No5 Canonical decomposition of a continuous mapping

Let $f: X \to Y$ be a cont. map of top. spaces. Define equivalence relation R on X by $x \sim x'$ if $fx = fx'$. Then we have a factorization of f as:

$$X \underset{\text{quotient map}}{\xrightarrow{\pi}} X/R \xrightarrow{g} f(X) \underset{\text{inclusion}}{\overset{i}{\hookrightarrow}} Y \quad \text{where } g \text{ is a bijection}$$

If X/R is given the quotient topology & $f(X)$ the subspace topology, then π, g, and i are continuous. Thus g is a continuous bijection. In general, g is not bicontinuous.

Propn TFAE ① g is a homeomorphism ② The image of every open set of X that is saturated with respect to R is open in $f(X)$ ③ Same as ② with "closed" in place of "open" at both places.

Proof: Under ② g would be an open map, under ③ a closed map. ☒

Glueing subspaces in a cover along common intersections

Example: Let X_i, $i \in I$, be a covering of a topological space X. On $\amalg X_i$, let R be the equivalence relation $x \sim y$ for x in X_i and y in X_j if x and y are the same point in X. Then we have a bijection $\amalg X_i/R \xrightarrow{\sim} X$. Let us give $\amalg X_i/R$ the final topology w.r.t. $X_i \hookrightarrow \amalg X_i/R$ (or, equivalently, by transitivity of final topologies, the quotient topology w.r.t. R of the direct sum $\amalg X_i$). Then $\amalg X_i/R \to X$ is continuous. It is not in general bicontinuous. (e.g. let X be a non-discrete space & $\{X_i\}$ consist of all singletons. In this case $\amalg X_i/R$ is discrete) A subset A of the set X is open as a subset of $\amalg X_i/R$ iff $A \cap X_i$ is open in X_i $\forall i$. Thus here are two sufficient conditions for $\amalg X_i/R \to X$ to be a homeomorphism:

① The interiors of X_i cover X, or ② $X_i, i \in I$, is a LF closed covering.

An important sufficient condition for the continuous bijection g to be a homeomorphism:

Propn: Suppose that $X \xrightarrow{f} Y$ is a continuous surjection. Define equivalence relation R on X by $x \sim x'$ if $fx = fx'$. We then have a continuous bijection $X/R \xrightarrow{\tilde{f}} Y$ induced by f, where X/R is given the quotient topology. If $\exists$ a continuous section $X \xleftarrow{\sigma} Y$ of f (ie, a continuous map s/t $f \circ \sigma = id_Y$) then $X/R \xrightarrow{\tilde{f}} Y$ is a bijection. In this case $\sigma(Y)$ is homeomorphic (as a subspace of X) to Y.

Proof: $Y \xrightarrow{\sigma} X \xrightarrow{\pi} X/R$ is continuous ($\because$ σ and π are) and $\sigma \circ \pi$ is the inverse of $\tilde{f}$.
$\tilde{f} \circ \pi|_{\sigma(Y)}$ is the continuous inverse of $\sigma: Y \to \sigma(Y)$. QED

Remark: A section σ is determined by its image $\sigma(Y)$. By abuse of language, $\sigma(Y)$ itself is called a section.