I §8 Compact spaces & Locally Compact spaces No.2. Regularity of a compact space

Propn: X^{compact}, $x \in X$. TFAE for a filter base $\mathcal{B}$ consisting of closed nbhds of x. ① $\mathcal{B}$ is a FSN for x ② $\bigcap_{B \in \mathcal{B}} B = \{x\}$.

Proof: $\Rightarrow$ by Axiom (H^1) of Hdfness. $\Leftarrow$: ETS $\mathcal{B} \to x$. In turn, ETS x is the only cluster point. But this is clear from hypothesis. ▨

CoR: Compact $\Rightarrow$ Regular. **Proof:** Let $\mathcal{B}$ be the set of closed nbhds of a point x. By Hdf axiom (H^1), $\bigcap_{B \in \mathcal{B}} B = \{x\}$. Now (O_{III}') holds by Propn. ▨

[**Propn:** Compact $\Rightarrow$ Normal. **Proof:** Suppose A, B be closed. ~~s/t $A \cap B = \emptyset$~~
Suppose $\nexists$ $U^{\text{open}} \supseteq A$ & $V^{\text{open}} \supseteq B$ s/t $U \cap V = \emptyset$. Then the collection of nbhds of A and nbhds of B together form a filter base, say $\mathcal{B}$. Let x be a cluster point of $\mathcal{B}$. By regularity (axiom (O_{III}')), $x \in A$ lly $x \in B$. Thus $A \cap B \neq \emptyset$. QED

"has important consequences"

Remark: X^{infinite} with finite complement topology. Then any two non-empty open sets intersect. Thus X is not Hdf, does not satisfy (O_{III}), is not normal.

I§8. No.3 Quasi-Compact Sets; Compact sets; Relatively compact sets.

A subset of a top space is QC/Compact if ~~its~~ w.r.t the subspace top it is QC/Compact.
A subset is QC iff $\forall$ every cover of A by open sets of X, there exists a finite subcover.

Examples:
- Finite subsets are QC (in any top. space). Singletons and the empty set are compact.
- Let $\{x_n\} \to x$ in X. Then $\{x_n\} \cup \{x\}$ is QC.

Propn: Every closed subset of a QC/Compact space is QC/Compact.
Proof: ~~Easy~~ (C'').

Propn: Every compact subset of a Hausdorff space is closed.
Proof: $A^{\text{compact}} \subseteq X^{\text{Hausdorff}}$. Let $x \in \bar{A}$. The nbhd filter $\mathcal{N}_x$ of x leaves a trace on A.
Let $y \in A$ be a cluster point of the trace. ~~(AAA)~~ Since the trace is finer than $\mathcal{N}_x$, y is also a cluster point of $\mathcal{N}_x$. But by (H^5) applied to N_x, we conclude ~~...~~ $y = x$, so $x \in A$. ▨

CoR: In a compact space X, a subset A is compact iff it is closed.

Propn5: ~~A~~ A finite union of ~~compact~~ QC subsets is QC.

Def: Relatively QC/Compact: a subset which is contained in a QC/Compact subset.