Comprehensive Examination: Paper-I

January 3, 2014 09:30 - 12:30

---

(1) *Please use separate notebooks for Classical Mechanics and Electromagnetism.*

(2) *In each notebook, at the beginning, please write your name and roll number clearly.*

(3) *You may use loose sheets, available in the exam hall, for rough work.*

Classical Mechanics: All problems carry equal marks. Solve any *three* problems.

---

1. Write down the Lagrangian, and Lagrange's equations of motion, for a pendulum consisting of a light rod of length ℓ pivoted at one end with a mass m attached to the other end, free to swing in three dimensions under gravity (acceleration due to gravity $= g$) with no other forces or constraints, in coordinates of your choice. What are the conserved quantities?

2. From the Lagrangian
$$L(t; x, y; \dot{x}, \dot{y}) = x\dot{x}^2 + 3\dot{x}\dot{y} + y\dot{y}^2$$
derive a Hamiltonian using the Legendre transform on the $\dot{x}$ and $\dot{y}$ arguments of L.

3. Consider the following time-independent transformation from q_1, q_2, p_1, p_2 to Q_1, Q_2, P_1, P_2:
$$Q_1 = (q_1)^2, \; Q_2 = (q_1 + q_2), \; P_i = P_i(q, p) \quad i = 1, 2$$
   (a) Find a general expression for P_i that ensures the transformation is canonical. You can use a time-independent F_2-type generating function. Reminder: this is a function of q_i, P_i, as follows:
$$\begin{aligned} F_2 &= F_2(q_i, P_i) \\ p_i &= \frac{\partial F_2}{\partial q_i} \\ Q_i &= \frac{\partial F_2}{\partial P_i} \end{aligned}$$
   (b) Find a particular choice of canonical transformation under which the Hamiltonian
$$H(q, p) = \left(\frac{p_1 - p_2}{2q_1}\right)^2 + p_2 + (q_1 + q_2)^2$$
   gets transformed into
$$H'(Q, P) = P_1^2 + P_2.$$

(c) Solve for the q_i (original coordinates).

4. Lotka-Volterra equations for a predator-prey system: Consider the pair of equations

$$\begin{aligned} \frac{dx}{dt} &= x(\alpha - \beta y) \\ \frac{dy}{dt} &= -y(\gamma - \delta x) \end{aligned}$$

where x is the number of prey, y is the number of predators, and the other symbols are parameters governing interactions.

(a) In the absence of predators, what is the solution for prey? In the absence of prey, what is the solution for predators?

(b) What are the fixed points of the system? (A fixed point is a choice of x, y such that $dx/dt = dy/dt = 0$).

(c) Take the non-trivial fixed point $(x_0, y_0 \neq 0)$, linearise the equations about this point (i.e. write $x = x_0 + \xi$ and $y = y_0 + \eta$ with ξ, η assumed small, and make a linear approximation to the equations). Solve the linearized equations. How does the system behave? Is this fixed point stable or unstable (ie, do ξ and η remain small, or do they grow with time)?

Comprehensive Examination: Paper-I

January 3, 2014 09:30 - 12:30

Electromagnetism: All problems carry equal marks. Solve any *three* problems.

---

1. Consider an electrostatic potential Φ given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\, \frac{e^{-ar}}{r}\, (1 + br)$$

where a and b are constants. Find the distribution of charge (both continuous and discrete) that will give this potential.

(a) What, if anything, is special when $a = 2b$?

(b) Interpret your results physically.

2. Consider two straight parallel line charges separated by a distance R with equal and opposite charge densities λ and $-\lambda$. (a) Show by direct construction that the surface of constant potential V is a circular cylinder (circular in the transverse dimensions) and find the coordinates of the axis of the cylinder and its radius in terms of $(R,\ \lambda,\ V)$. (b) Use the results in part (a) above to show that the capacitance per unit length C of two right circular cylindrical conductors, with radius a and b, separated by a distance $d > a + b$, is

$$C = \frac{2\pi\epsilon_0}{cosh^{-1}\left(\frac{d^2 - a^2 - b^2}{2ab}\right)} \quad .$$

3. Consider three charges: a charge $(-2q)$ is placed at $(0, 0, 0)$; a charge $(+q)$ is placed at $(0, 0, a)$; and, the third charge $(+q)$ is placed at $(0, 0, -a)$. Find the point(s), if any, at finite distance(s) where the net force acting on a test charge vanishes.

Such points are stationary points. Infinitesimal motion of the test charge away from such points in any of the three independent directions may be stable or unstable in some or all directions. The possibilities are: (1) stable in all three directions; (2) stable in two and unstable in the other one direction; (3) stable in one and unstable in the other two directions; (4) unstable in all three directions.

(a) Which of the possibilities are realised by the stationary point(s) of the three charge distribution given above.

(b) Note that, for static charge distributions, not all the above four possibilities may be realised. If you assert that some of the possibilities cannot be realised by any static charge distributions then: which possibilities or they? Prove your assertion.

(c) By giving at least one example of charge distribution (besides the example given above), show that each of the remaining possibilities is realised. (That is, in your examples, you must find all the stationary points and analyse explicitly the infinitesimal motions near them.)

4. Let $x^a : (x^1, x^2, x^3) = (x, y, z)$ be the Cartesian coordinates. Let the semi infinite space $x \geq 0$ be filled with a medium where the components of the electric displacement D_a are related to those of the electric field E_b as

$$D_a = \sum_b \epsilon_{ab} \, E_b \quad .$$

Consider the case: $\quad \epsilon_{ab} = \delta_{ab} \, \epsilon \quad$ where

$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{A}{\omega_0^2 - \omega^2 - i\omega\gamma} \quad \textit{and} \quad \omega \simeq \omega_0 \quad .$$

Consider a transverse plane wave travelling in this medium in the positive x direction. Using Maxwell's equations in this medium, describe the properties of such a plane wave.

That is, obtain the complete expressions for the electric ($\vec{E}$), magnetic ($\vec{B}$), and displacement ($\vec{D}$) fields; and the time averaged Poynting vector $\vec{S}$; the phase and the group velocities; and describe briefly the salient features of these waves.

Comprehensive Examination: Paper-II

January 6, 2014 09:30 - 12:30

(1) *Please use separate notebooks for Quantum Mechanics and Statistical Mechanics.*

(2) *In each notebook, at the beginning, please write your name and roll number clearly.*

(3) *You may use loose sheets, available in the exam hall, for rough work.*

Quantum Mechanics: All problems carry equal marks. Solve any *three* problems.

1. In 1956, parity violation in weak interactions was confirmed experimentally by observing emission of electrons in the β decay of spin-polarised Co^{60} nuclei cooled in the millikelvin range. Ignoring experimental subtleties, assume that the directions of the emitted electrons relative to the average direction of the nuclear spins, can be determined. Using this experimental information,

   (a) can parity (mirror reflection or space inversion) violation be tested?

   (b) what would you expect if parity invariance holds?

   (c) Is there any difference between mirror reflection invariance and space inversion invariance? Explain your answer in either case.

   (*Hint: Use the transformation properties under parity/inversion. The nuclei may be viewed as small spinning bodies.*)

2. The classical Lagrangian for a particle moving in a two dimensional plane minus the origin, is given by $L(\vec{r}, \dot{\vec{r}}) = \frac{1}{2}m\dot{\vec{r}}^2 - \frac{1}{2}m\omega^2 r^2 - \alpha\dot{\theta}$, $\theta = \tan^{-1}(y/x)$. Derive the *quantum mechanical Hamiltonian* and obtain the energy *eigenvalues*. Be careful about the non-commuting nature of operators in making passage from classical expressions to quantum operators.

3. Consider a trial wave function of the form, with a as the variational parameter,

$$\psi(x) = \begin{cases} C(1 - \frac{|x|}{a}) & \text{for} \quad |x| \le a \\ 0 & \text{for} \quad |x| > a \end{cases}$$

   for one dimensional harmonic oscillator. Find the optimum value of a and estimate the corresponding upper bound on the ground state energy.

4. Consider a particle of mass m moving in a spherical square well potential, $V(r) = -\,V_0$ for $0 < r \leq R$ and $V(r > R) = 0$. Obtain the conditions on V_0 , R such that there is *exactly one* bound state of zero angular momentum.

   For any normalised wave function $\Psi(r, \theta, \phi) = f(r)$ for $r \leq R$ and zero outside, show that $\langle p_i \rangle = 0$, $i = x, y, z$. Using the position-momentum uncertainty relation, estimate the average energy in such a state.

Comprehensive Examination: Paper-II

January 6, 2014 09:30 - 12:30

Statistical Mechanics: All problems carry equal marks. Solve any *three* problems.

---

1. The internal energy $U(P, V)$ of one mole of a substance is given by $U = cP^2V$, c is a constant. Find out the relation between P and V for a reversible adiabatic path on the $P - V$ plane.

2. Consider $N(>> 1)$ non-interacting (non-identical) particles. There are only two levels ϵ_1 and ϵ_2 ($\epsilon_2 > \epsilon_1$) accessible to each of these particles. Let n_1 and n_2 be the number of particles in the ground state and excited state respectively.

   Using microcanonical ensemble find the entropy, temperature and ratio (n_2/n_1). Consider the energy of the system is such that the corresponding $n_1,\ n_2 >> 1$.

3. Consider a gas of non-interacting, non-relativistic, identical bosons. Explain whether and why the Bose-Einstein condensation effect that applies to a three-dimensional gas applies also to a two-dimensional gas and to a one-dimensional gas.

   Hint: A fixed N number of particles are distributed according to the Bose-Einstein distribution. For 3-D the capacity of the excited states falls below N at non-zero temperature. The excess particles go to the ground state forming the Bose-Einstein condensate.

4. Consider a $1-$dimensional Ising chain of $N-$spins, the Hamiltonian is given by

$$H = -J \sum_{i=1}^{N} S_i S_{i+1}, \qquad S_{N+1} = S_1, \qquad S_i = \pm 1, \tag{1}$$

   Consider the system in the canonical ensemble and Calculate the leading order temperature dependence of the specific heat in the $T \to 0$ limit.

   Hint: Exact calculation of the partition function is not essential.