Linear Algebra Instructor: K N Raghavan
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These are notes for the lectures on Linear Algebra in the first week of SWIM-2026 at Kannur. The following remarks should provide context:

1. The proof of the basic fact mentioned in item 3 in §1 should be clear, although writing it down formally seems like it would be messy.[1] The first two items in §1 are just definitions.

2. Items 3, 7, and 8 in §2 are good exercises in "University Mathematics". The remaining items merely recall definitions.

3. In §3, the first item merely recalls the definition of row space and column space. Items 2 and 3 were discussed in class. The proof of the claim in item 5 should not be difficult: in fact, it should be visibly evident. The proof of the claim in item 4 could be more challenging.

4. As for items in §§4–6, they will likely require time to digest. The items in §4 except for the last one were discussed briefly in class. So was the result in item 2 in §6.

5. On the use of AI-assistants, see 7.

1 Elementary Row Operations and Row Reduced Echelon Forms

1. The <u>elementary row operations</u> on a matrix are of three types:

   (a) Interchange any two rows.[2]

   (b) Multiply any row by a non-zero scalar.

   (c) Add a multiple of some row to another. (Here "another" means that the row to which the addition is done is distinct from the row whose multiple is being added.)

   The inverse of an elementary row operation is also an elementary row operation of the same type.

2. A matrix is in <u>Row Reduced Echelon Form (RREF)</u> if the following three conditions are <u>all</u> satisfied:

   (a) The leading entry in any non-zero row is 1. ("Leading entry" means that the first non-zero entry from the left.)

   (b) (The "Echelon" Condition) For any two distinct rows, the leading entry of the lower row occurs in a column that is strictly to the right of the column in which the leading entry of the higher row occurs. This is interpreted to also mean that any zero rows appear below non-zero rows.

[1] I am not advising attempting a formal proof. It would mean spending a lot of time for very little gain.

[2] We could make this operation either more expansive by allowing an arbitrary permutation of rows, or more restrictive by allowing interchanges only of adjacent rows. It would not make a difference. It is a basic fact about symmetric groups that they are generated by transpositions, or even by simple transpositions. In our context, this means that, by repeatedly doing interchanges of adjacent rows, we can achieve any permutation of the rows.

(c) If a column contains the leading non-zero entry of any row, then all the other entries in that column are zero.

3. **Theorem**: *Any matrix can be brought to RREF by performing elementary row operations.*
Note: This fact justifies the nomenclature "row reduced". The RREF is in fact unique (see item 4 in §3) in spite of there being many ways to get to it from the given matrix. The nomenclature "form" is meant to convey this uniqueness.

2 Subspaces, Linear Combinations, Spanning Sets, and Bases

1. A subset of $\mathbb{R}^n$ is a subspace if it is non-empty and closed under the operations of addition and scalar multiplication. (Note: Any non-empty subset that is closed under scalar multiplication contains the zero vector.)

2. Given a subset S of $\mathbb{R}^n$, a linear combination of elements of S is an expression—or the corresponding vector—of the form $a_1x_1 + \ldots + a_rx_r$, where $x_1, \ldots, x_r$ are (finitely many) elements of S and $a_1, \ldots, a_r$ are real numbers. (There could be repetitions among the vectors $x_1, \ldots, x_r$.)

3. Given any set S of vectors in $\mathbb{R}^n$, the set W of all linear combinations of elements of S is a subspace of $\mathbb{R}^n$. The space W is said to be the span of S. (Note: S is allowed to be empty, in which case $W = \{0\}$. Any linear combination is of finitely many elements at a time.)

4. Let W be a subspace of $\mathbb{R}^n$. A set of S of vectors in $\mathbb{R}^n$ is said to be a spanning set for W—we also say that S spans W—if $S \subseteq W$ and every element of W is a linear combination of elements of S.

5. A set S of vectors in $\mathbb{R}^n$ is said to be linearly independent if no non-trivial linear combination of elements of S equals zero: a non-trivial linear combination means an expression of the form $a_1v_1 + \ldots + a_rv_r$, where $v_1, \ldots, v_r$ are distinct elements of S and not all the coefficients $a_1, \ldots, a_r$ are zero.

6. Given a subset S of a subspace W of $\mathbb{R}^n$, we say that S is a basis for W if the following conditions both hold:

   - S spans W;
   - S is linearly independent.

   The "standard basis" elements $e_1, \ldots, e_n$ form a basis for $\mathbb{R}^n$.

7. Let S be a linearly independent set of vectors in some $\mathbb{R}^n$ and w be an element of the same $\mathbb{R}^n$. Then the set $S \cup \{w\}$ is linearly dependent if and only if w belongs to the span of S. In particular:

   (a) Given a subspace W of $\mathbb{R}^n$, any maximally linearly independent subset of W forms a basis for W.

   (b) Given a subset S of $\mathbb{R}^n$, any maximally linearly independent subset of S forms a basis for the span of S.

8. Let $v_1, \ldots, v_m$ be m vectors in $\mathbb{R}^n$. Think of them as $1 \times n$ row vectors and let A be the $m \times n$ matrix whose rows are $v_1, \ldots, v_m$. Let R be the RREF of A. Then, the vectors $v_1, \ldots, v_m$ are linearly independent if and only if R has no zero rows. In particular, any subset of $\mathbb{R}^n$ with more than n vectors is linearly dependent.

3 Row Space and Column Space of a matrix

1. Let A be an $m \times n$ matrix.

   - The rows of A can be thought of as vectors in $\mathbb{R}^n$ (here we think of $\mathbb{R}^n$ as row matrices of size $1 \times n$). The span of the rows of A is called the row space of A.
   - The columns of A can be thought of as vectors in $\mathbb{R}^m$ (here we think of $\mathbb{R}^m$ as column matrices of size $m \times 1$). The span of the columns of A is called the column space of A.

2. Performing elementary row operations on a matrix does not change its row space.

3. Show by means of an example that the column space does not remain the same when elementary row operations are performed on a matrix. Note: However, the dimension of the column space stays the same, as explained in §6.

4. **Theorem**: *If two matrices with the same number of columns are both in RREF and have the same row space, then they coincide after deletion of zero rows.*

   In particular, this means the following: Consider the equivalence relation on matrices of a fixed size—say $m \times n$—given by $A \sim B$ if A and B have the same row space. Then, in any equivalence class, there is a unique matrix in RREF. We are thus justified in talking about the RREF of any given matrix.

5. **Theorem**: *The non-zero rows of the RREF of a matrix A form a basis for the row space of A.*

   Hint: The row space of A is the same as that of R (item 2 above). The non-zero rows of R span the row space of R. That they are linearly independent should be visibly evident.

6. **Theorem**: *For any subspace W of $\mathbb{R}^n$ (here n is arbitrary), any two bases have the same cardinality. This number, which is finite (in fact, at most n, by item 8 of §2) is called the dimension of W.*

   Hint: Consider two bases $\mathscr{B}_1$ and $\mathscr{B}_2$ of W. Let m_1 and m_2 be their respective cardinalities. Use these two sets of vectors to form matrices A_1 and A_2 of sizes $m_1 \times n$ and $m_2 \times n$ respectively. Let R_1 and R_2 be their respective RREFs. Observe that the row spaces of A_1, A_2, R_1, and R_2 are all equal to W. Invoking item 8 of §2, we conclude that there are no zero rows in either R_1 or R_2. Now, by item 4 above, we have $R_1 = R_2$, and in particular $m_1 = m_2$.

7. True or False? If two matrices of the same size have the same row space, then one can be transformed to the other by means of elementary row operations.

   Hint: True. Convert each to RREF by means of elementary row operations. These must be equal, by an application of item 4 above.

4 The Solution Space of a Homogeneous System $Ax = 0$

1. The set of solutions of a homogeneous system $Ax = 0$ of linear equations, where A is an $m \times n$ matrix, forms a subspace of $\mathbb{R}^n$, called the solution space of the system $Ax = 0$.

2. (Orthogonal Subspace) Given a subset S of $\mathbb{R}^n$, the set $S^\perp := \{y \in \mathbb{R}^n \mid y \cdot x = 0 \ \forall \ x \in S\}$ is a subspace, called the orthogonal subspace to S.

   (a) Letting W be the span of S, we have $W^\perp = S^\perp$.

   (b) The solution space of a homogeneous system $Ax = 0$ of linear equations is the orthogonal subspace to the row space of A. (Note: Thus, solving $Ax = 0$ amounts to finding vectors orthogonal to every row of A.)

   (c) As a particular case of the previous item, if matrices A and B have the same number of columns and the same row space, then the solution spaces of $Ax = 0$ and $Bx = 0$ are the same.

3. Consider a homogeneous system $Ax = 0$ of linear equations. We outline a construction of a basis for its solution space:

   Let R be the RREF of A. Recall the algorithm outlined in class for writing down the solution set. The variables corresponding to the columns of R not containing any leading 1 are considered "free". The other variables are "bound". The bound variables can all be expressed in terms of the free variables. To form a basis of the solution space, we do the following. For each one of the free variables, put it equal to 1 and the rest of the free variables to be 0. Work out the corresponding values of the bound variables. In this way we get as many vectors as there are free variables. Prove that they form a basis for the solution space of $Ax = 0$.

4. **Rank-Nullity Theorem**: *Let A be an $m \times n$ matrix. Then the dimension of the row space of A plus the dimension of the solution space of $Ax = 0$ equals n.*

   Hint: Let R be the RREF of A. The dimension of the row space of A is the number of non-zero rows of R (item 5 of §3). From the defining properties of RREF, this clearly equals the number of columns in R containing the leading non-zero entry (which is equal to 1) of some non-zero row of R. On the other hand, by the previous item, the dimension of the solution space of $Ax = 0$ equals the number of "free variables", or in other words the number of columns in R that do not contain the leading non-zero entry of any row. The sum of these dimensions therefore equals the total number of columns in R, which is n.

5. Let W be a subspace of $\mathbb{R}^n$. Then the dimension of W plus the dimension of the orthogonal subspace $W^\perp$ equals n.

   Hint: Choose a basis for W. We can do this, for example, by building step-by-step a maximally linearly independent subset of W. Letting the basis elements be the rows of a matrix, we get a matrix with n columns, say A. The row space of A is W and the solution space of the homogeneous system $Ax = 0$ is $W^\perp$. Now apply the Rank-Nullity Theorem.

5 The Solution Set of a General System $Ax = b$ of Linear Equations

1. A system $Ax = b$ of linear equations admits a solution if and only if b belongs to the column space of A.

2. Let x_0 be a fixed particular solution of a system $Ax = b$ of linear equations. If W is the solution space of the homogeneous system $Ax = 0$, then the complete solution set of $Ax = b$ equals $x_0 + W := \{x_0 + w \mid w \in W\}$. Thus the solution set is a "shifted subspace" (if it is non-empty).

6 Rank of a Matrix

1. Suppose matrices A and B have the same number of columns and the same row space (for example, we could take B to be the RREF of A). Then the dimensions of their column spaces are also the same.

   Hint: As seen in item 2 of §4, the solution spaces of $Ax = 0$ and $Bx = 0$ are the same. This means that a given set of columns of A is linearly independent if and only if the corresponding set of columns of B is linearly independent. Indeed, a linear relation among selected columns corresponds to a solution of the homogeneous system whose remaining coordinates are zero. In particular, a given set of columns of A is maximally linearly independent if and only if the corresponding columns of B is maximally linearly independent. Such maximally linearly independent subsets form bases (item 7 of §2).

2. **Theorem**: *For any matrix A (of some finite size $m \times n$), the dimension of its row space equals the dimension of its column space. This common number is called the rank of A.*

   Hint: Let R be the RREF of A. We first prove that the result holds for R and then use that to argue that it holds also for A. The number of non-zero rows of R is the dimension of the row space of R (item 5 of §3). It is easy to see that the columns in which the leading entries of the non-zero rows of R occur form a basis for the column space of R, which in fact is the co-ordinate subspace of $\mathbb{R}^m$ defined by the vanishing of all the co-ordinates corresponding to the zero rows of R. Thus the result holds for R.

   Passing now to A, the row spaces of A and R are the same (item 2 of §3); and, although the column spaces of A and R are in general very different, they have the same dimension (see the previous item).

7 On the Use of ChatGPT and Other AI-Assistants

1. ChatGPT (free version) was consulted on this document (including this section!). Suggestions and critiques generated through interaction with it helped improve these notes. Thank you, ChatGPT.

2. One of the suggestions made by ChatGPT was to include "anchoring" examples. But I have resisted that temptation, since these notes are meant as "lecture companions", not as "substitute textbooks" (to repurpose some of ChatGPT's own phrases!). Interested readers could readily ask ChatGPT (or another AI assistant) to generate examples, additional exercises, or alternate explanations.

3. **Important Caveat**: AI-generated mathematical arguments can be persuasive without being correct; they should therefore be read critically.

4. ChatGPT is of the opinion that these notes are "compressed" and they demand a lot from the students. I plead guilty to that charge.