Comprehensive Examination - Paper I

Instructions

February 5, 2016 — 09:00 - 13:00

• *Please use separate notebooks for Classical Mechanics and Electrodynamics.*

• *In each notebook, at the beginning, please write your name and roll number clearly.*

• *You may use loose sheets, available in the exam hall, for rough work.*

• *All problems carry equal marks.* In each secion, you have to do *any three* out of the five problems.

• Passing criterion: **minimum of 10/30** marks in each section *and* a **total of 27/60** in both sections together.

• Duration of examinataion for both the parts together: **09:00 hours to 13:00 hours**

Paper-I: Part A

Classical Mechanics

All problems carry equal marks. Solve any *three* problems.

1].

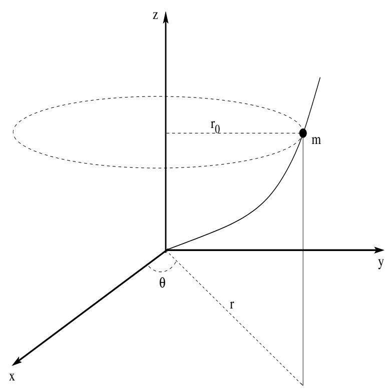

Figure 1: Figure for Q1.

(A) A bead of mass m slides without friction along a curved wire with shape $z = f(r)$, as indicated in Fig. 1, with $r^2 = x^2 + y^2$.
(i) Identify (and write down the expressions for) generalized coordinates, velocities, and momenta.
(ii) Write down the expressions for the Lagrangian (L) and the Hamiltonian (H).
(iii) Get the equation of motion from the Euler-Lagrange equations.

[full marks of (A) = 3]

(B) Now rotate the wire with a constant angular speed ω around the z-axis and
(i) Identify (and write down the expressions for) generalized coordinates, velocities, and momenta.
(ii) Write down the expressions for the Lagrangian (L) and the Hamiltonian (H).
(iii) Get the equation of motion from the Euler-Lagrange equations.
(iv) What should be the value of ω to stop the slide of the bead and keep it at a fixed $r = r_0$?

[full marks of (B) = 4]

(C) Now, from *case B* (i.e. the bead balanced at $r = r_0$ by rotation of the wire), you displace the bead slightly and release it at $r = r_0 + \epsilon$. Derive a condition on the function f, so that the bead comes back to r_0.

hint: ϵ is so small that you can neglect higher powers of ϵ, $\dot{\epsilon}$ etc.

[full marks of (C) = 3]

2].

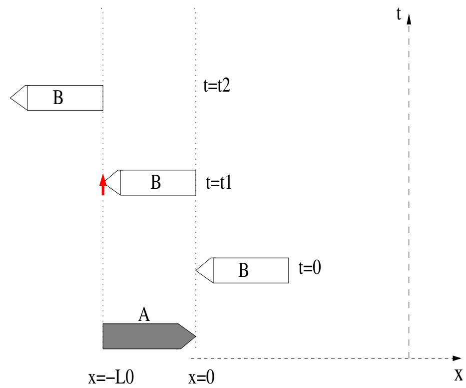

Figure 2: Figure for Q2A.

(A) There are two battle spaceships A and B of the same rest length L_0 of two opponent armies. Both of them can fire missiles from their tails perpendicular to the body of the spaceship. They move in opposite directions (along x-axis) with constant speed u (with respect to an observer at rest on the x-axis). There is no separation between them along y-direction and a very small constant separation 'h' along the z-axis so that the time taken by a missile to travel h is negligible. Assume A is below B by this tiny height h. Motion of B as viewed from A has been shown in Fig. 2 in the $x-t$ plane when u is non-relativistic. A fires a missile towards B (shown as a red arrow) when A's pilot sees the tail of B over its (A's) tip.

(i) Will the missile hit B if u is non-relativistic? Explain your answer.
(ii) For u relativistic, draw the motion of B as viewed from A as well as the motion of A as viewed from B in $x-t$ plane. This will actually help you to answer (iii) and (iv).
(iii) Will the missile hit B if u is relativistic? Explain your answer.
(iv) What is the difference between the times when B's pilot sees the missile fire and he sees the tip of A below B's tail when u is relativistic?

[full marks of (A) = 5]

(B) (i) A π^0 at rest decays into two photons. Calculate the energy and the momentum of each photon. Rest mass of π^0 is $m_{\pi^0,0} = 135$ Mev/c^2.
(ii) A K^0 at rest decays into two π^0s. Calculate the energy and the momentum of each π^0. Rest mass of K^0 is $m_{K^0,0} = 498$ Mev/c^2.
(iii) Two particles of rest mass $m_0 = 0.5$ Mev/c^2 collide head-on with the speed in the laboratory frame of $0.5\,c$, and stick together after the collision. The agglomerated particle is at rest in the laboratory frame. Calculate the mass of the agglomerated particle.

[Write the expressions in units of c whenever applicable.]

[full marks of (B) = 5].

3]. Orbits around a black-hole can be described in terms of the effective potential:

$$V_{\text{eff}}(r) = -\frac{1}{r} + \frac{l^2}{2r^2} - \frac{l^2}{r^3},$$

and the Lagrangian of a point particle in an orbit around the black-hole is:

$$L = \frac{1}{2}\dot{r}^2 - V_{\text{eff}}(r)$$

For the sake of simplicity we have assumed $G = 1$ and the reduced mass $\mu = 1$, so you don't need to worry about units and dimensions. l is the orbital angular momentum. (Note that the third term in the expression of $V_{\text{eff}}(r)$ makes it different from the conventional Newtonian expression, so we can call it a 'pseudo-Newtonian' term used to approximate GR effects. Although you don't need this information to solve the problems below.)

(A) Show that no orbit is possible for $l^2 < 12$.

[full marks of (A) = 2]

(B) Derive the condition for stable orbit.

[full marks of (B) = 2]

(C) Show whether orbits are stable or unstable at $r = 2, 5,$ and 15?

[full marks of (C) = 6]

4].

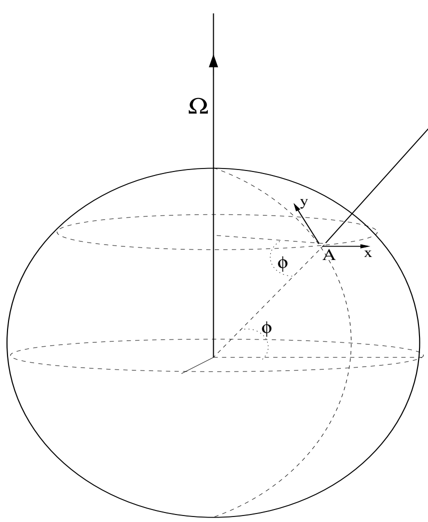

Figure 3: Figure for Q4A. $\vec{\Omega}$ is the angular velocity of the earth. A is a point on earth with latitude ϕ. $x-y-z$ is a local right-handed Cartesian coordinate system at A where z points to the zenith, x points to the east, and y points to the north.

(A) With the help of Fig. 3, show that a river flowing south in northern hemisphere tends to break its west bank due to Coriolis effect.

[full marks of (A) = 3]

(B) For a system of three identical mass points located at $(a, 0, 0)$, $(0, a, 2a)$, and $(0, 2a, a)$; write down the inertia tensor and find the principal moments of inertia (with respect to the origin).

[full marks of (B) = 3]

(C) Find the frequencies of the normal modes the coupled pendulum as shown in Fig. 4. Use small angle approximations and assume that $s - s_0 = l(\theta_2 - \theta_1)$, where s_0 is the length of the spring when the two bobs are at equilibrium (i.e. $\theta_1 = 0,\ \theta_2 = 0$) and s is the length of the spring when the bobs are at θ_1 and θ_2 (i.e. at the positions depicted in the figure).

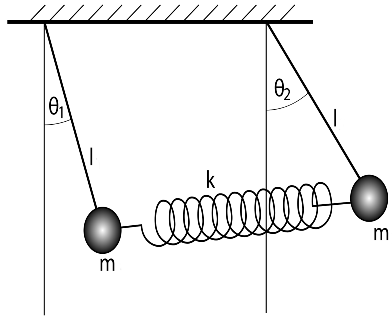

Figure 4: Figure for Q4C.

[full marks of (C) = 4]

5]. (A) Consider the following infinitesimal transformation parametrized by a,

$$\delta x = \xi(x,t)\delta a \ , \quad \delta t = \eta(x,t)\delta a \ , \tag{1}$$

Show that a first order differential equation, $f(x,t,\dot{x}) = 0$ ($\dot{x} \equiv \frac{dx}{dt}$) is invariant under the above transformation if,

$$\xi(x,t)\frac{\partial f}{\partial x} + \eta(x,t)\frac{\partial f}{\partial t} + \chi(x,t,\dot{x})\frac{\partial f}{\partial \dot{x}} = 0 \ ,$$

where,

$$\chi(x,t,\dot{x}) \ = \ \frac{\partial \xi}{\partial t} + \left(\frac{\partial \xi}{\partial x} - \frac{\partial \eta}{\partial t}\right)\dot{x} - \frac{\partial \eta}{\partial x}\dot{x}^2 \ .$$

[full marks of (A) = 3]

(B) The above statement can be easily generalized to second order differential equations. In this case the condition for invariance of $f(x,t,\dot{x},\ddot{x}) = 0$ under Eqn. (1) is given by,

$$\xi(x,t)\frac{\partial f}{\partial x} + \eta(x,t)\frac{\partial f}{\partial t} + \chi(x,t,\dot{x})\frac{\partial f}{\partial \dot{x}} + \kappa(x,t,\dot{x},\ddot{x})\frac{\partial f}{\partial \ddot{x}} \ = \ 0 \ , \tag{2}$$

where χ is as given above and,

$$\begin{aligned}\kappa(x,t,\dot{x},\ddot{x}) \ = \ & \frac{\partial^2 \xi}{\partial t^2} + \left(2\frac{\partial^2 \xi}{\partial x \partial t} - \frac{\partial^2 \eta}{\partial t^2}\right)\dot{x} + \left(\frac{\partial^2 \xi}{\partial x^2} - 2\frac{\partial^2 \eta}{\partial x \partial t}\right)\dot{x}^2 - \frac{\partial^2 \eta}{\partial x^2}\dot{x}^3 \\ & + \left(\frac{\partial \xi}{\partial x} - 2\frac{\partial \eta}{\partial t}\right)\ddot{x} - 3\frac{\partial \eta}{\partial x}\dot{x}\ddot{x}\end{aligned}$$

By specializing to harmonic oscillator for which,

$$f(x,t,\dot{x},\ddot{x}) \;=\; \ddot{x}+x \ ,$$

show that the invariance condition (2) reads as follows,

$$\begin{aligned}\xi + \frac{\partial^2\xi}{\partial t^2} - \left(\frac{\partial\xi}{\partial x} - 2\frac{\partial\eta}{\partial t}\right)x + \left(2\frac{\partial^2\xi}{\partial x\partial t} - \frac{\partial^2\eta}{\partial t^2} + 3x\frac{\partial\eta}{\partial x}\right)\dot{x} & \\ + \left(\frac{\partial^2\xi}{\partial x^2} - 2\frac{\partial^2\eta}{\partial x\partial t}\right)\dot{x}^2 - \frac{\partial^2\eta}{\partial x^2}\dot{x}^3 &= 0 \ . \end{aligned} \tag{3}$$

[full marks of (B) = 2]

(C) The generator of the infinitesimal transformation (1) is given by,

$$X \;=\; \xi(x,t)\frac{\partial}{\partial x} + \eta(x,t)\frac{\partial}{\partial t} \ .$$

It is easy to check that the invariance condition (3) admits the following generators as solutions,

$$\begin{aligned} X_1 &= (1+x^2)\sin t\frac{\partial}{\partial x} - x\cos t\frac{\partial}{\partial t} \ , \\ X_2 &= (1+x^2)\cos t\frac{\partial}{\partial x} + x\sin t\frac{\partial}{\partial t} \ , \\ X_3 &= \frac{\partial}{\partial t} \ . \end{aligned}$$

Show that the above generators form the Lie algebra $so(3)$.

[full marks of (C) = 5]

[Electrodynamics question paper starts on the next page]

Paper-I: Part B

Electrodynamics

All problems carry equal marks. Solve any *three* problems.

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1. Consider a parallel plate capacitor having perfectly conducting plates with plate separation d. It is filled with two layers (1 & 2) of material with dielectric constants ε_1, ε_2, conductivities σ_1, σ_2 and thicknesses d_1 and d_2, respectively. A potential V is placed across the capacitor. (Neglect edge effects). Determine

   (a) the electric field in material (1) and (2).

   (b) the current flowing through the capacitor.

   (c) the total surface charge density on the interface between (1) and (2).

   (d) the free surface charge density on the interface between (1) and (2).

   [3+2+2.5+2.5]

2. Consider a spherical shell of zero thickness, inside and outside the sphere there being empty space. Assume that the potential on it depends only on the polar angle θ and is given by V(θ).

   (a) Find out the potential inside and outside the sphere and the charge distribution on the sphere in terms of V(θ).

   (b) Compute the same for $\mathrm{V}(\theta) = \mathrm{V}_0 \cos^2\theta$.

   Hint:

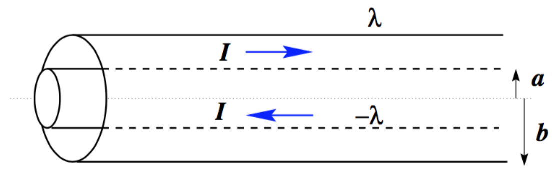
$$P_0(\cos\theta) = 1, P_1(\cos\theta) = \cos\theta, P_2(\cos\theta) = \frac{3\cos^2\theta - 1}{2} \tag{4}$$

$$\int_0^\pi P_m(\cos\theta)P_n(\cos\theta)\sin\theta d\theta = 0 \qquad m \neq n \tag{5}$$

$$\int_0^\pi P_n^2(\cos\theta)\sin\theta d\theta = \frac{2}{2n+1} \tag{6}$$

   [5+5]

3. A circular cylindrical transmission line is made of two coaxial cylinders with the radii a and b, $a < b$ and linear charge densities λ and $-\lambda$, respectively. The outer cylinder carries current I, that returns along the inner cylinder.



   (a) Compute the electric and magnetic field, and as well as the energy flux S (Poynting vector).
   (b) Obtain the total energy flux in the transmission line.
   (c) Show that, if a resistor R is connected to an end of the line, the power dissipated equals the total energy flow due to S.

[5+2+3]

4. Consider a square loop of side a, carrying a current I.

   (a) Compute the magnetic field at a distance z above the center of the square loop.
   (b) Can you identify the field with a known object when $z >> w$? Show it by explicit computation.

[5+5]

5. Consider two long, straight wires, separated by a distance d and carrying currents I in opposite directions.

   (a) Compute the magnetic field $\mathbf{H}$ and describe it in terms of a magnetic scalar potential Φ_M, with $\mathbf{H} = -\nabla\Phi_M$.
   (b) Do you find any unusual behavior of Φ_M at the origin? Why?
   (c) If the wires are parallel to the z axis with positions, $x = \pm\frac{d}{2}$, $y = 0$, show that in the limit of small spacing, the potential is approximately that of a two-dimensional dipole,

$$\Phi_M \approx -\frac{Id\sin\phi}{2\pi\rho} + O(\frac{d^2}{\rho^2}) \tag{7}$$

[4+2+4]

Comprehensive Examination

Instructions

February 8, 2016 **09:00 - 13:00**

• *Please use separate notebooks for Statistical Mechanics and Quantum Mechanics.*

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Comprehensive Examination: Paper-II

Quantum Mechanics

February 8, 2016 **09:00 - 13:00**

All problems carry equal marks. Solve any *three* problems.

1(a): Derive the continuity equation relating the rate of change of probability density $\Psi^*\Psi$ to the gradient of a probability current j and find expression for j.
[5 marks]

1(b): For plane wave solution $\Psi(x,t) = Ae^{ikx-i\omega t}$ find j and express your answer in terms of the particle velocity $\frac{p}{m}$ [A is in general complex]. **[5 marks]**

(2): A *coherent state* of a one-dimensional simple harmonic oscillator is defined to be an eigenstate of the (non-Hermitian) annihilator operator a:

$$a|\lambda\rangle = \lambda|\lambda\rangle,$$

where λ is, in general, a complex number.

2(a): Prove that

$$|\lambda\rangle = e^{-|\lambda|^2/2}\ e^{\lambda a^\dagger}|0\rangle$$

is a normalized coherent state. **[3 marks]**

2(b): Prove the minimum uncertainty relation for such a state. **[3 marks]**

2(c): Write $|\lambda\rangle$ as

$$|\lambda\rangle = \sum_{n=0}^{\infty} f(n)|n\rangle.$$

Show that the distribution of $|f(n)|^2$ with respect to n is of the Poisson form. Find the most probable value of n, hence of E. **[4 marks]**

(3): A particle of spin $\frac{1}{2}$ and magnetic moment μ moves in the homogeneous magnetic field

$$\mathcal{H}_z = \mathcal{H}_0 + kz\,, \quad \mathcal{H}_y = -ky\,, \quad \mathcal{H}_x = 0 \quad (div\mathcal{H} = 0).$$

3(a): Express the time-dependence of the position-coordinate operators $\hat{x},\ \hat{y},\ \hat{z}$.

[5 marks]

3(b): Find the mean value of the position coordinates and the time-dependence of the dispersion of the position coordinates if the wave function of the particle at the time $t = 0$ has the form

$$\psi = \varphi(x,\ y,\ z)e^{\frac{ip_0 x}{\hbar}}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}.$$

[5 marks]

(4): A system consists of two distinguishable particles, each with intrinsic spin $\frac{1}{2}$. The spin-spin interaction of the particles is $J\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}$, where J is a constant. An external magnetic field $\mathcal{H}$ is applied. The magnetic moments of the two particles are $\alpha\boldsymbol{\sigma_1}$ and $\beta\boldsymbol{\sigma_2}$.

4(a): Choose the magnetic field along the z axis and write the complete Hamiltonian. **[3 marks]**

4(b): Find the exact energy eigenvalues of this system. **[7 marks]**

(5): A quantum-mechanical system in the absence of perturbations can exist in either of the two states 1 or 2 with energies E_1 or E_2. Suppose that it is acted upon by a time-independent perturbation

$$V = \begin{pmatrix}0 & V_{12}\\ V_{21} & 0\end{pmatrix},$$

where $V_{21} = V_{12}^*$. If at time $t = 0$, the system is in state 1, determine the amplitudes for finding the system in either state at any later time.

[10 marks]

Paper-II: Part B

Statistical Mechanics

All problems carry equal marks. Solve any *three* problems.

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(1): (A) Starting from $dE = TdS - PdV$ and the equation of state for an ideal gas, show that E can only depend on temperature. **[6 marks]**

(B) What is the most general equation of state consistent with an internal energy that only depends on temperature? **[4 marks]**

(2): (A) Consider particles in a three-dimensional harmonic oscillator potential. What is the density of states? **[4 marks]**

(B) These levels are occupied by $N \gg 1$ distinguishable particles; this picture is justified for quantum particles at low densities and high temperatures. The system is allowed to exchange energy with a bath at temperature T. Write an expression for the energy of the system. Thereby, deduce the specific heat. **[5 marks]**

(C) Show that the specific heat obtained in (b) is consistent with the equipartition theorem. **[1 mark]**

(3): Consider a simple one-dimensional model for polymers such as rubber. The polymer is composed of N links, each of length l. Each link has two possible states, pointing either left or right. There is no difference in energy between the two states. The total length of the rubber band is the net displacement from the first link to the last, denoted by Ll. See Fig. 1 for a sample configuration.

(A) Find the number of possible configurations for given (N, L). Find the entropy $S(N, L)$. Assume $N \gg 1$ and use Stirling's approximation to simplify the expression. *Hint: take N_R to be the number of links that point right. Find entropy $S(N, N_R)$ first. Stirling's approximation:* $\ln N! \sim N \ln N - N$. **[3 marks]**

(B) Find the free energy of the system as a function of N, L and temperature T. **[1 mark]**

(C) Find the tension in the string (L here is analogous to the volume of a gas in three dimensions, and the tension F is analogous to pressure). The answer may be simplified by assuming $L/N \ll 1$. Show that the polymer has negative tension (it tends to implode). **[4 marks]**

(D) As temperature increases, does the tension increase or decrease? Rationalise your answer. **[2 marks]**

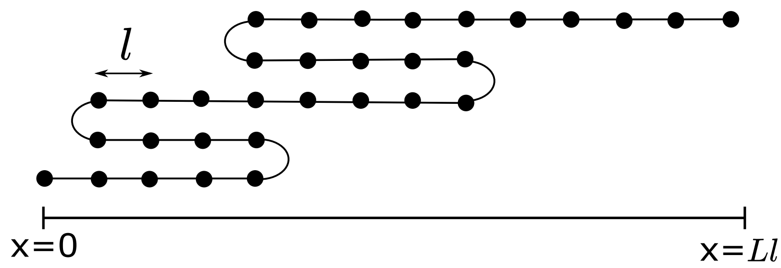

Figure 1: Simplified model for a rubber polymer

(4): Consider a non-relativistic Fermi gas at absolute zero temperature in *two* dimensions.

(A) Find the Fermi momentum as a function of system area and density. [**3 marks**]

(B) Show that the average energy per particle is $\epsilon_F/2$, where ϵ_F is the Fermi energy. [**3 marks**]

(C) Show that the Fermi pressure is E/A, the energy per area of the system. [**3 marks**]

(D) Find the bulk modulus $(-A\partial P/\partial A)$ is given by $2E/A$. [**1 mark**]

(5): Consider N bosons in a *two dimensional* harmonic oscillator potential.

(A) The bosons occupy the discrete eigenstates of harmonic oscillator potential. Show that there will be no Bose condensation in any such discrete system as temperature is lowered. [**4 marks**]

(B) Suppose the number of particles goes to infinity and the spacing between the energy levels goes to zero, so that the energy levels can be thought of as a continuous spectrum.

- What is the density of states in this case? [**2 marks**]
- As temperature is lowered, will we have Bose condensation? [**4 marks**]