Comprehensive Examination

Instructions

January 2, 2015 **09:00 - 13:00**

• *Please use separate notebooks for Classical Mechanics and Electromagnetism.*

• *In each notebook, at the beginning, please write your name and roll number clearly.*

• *You may use loose sheets, available in the exam hall, for rough work.*

• *All problems carry equal marks.* In each secion, you have to do *any three* out of the five problems.

• Passing criterion: **minimum of 10/30** marks in each section *and* a **total of 27/30** in both sections together.

• Duration of examintaion for both the parts together: **09:00 hours to 13:00 hours**

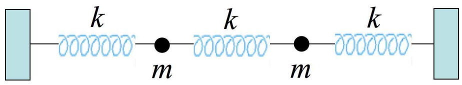
Comprehensive Examination: Paper-I

Classical Mechanics

January 2, 2015 **09:00 - 13:00**

All problems carry equal marks. Solve any *three* problems.

1(a): Consider the system of two masses m and three springs, all with identical spring constants k and equilibrium length a (as shown in the figure below). The masses can only move longitudinally. Write down expressions for the kinetic and potential energies of the system. What are the normal frequencies ω_k and the normal modes a_k of the system? **[7 marks]**



1(b): If the system starts at rest, with the first mass at a distance $(a + \Delta$ from the left wall, and the second mass at a distance $2a$ from the wall, what are the displacements $x_1(t)$, $x_2(t)$ as a function of time for each mass, and $x_{CM}(t)$ for the center of mass of the system? **[3 marks]**

2(a): A hoop is rolling, without slipping, on an inclined plane with angle of slope ϕ. Write the equation of constraint. **[2 marks]**

2(b): Write expressions for the kinetic and potential energies in terms of polar coordinates. Show that the motion separates into the kinetic energy of the centre of mass plus the kinetic energy of motion about the centre of mass. **[4 marks]**

2(c): Write down the total Lagrangian and the equations of motion and compute the acceleration of the hoop. **[4 marks]**

3(a): Consider a simple harmonic oscillator in one dimension. **[4 marks]**

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right), \ \omega = k/m.$$

Transform the variables to

$$p = f(P)\cos Q, \quad q = \frac{f(P)}{m\omega}\sin Q.$$

Find $f(P)$ so that Q is cyclic using a generating function of the first kind,

$$F_1 = \frac{m\omega q^2}{2}\cot Q.$$

3(b): Solve the equations of motion for Q and P and hence solve for q and p.

[3 marks]

3(c): Plot the time dependence of the new and old variables. What do you infer from these? **[3 marks]**

4(a): A rocket of length l_0 in its rest frame is moving with constant speed along the z axis of an inertial system. An observer standing at the origin of this system observes the apparent length of the rocket at any time by noting the z coordinates of the head and tail of the rocket. How does this apparent length vary as the rocket moves past the observer from the extreme left to the extreme right? **[8 marks]**

4(b): How do these results compare with measurements in the rest frame of the observer? (Note: observe, not measure). **[2 marks]**

5(a): What is a point transformation? Show that all point transformations are canonical. **[4 marks]**

5(b): For a 1-d Hamiltonian system with **[4 marks]**

$$H = \frac{p^2}{2} + \frac{1}{2q^2}$$

show that there is a constant of motion

$$D = \frac{pq}{2} - Ht$$

5(c): Show that the transformation $Q = \lambda q$, $p = \lambda P$ is canonical. **[2 marks]**

Comprehensive Examination: Paper-I

Electromagnetism

January 2, 2015 **09:00 - 13:00**

All problems carry equal marks. Solve any *three* problems.

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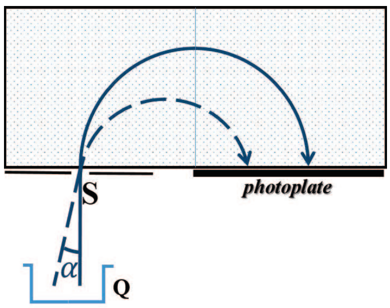
(1): Four charges of magnitude $-q$ and one of magnitude $4q$ are placed in the $z = 0$ plane, such that $4q$ is at the origin and $-q$ are placed at $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$ and $(0, -1, 0)$. Find the components of the quadrupole moment of the distribution.

[10 marks]

(2): Particle of mass M are singly ionized in an ion source Q and accelerated by the voltage V. They are entering the magnetic field B that is perpendicular to the plane of the paper through a slit S, as in the figure given below.



2(a): Where do they hit the photoplate?

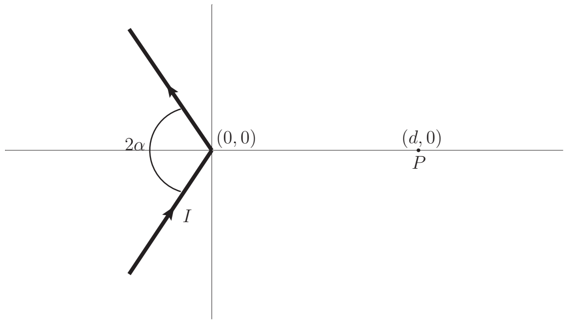
2(b): Were do the particles entering the magnetic field at an angle $\alpha \ll 1$ with respect to the axis hit the photoplate?

2(c): This setup is called a mass-spectrometer. Explain, how the mass of the ion can be measured.

[10 marks]

(3): An infinitely long wire carrying current I along the y axis is bent at the origin so that it forms an angle 2α and lies symmetrically about the x axis in the second and third quadrants (see the figure given below). Show that the field at a symmetric point from both the bent legs and at a distance d from the bend point, on the plane of the wire along the positive axis has the magnitude

$$B = \frac{2\,I}{c\,d\sin\alpha}(1 - \cos\alpha)\ .$$



[10 marks]

(4): A long coaxial cable carries current I such that the current flows down the surface of the inner cylinder, radius a and back along the outer cylinder of radius b. Find the magnetic energy stored in a section of length l. **[10 marks]**

(5): The time averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}(1+\frac{\alpha r}{2})$$

where q is the magnitude of the electron charge and $\alpha^{-1} = a_0/2$ is the Bohr radius. Find the distribution of charge both continuous and discrete that will give the potential and interpret your result physically.

[10 marks]

Comprehensive Examination

Instructions

January 5, 2015 09:00 - 13:00

• *Please use separate notebooks for Quantum Mechanics and Statistical Mechanics.*

• *In each notebook, at the beginning, please write your name and roll number clearly.*

• *You may use loose sheets, available in the exam hall, for rough work.*

• *All problems carry equal marks.* In each secion, you have to do *any three* out of the five problems.

• Passing criterion: **minimum of 10/30** marks in eaxh section *and* a **total of 27/30** in both sections together.

• Duration of examintaion for both the parts together: **09:00 hours to 13:00 hours**

Comprehensive Examination: Paper-II

Quantum Mechanics

January 5, 2015 09:00 - 13:00

All problems carry equal marks. Solve any *three* problems.

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(1): Consider a particle whose Hamiltonian H is given by:

$$H = \frac{1}{2m} P^2 - \alpha \delta(x)$$

where α is a positive constant. We would like to understand the bound state with energy $E < 0$.

1(a): Integrate the Schrodinger equation between $-\epsilon$ and plus ϵ and let ϵ approach zero. Obtain an expression for the discontinuity in the derivative $\psi'(x)$ at $x = 0$ (in terms of $\psi(0), \alpha, m$). **[4 marks]**

1(b): Assume that the energy E, of the particle is negative (bound state). The $\psi(x)$ can be written as

$$\begin{aligned} \psi(x) &= A_1 e^{\rho x} + A_1' e^{-\rho x} \quad & x < 0 \\ \psi(x) &= A_2 e^{\rho x} + A_2' e^{-\rho x} \quad & x > 0 \end{aligned}$$

Express ρ in terms of E and m. **[2 marks]**

1(c): Calculate the matrix M defined by

$$\begin{pmatrix} A_2 \\ A_2' \end{pmatrix} = M \begin{pmatrix} A_1 \\ A_1' \end{pmatrix}$$

Use the requirement of square integrability to find possible values of E (in terms of the parameters in H). **[4 marks]**

(2): Consider a spin 1/2 particle and basis states $|+\rangle, |-\rangle$ denoting eigenvalues $\pm\frac{\hbar}{2}$ of S_z.

2(a): At time $t = 0$ S_y is measured to be $\frac{\hbar}{2}$. What is the state vector $|\psi(0)\rangle$ immediately after the measurement? **[2 marks]**

2(b): Immediately after this measurement a uniform time dependent magnetic field parallel to $\hat{z}$ is applied. The Hamiltonian is then given as

$$H(t) = \omega_0(t) S_z$$

Assume that $\omega_0(t)$ is zero for $t < 0$ and $t > T$ and increases linearly in time from 0 to ω_0 during $0 \leq t \leq T$. Show that at time t the state vector can be written as

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}[e^{i\theta(t)}|+\rangle + ie^{-i\theta(t)}|-\rangle]$$

and calculate $\theta(t)$. **[6 marks]**

2(c): At a time $t = \tau > T$, S_y is measured. What results can we find and with what probabilities? Determine the relation between ω_0 and T such that we can be sure of the result. **[2 marks]**

(3): Consider a system of angular momentum $l = 1$. A basis of its state space is formed by the three eigenvalues of L_z: $|+1\rangle, |-1\rangle, |0\rangle$ whose eigenvalues are $\pm\hbar, 0$ and which satisfy ($L_\pm = L_x \pm iL_y$):

$$L_\pm|m\rangle = \hbar\sqrt{2}|m \pm 1\rangle; \; L_+|+1\rangle = L_-|-1\rangle = 0.$$

Consider a Hamiltonian gL_x^2.

3(a): Show that time evolution keeps the system in the state space with $l = 1$.

[2 marks]

3(b): Write the 3×3 matrix representing H in this subspace. **[3 marks]**

3(c): Diagonalize the matrix and find the energy eigenvalues and eigenvectors. (Time saving hint: Only a 2×2 submatrix needs to be diagonalized.) **[3 marks]**

3(d): At time $t = 0$ the system is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}[|+1\rangle - |-1\rangle]$$

Find the state at time t. **[2 marks]**

(4): Consider the anharmonic oscillator

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 + \lambda X^3$$

with small λ.

4(a): Find the energy of the ground state to first order in λ. **[2 marks]**

4(b): Find the ground state energy to second order in λ. Does it increase or decrease? **[5 marks]**

4(c): What does "small λ" mean precisely? **[3 marks]**

(5): Let H be the Hamiltonian of a physical system. Denote by $|\phi_n\rangle$ the eigenvectors of H with eigenvalues E_n:

$$H|\phi_n\rangle = E_n|\phi_n\rangle$$

5(a): For an arbitrary operator A calculate **[1 mark]**

$$\langle\phi_n|[A, H]|\phi_n\rangle$$

5(b): Consider a one-dimensional problem, where the physical system is a particle of mass m and of potential energy $V(X)$. In this case H is written:

$$H = \frac{P^2}{2m} + V(X)$$

5(b.1): In terms of P, X and $V(X)$, find the commutators: $[H, P], [H, X]$.

[2 marks]

5(b.2): Show that the mean value $\langle\phi_n|P|\phi_n\rangle$ is zero.

[1 mark]

5(b.3): Establish a relation between $E_k^{(n)} = \langle\phi_n|\frac{P^2}{2m}|\phi_n\rangle$ (the mean value of kinetic energy in the state $|\phi_n\rangle$) and $\langle\phi_n|X\frac{dV}{dX}|\phi_n\rangle$: **[4 marks]**

$$\langle\phi_n|X\frac{dV}{dX}|\phi_n\rangle = 2E_k^{(n)}$$

5(b.4): Since the mean value of the potential energy is $\langle\phi_n|V(X)|\phi_n\rangle$, how is it related to the mean value of the kinetic energy when: **[2 marks]**

$$V(X) = V_0 X^\lambda$$

$(\lambda = 2, 4, 6, \ldots; V_0 > 0)$

Comprehensive Examination: Paper-II

Statistical Mechanics

January 5, 2015 09:00 - 13:00

All problems carry equal marks. Solve any *three* problems.

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(1): The Landau free energy for a system is:

$$\mathcal{L}(\phi) = \frac{a}{2}\phi^2 + \frac{d}{6}\phi^6 - h\phi$$

with $a \propto (T - T_c) = \epsilon$, and $d > 0$. Determine the critical exponents:

1(a): β, $[m = \langle\phi\rangle \sim \epsilon^\beta]$

1(b): γ, [susceptibility $\sim \epsilon^{-\gamma}$], and

1(c): δ, $[m \sim h^{1/\delta}$, $\epsilon = 0]$.

[10 marks]

2(a): Show that the average energy per particle in a non-relativistic Fermi gas at absolute zero temperature in three dimensions is

$$U = \frac{3E_F}{5},$$

where E_F is the Fermi energy.

2(b): Hence deduce the Fermi pressure:

$$P_F = \frac{2E_F}{5}\frac{N}{V}.$$

[10 marks]

(3): An equation of state for a rubber band is either

$$S = L_0\gamma\left[\sqrt{\frac{\theta E}{L_0}} - \frac{1}{2}\left(\frac{L}{L_0}\right)^2 - \frac{L_0}{L} + \frac{3}{2}\right]$$

or

$$S = L_0\gamma\left[e^{\theta n E/L_0} - \frac{1}{2}\left(\frac{L}{L_0}\right)^2 - \frac{L_0}{L} + \frac{3}{2}\right]$$

where $L_0 = nl_0$, γ, l_0 and θ are constants, and L the length of the rubber band.

3(a): Which of the two possibilities are acceptable? Why?

3(b): For the acceptable choice, deduce the dependence of the tension f upon T and L/L_0; that is determine $f(T, L/L_0)$.

[10 marks]

(4): Consider a gas of N hard spheres in a three dimensional box. A single sphere excludes a volume ω around it, while its center of mass can explore a volume V [if box is otherwise empty]. There are no other interactions between the spheres, except for constraints of hard core exclusion. Assume low density.

4(a): Calculate the entropy S, as a function of the total energy E.
You can use the approximation

$$(V - a\omega)(V - (N - a)\omega) \approx (V - N\omega/2)^2$$

.

4(b): Calculate the equation of state of this gas.

4(c): Show that the isothermal compressibility

$$\kappa_T = -V^{-1}\frac{\partial V}{\partial P}$$

is always positive.

[10 marks]

(5): Consider a lattice gas model with L lattice sites and N particles. Calculate the equation of state when:

5(a): particles are indistinguishable and a lattice site may be occupied by utmost one particle.

5(b): particles are distinguishable and a lattice site may be occupied by utmost one particle.

5(c): particles are indistinguishable and a lattice site may be occupied by any number of particles.

5(d): In what limiting case does the equation of state become identical in (a)-(c)

[You may use $\ln(n!) \approx n \ln n - n$ for $n \gg 1$.]

[10 marks]