"EVERY uniformity is generated by the family of its u. cont pseudo metrics"

Let $X, \mathcal{U}$ be a uniformity. Let $\{d_\alpha\}$ be the collection of all pseudo-metrics d_α s/t $d_\alpha : X \times X \to \mathbb{R}$ is uni. cont (w.r.t. the product uniformity on $X \times X$).
Let X_α denote X with the uniformity induced by d_α. Then as seen already $X \xrightarrow{id} X_\alpha$ is u.cont. (In fact, this is equivalent to u.continuity of d_α.)

<u>Theorem</u>: $(X, \mathcal{U})$ is the initial ~~top~~ uniformity w.r.t. the collection $X \xrightarrow{id} X_\alpha$.

<u>Proof</u>: Since $X \xrightarrow{id} X_\alpha$ is u. cont $\forall \alpha$, clear that $\mathcal{U}$ is finer than the initial top.
For the converse, we use the metrization lemma (and make the following construction).
~~ETS every by~~ Let $V \in \mathcal{U}$. Let $U_0 = X \times X$. Let U_1 be symmetric in $\mathcal{U}$ s/t $U_1 \subseteq U$. Choose $U_2^{symm} \in \mathcal{U}$ s/t $U_2 \circ U_2 \circ U_2 \subseteq U_1$, U_3^{symm} in U s/t $U_3 \circ U_3 \circ U_3 \subseteq U_2$ etc. By the metrization lemma, $\exists$ a pseudometric —call it d_U— such that

$$U_n \subseteq \{(x,y) \mid d_U(x,y) < \tfrac{1}{2^n}\} \subseteq U_{n-1}$$

Thus d_U is u.cont w.r.t. $\mathcal{U}$ (on the one hand). And, on the other,
$\{(x,y) \mid d_U(x,y) < \frac{1}{4}\} \subseteq U_1 \subseteq U$, so U is an entourage for the uniformity of d_U QED

<u>COR</u>: Any uniform space is (uniformly isomorphic to) ~~the~~ a subspace of a product of pseudo-metric spaces. Any Hdf uniform space is ~~the~~ a subspace of a product of metric spaces.

<u>Proof</u>: With notation as in the theorem, we have $X \hookrightarrow \prod_\alpha X_\alpha$. By the transitivity of initial uniformities, X is the subspace of the product.

For the second assertion, we make the following observation.
For every pseudo-metric space Y we can take the quotient metric space Y'.
The uniformity of the pseudo-metric on Y is the initial uniformity induced from $Y \twoheadrightarrow Y'$.

If X is a Hdf uniform space then we have $X \hookrightarrow \prod_\alpha X'_\alpha$ (where again notation is as above). Once again the result follows from transitivity of initial uniformities. ▨

<u>CoR</u>: (Criterion for when a topology is a uniform topology) Answer: iff it is completely regular (taken ~~as~~ without Hdf). <u>Proof</u>: $\because$ a top space is completely regular iff it is ~~the~~ a sub of a product of pseudo-metric spaces. QED