No.5 Pasting together of Topological spaces: Let $X = \coprod X_i$ as top spaces. Let R be an equivalence relation on X s/t each equivalence class has at most one element in each X_i. We may define: $X_{ij} := \{x_i \in X_i \mid \exists x_j \in X_j \text{ s/t } x_i \sim x_j\}$ and $h_{ij} : X_{ij} \to X_{ji}$ by $x_i \mapsto x_j$ where x_j is the ! elt of X_j s/t $x_i \sim x_j$. We then have:

① $X_{ii} = X_i$ and $h_{ii} =$ identity on X_i ② for i,j,k $h_{ij}(X_{ij} \cap X_{ik}) \subseteq X_{jk}$ and $h_{jk} \circ h_{ij} = h_{ik}$ on $X_{ij} \cap X_{ik}$.

Conversely, given $X_{ij} \subseteq X$ and $h_{ij} : X_{ij} \to X_{ji}$ $\forall i,j$ s/t ① and ② hold, we may construct an R as above (uniquely).

We have injections $X_i \hookrightarrow X/R$.

Let X/R be the quotient space. By the transitivity property of final topologies, X/R is also the final topology for $X_i \hookrightarrow X/R$ (as i varies). The space X/R is said to be obtained from <u>pasting the spaces X_i along X_{ij} by means of the bijections h_{ij}</u>.

The original topology on X_i is clearly finer than the subspace top. induced from X/R (since $X_i \hookrightarrow X/R$ is continuous). It could in general be strictly finer. However, we have

<u>Propn:</u> Suppose that ① X_{ij} are open (resp. closed) in X_i and ② h_{ij} are homeomorphisms. Then each X_i is open (resp. closed) in X/R and the original topology on X_i coincides with the subspace topology induced from X/R (so $X_i \hookrightarrow X/R$ is a homeomorphism onto its image).

<u>Proof:</u> See Example 4 in the paragraph on final topologies. ◪