ROOT SYSTEMS

JULY 4–8, 2011

(1) (a) If R is a root system in V, then $R^\vee := \{\alpha^\vee : \alpha \in R\}$ is a root system in V^*.
(b) $R^{\vee\vee} = R$ (under the natural identification between V and V^{**}).
(c) Recall we have a natural isomorphism $\phi : GL(V) \to GL(V^*)$ defined by $g \mapsto ({}^t g)^{-1}$ where for $g : V \to V$, ${}^t g$ is the induced map defined by ${}^t g(f)(v) := f(gv)$ for $f \in V^*, v \in V$. If $\alpha \in R$ and $g \in A(R)$, then $(g\alpha)^\vee = \phi(g)(\alpha^\vee)$.
(d) $\phi(s_{\alpha,\alpha^\vee}) = s_{\alpha^\vee,\alpha}$.
(e) $W(R) \cong W(R^\vee)$ and $A(R) \cong A(R^\vee)$.

(2) If (V, R) is a root system, prove that $(x|y) := \sum_{\alpha \in R} \langle \alpha^\vee, x\rangle \langle \alpha^\vee, y\rangle$ defines a positive definite symmetric bilinear form on V that is $A(R)$-invariant.

(3) If $R = \oplus_i R_i$ is a direct sum of root systems, prove that $W(R) \cong \prod_i W(R_i)$.

(4) (a) If (V, R) is an irreducible root system, recall that V is an irreducible $W(R)$-module. Prove that a linear map $T : V \to V$ which commutes with every element of $W(R)$ must be a scalar operator; in other words, V is an *absolutely irreducible* representation of $W(R)$. *Hint: This is the statement of Schur's lemma, but the field is now* $\mathbb{R}$, *rather than* $\mathbb{C}$. *But try to tweak the proof of Schur's lemma.*
(b) Prove that when R is irreducible, any $W(R)$-invariant form on V must be a multiple of the form defined in problem **??**.

(5) Let α, β be non-proportional roots. Prove that s_α and s_β commute $\iff$ $n(\alpha, \beta) = n(\beta, \alpha) = 0$. More generally, determine the order of the element $s_\alpha s_\beta \in W(R)$ as a function of $n(\alpha, \beta)n(\beta, \alpha)$.

(6) Let α, β be non-proportional roots in R. Then
(a) The set J of integers j such that $\beta + j\alpha \in R$ is an interval $[-q, p] \cap \mathbb{Z}$.
(b) If $S = \{\beta + j\alpha : j \in J\}$, then $s_\alpha(S) = S$ and $s_\alpha(\beta + p\alpha) = \beta - q\alpha$.
(c) $p - q = -n(\beta, \alpha)$.

(7) (a) If α, β are roots such that $n(\alpha, \beta) = n(\beta, \alpha) = -1$, then $\exists\, w \in W(R)$ such that $\beta = w\alpha$.
(b) If R is an irreducible root system, and α, β are roots of the same length, then $\exists\, w \in W(R)$ such that $\beta = w\alpha$.

(8) If R is a reduced, irreducible root system in the Euclidean space $V, (,)$, show that $\frac{(\beta,\beta)}{(\alpha,\alpha)}$ can only take one the values $1, 2, 1/2, 3, 1/3$. Further, at most two root lengths occur in R.

(9) Let R be an irreducible, non-reduced root system of rank ≥ 2.
(a) Let R_0 be the set of *indivisible* roots of R (i.e $\alpha \in R$ for which $\frac{1}{2}\alpha \notin R$). Prove that R_0 is a reduced irreducible root system and $W(R_0) = W(R)$.
(b) Let A be the set of roots α for which (α, α) is minimal ($= \lambda$, say). Then any two distinct positive roots in A are orthogonal.
(c) Let B be the set of $\beta \in R$ such that $(\beta, \beta) = 2\lambda$. Then $B \neq \emptyset$, $R_0 = A \cup B$, $R = A \cup B \cup 2A$.

for more problems, see Bourbaki's *Lie Groups and Lie algebras, Chapters 4-6.*

(10) (a) Prove that every symmetric bilinear form on V which is $W(R)$-invariant is also $A(R)$-invariant.
(b) Prove that $W(R)$ is a normal subgroup of $A(R)$.
(c) Let B be a basis of R and G be the subgroup of $A(R)$ which preserves B. Prove that $A(R)$ is the semidirect product of $W(R)$ and G.

(11) Let R be a reduced root system with basis $\{\alpha_i : i = 1 \cdots l\}$. If $\alpha = \sum_i c_i \alpha_i$ is a root, prove that $c_i(\alpha_i, \alpha_i)/(\alpha, \alpha) \in \mathbb{Z}$ for all i.

(12) Construct the root systems of exceptional types E_6, E_7, E_8, F_4, G_2.

(13) Let R be an irreducible reduced root system, C be a chamber of R, and $B(C)$, the corresponding basis. Prove that:
(a) There is a unique root θ such that $\theta \geq \alpha$ for every $\alpha \in R$, where the partial ordering is defined by $B(C)$.
(b) $\theta \in \bar{C}$.
(c) $(\theta, \theta) \geq (\alpha, \alpha)$ for all $\alpha \in R$.
(d) For every positive root $\alpha \neq \theta$, we have $n(\alpha, \theta) = 0$ or 1.
(e) For each root system $A - G$, find θ.
(f) The *Coxeter number* of R is defined to be $h := \text{height}(\theta) + 1$, where the height of a root is just the sum of the coefficients obtained when the root is written in terms of the basis $B(C)$.

(14) Let R be an irreducible reduced root system, C be a chamber of R, and B the corresponding basis. Let $\rho := \frac{1}{2} \sum_{\alpha \in R^+} \alpha$. Prove that:
(a) $s_\alpha(\rho) = \rho - \alpha$ for all $\alpha \in B$.
(b) $\langle \rho, \alpha^\vee \rangle = 1$ for all $\alpha \in B$.
(c) $\rho \in C$.

(15) Verify the following relation case-by-case. If R is an irreducible reduced root system of rank l and Coxeter number h, then $|R| = lh$.