D: Statistical Mechanics

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All problems carry equal marks. Solve any *three* problems.

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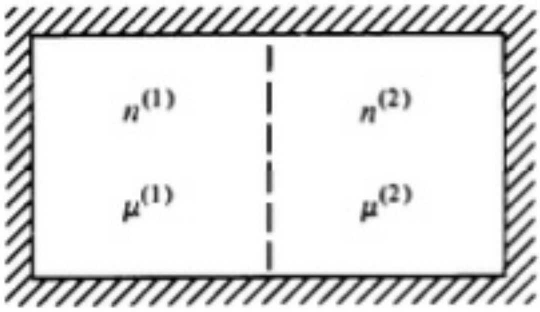
1. The condition for stable equilibrium can be written in terms of quadratic term as $(\delta^2 E)_{S,V,n} \geq 0$. Consider a composite system with two compartments with the following fluctuations:
$\delta S = 0 = \delta S^{(1)} + \delta S^{(2)}$
$\delta V^{(1)} = \delta V^{(2)} = \delta n^{(1)} = \delta n^{(2)} = 0$
Using the thermodynamic definition of $\frac{1}{C_v} = \frac{1}{T}(\frac{\delta^2 E}{\delta S^2})_{V,n} = \frac{1}{T}(\frac{\delta T}{\delta S})_{V,n}$,




   (a) Show that for a stable system $C_v$ has to be positive. **(7 marks)**

   (b) Explain how the instability on the system will be manifested if this condition is not satisfied. **(3 marks)**

2. Consider a hypothetical equation of state for a substance near liquid-solid phase transition. The dependency of Helmholtz free energy per unit volume ($A$) on molar density ($\rho$) in liquid($L$) and solid ($S$) phases are characterized by the following:
$\frac{A_L}{V} = (1/2)\frac{a}{T}\rho^2$ and
$\frac{A_S}{V} = (1/3)\frac{b}{T}\rho^3$, where $\rho = n/V$ for $n$ moles.
Now consider that the molar densities of this system before and after solidification are $\rho_L$ and $\rho_S$ respectively and at the co-existence, $\mu_L = \mu_S$ and $P_L = P_S$.

   (a) determine $\rho_L$ and $\rho_S$ and comment on their dependence on temperature **(4 marks)**

(b) determine $P_S$ as a function of temperature **(3 marks)**

(c) calculate change in entropy per mole during solidification **(3 marks)**

3. Consider a system of two identical particles which may occupy any of the three energy levels
$\epsilon_n = n\epsilon, n = 0, 1, 2, ......$
The lowest energy state, $\epsilon_0 = 0$ is doubly degenerate. The system is in thermal equilibrium at temperature $T$. For each of the following cases determine the (1) partition function, (2) the energy and, (3) carefully enumerate the configurations.

   (a) The particles obey Fermi statistics **(2 marks)**

   (b) The particles obey Bose statistics **(2 marks)**

   (c) If they are distinguishable and obey Boltzmann statistics **(2 marks)**

   (d) Calculate the specific heat at constant volume when the particles obey Fermi statistics **(2 marks)**

   (e) Discuss the conditions under which Fermions and Bosons may be treated as Boltzmann particles. **(2 marks)**

4. A system consists of three spins in a line, each having $s = \frac{1}{2}$, coupled by the nearest neighbor interactions. Each spin has a magnetic moment pointing in the same direction as the spin, $\vec{\mu} = 2\mu s$. The system is placed in an external magnetic field $B$ in the $z$ direction and is in thermal equilibrium at temperature $T$. The Hamiltonian for the system is approximated by an Ising model, where the true spin-spin interaction is replaced by a term of the form $JS_z(i)S_z(i+1)$:
$H = JS_z(1)S_z(2) + JS_z(2)S_z(3) - 2\mu B[S_z(1) + S_z(2) + S_z(3)]$
where $J$ and $\mu$ are positive constants.

   (a) List each of the possible microscopic states of the system and its energy and indicate any degeneracies. **(2 marks)**

   (b) For each of the following conditions, write down the limiting values of the internal energy $U(T, B)$, the entropy $S(T, B)$, and the magnetization $M(T, B)$.

      i. $T = 0$ and $B = 0$,

      ii. $T = 0$ and $0 < B \ll J/\mu$

      iii. $T = 0$ and $J/\mu \ll B$

      iv. $J \ll kT$ and $B = 0$.

   **(4 marks)**

   (c) Calculate the partition function $Z(T, B)$ and find its closed form **(2 marks)**

(d) Find the magnetization $M(T,B)$. Find an approximate expression for $M(T,B)$ which is valid when $kT \gg \mu B$ or $kT \gg J$. **(2 marks)**

5. Consider a dilute gas of $N$ hard spheres with the 2-body interactions

$$\begin{aligned} V(|\vec{r_i} - \vec{r_j}|) &= 0 \qquad |\vec{r_i} - \vec{r_j}| > a, \\ &= \infty \qquad |\vec{r_i} - \vec{r_j}| < a. \end{aligned}$$

Using the definition of classical partition function and dilute concentration approximation, compute:

(a) total partition function **(3 marks)**

(b) average energy **(3 marks)**

(c) entropy as a function of energy. You may use the approximation: $(V - a\alpha)(V - (N - a)\alpha) \approx (V - N\alpha/2)^2$, where $\alpha$ is the volume occupied by each hard sphere. **(3 marks)**

(d) if the partition function was quantum mechanical in nature, would the answer for average energy change? Explain. **(1 marks)**