I §9. No. 8. One-point (or Alexandroff Compactification) of a LC space.

__Theorem__ (Alexandroff) X LC space. Then $\exists$ a homeomorphism say f of X into the complement of a point of a compact space $\tilde{X}$. Uniqueness: if $\tilde{X}'$ is another compact space with complement of a point being homeo to X, ~~[illegible]~~ via $g: X \to \tilde{X}'$, then the bijection from $\tilde{X}$ to $\tilde{X}'$ that is $g\circ f^{-1}$ on fX is a homeo.

__Proof:__ __Uniqueness.__ Let φ be the bijection from $\tilde{X}$ to $\tilde{X}'$ described in the assertion. ETS φ is open (for by switching the roles of $\tilde{X}$ and $\tilde{X}'$ we see that φ^{-1} is also open, so that φ is a homeo. Let V be open in $\tilde{X}$. If $V \subseteq X$, then φV is clearly open in gX, and so open in $\tilde{X}'$ (since gX is open in $\tilde{X}'$, because it is the complement of a point in a Hdf space). Now suppose that $V \not\subseteq X$, ~~so that~~ Then $CV \subseteq X$ and CV being closed ~~is comp~~ in $\tilde{X}$, it is compact. $\varphi CV = (g\cdot f^{-1})CV$ and since $g\circ f^{-1}$ is continuous ~~[illegible]~~ ($g\circ f^{-1}: fX \to \tilde{X}'$), φCV is compact and so closed. But $\varphi CV = C\varphi V$ (since φ is a bijection), so φV is open.

__Existence:__ As a set, take $\tilde{X} := X \amalg \{\omega\}$. Define a topology on $\tilde{X}$ by taking the open sets to be of the form $U \cup (\{\omega\} \cup (X \setminus K))$ where U is a open set in X and K is compact in X. We check that this defines a topology: E.g. $\bigcup_{i\in I} \{\omega\} \cup (X\setminus K_i) \underset{I\neq\emptyset}{=} \{\omega\} \cup (X \setminus \bigcap_{i\in I} K_i)$; ~~note that K_i being compact is closed in $\tilde{X}$ and so $\cap K_i$ is also closed in $\tilde{X}$ if the intersection $\cap K_i$ is non-empty~~ Since $I \neq \emptyset$, $\cap K_i$ is closed in K_i ($\because$ the K_i are closed in X and so is their intersection), for any choice of i, so compact ($\because$ closed in compact is compact). Thus $\{\omega\} \cup (X \setminus \bigcap_{i\in I} K_i)$ is open in $\tilde{X}$.

CHECK ~~Check~~:
- The topology induced on X from $\tilde{X}$ is the original top of X.
- $\tilde{X}$ Hdf [uses LC ness of X]
- $\tilde{X}$ compact. QED

__Remark:__
- If X is compact, then ω is ~~of little use~~ an isolated point of $\tilde{X}$
- $\tilde{X}$ is said to be constructed by adjoining "a point at infinity"
- $X = \mathbb{R}^2$. ~~[illegible]~~ Then $\tilde{X} \approx S^2$ and the inclusion of $\mathbb{R}^2 \hookrightarrow S^2$ is by the "stereographic projection".
- LC ness of X is used only in showing $\tilde{X}$ Hdf. ~~Otherwise~~ For the rest, we may as well take X to be just Hdf and proceed.