7. Simple and semisimple modules (Après Bourbaki Algebra Chapter 8 §3)

[s:ssmod]

In this section A denotes a ring with unity. Modules will mean left A-modules. Submodules will mean A-submodules. *Maximal* submodules will mean maximal proper submodules; *minimal* submodules will mean minimal non-zero submodules.

[ss:smod]

7.1. **Simple modules.**

A module is *simple* if it is non-zero and does not admit a proper non-zero submodule. Simplicity of a module M is equivalent to either of:

simple module

- $Am = M$ for every m non-zero in M.
- $M \simeq A/\mathfrak{m}$ for some maximal left ideal of A.

In particular, simple modules are cyclic; and the annihilator of any non-zero element of a simple module is a maximal left ideal.

primitive ideal
primitive ring

The annihilator of a simple module is called a *primitive* ideal. The ring A is *primitive* if the zero ideal is primitive, or, equivalently, if A admits a faithful simple module.[8]

- A module may have no simple submodules. Indeed, simple submodules of ${}_AA$ are minimal left ideals, but there may not be any such (e.g., in $\mathbb{Z}$).
- The module ${}_AA$ is simple if and only if A is a division ring. In this case, any simple module is isomorphic to ${}_AA$.
- The $\mathbb{Z}$-module $\mathbb{Z}/p^n\mathbb{Z}$ where p is a prime is indecomposable; it is simple if and only if $n = 1$.
- Let $A = \operatorname{End}_k V$ for k a field and V a k-vector space. The set $\mathfrak{a}$ of finite rank endomorphisms is a two-sided ideal of A. Let B be the subring A generated by the identity endomorphism and $\mathfrak{a}$. Then V is a simple B-module (in particular a simple A-module). And $B \subsetneq A$ if $\dim_k V$ is infinite. Let W be a codimension 1 subspace of V. The endomorphisms killing W form a minimal left ideal in A (and in B). Thus A and B when $\dim_k V$ is infinite give examples of primitive rings that admit non-trivial proper two-sided ideals.

An application of Zorn's lemma gives:

[p:maxexists]

Proposition 7.1. *Let M be a finitely generated A-module and $N \subsetneq M$ a proper submodule. Then there exists a maximal submodule of M containing N.*

[c:p:maxexists]

Corollary 7.2. *Let M be finitely generated non-zero. Then there exists a primitive ideal $\mathfrak{a}$ such that $\mathfrak{a}M \subsetneq M$.*

This corollary is essentially Nakayama's lemma.

Proof. Choose N maximal submodule and let $\mathfrak{a} = \operatorname{Ann} M/N$. □

[sss:faithful]

7.1.1. When faithful modules with strong properties exist.

If a ring admits faithful modules with strong properties (e.g., a primitive ring), then, as might be expected, the ring itself has strong properties.

[p:lidealsimple]

Proposition 7.3. *Let M be faithful simple and $\mathfrak{l}$ a minimal left ideal. Then $M \simeq \mathfrak{l}$.*

Proof. The submodule $\mathfrak{l}M$ is non-zero by faithfulness. Choose m in M such that $\mathfrak{l}m \neq 0$. By simplicity, $\mathfrak{l}m = M$. The homomorphism $\mathfrak{l} \to M$ defined by $a \mapsto am$ has zero kernel because $\mathfrak{l}$ is minimal. □

[p:jhfgfaith]

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[8] We should, strictly speaking, say *left primitive* (not just primitive), for there are rings that admit faithful simple left modules but not faithful simple right modules (and of course vice-versa). Similarly, we should distinguish between left and right primitive ideals although both kinds are two-sided ideals being annihilators of modules.

Proposition 7.4. *Let M be a faithful module admitting a composition series Σ. If the opposite of M is of finite type, then every simple A-module is a quotient in Σ.*

Proof. Let $\{m_i\}$ be a finite generating set of M over the commutant of A. Consider the map $a \mapsto (am_i)$ from ${}_AA$ into $\oplus_i M$. If $am_i = 0$ for all i, then $aM = 0$ and so $a = 0$ by the faithfulness of M. Thus ${}_AA$ imbeds into a finite number of copies of M. Every simple module being a quotient of ${}_AA$, we are done. □

[ss:ssmod]

7.2. **Semisimple modules.** A module is *semisimple* if it satisfies any of the following equivalent conditions:

- it is a sum of simple submodules.
- it is a direct sum of simple submodules.
- every submodule has a complement.

Before turning to the proof of the equivalence of the three conditions, let us observe that the third condition passes to submodules and quotient modules. Indeed, every quotient is isomorphic to a sub ($M/N \simeq Q$, where Q is a complement of N), so it is enough to observe the passage for submodules. If $P \subseteq N$ are submodules, and Q is a complement of P in M, then $Q \cap N$ is a complement of P in N as is easily verified.

Now we prove the equivalence of the three conditions. The second clearly implies the first. Now suppose that the first holds and let N be a submodule. Choose, by Zorn, a submodule maximal P with respect to the following two properties: it is a sum of simple submodules; it intersects N trivially. If $N \oplus P \subsetneq M$, then there is a simple submodule S of M that is not contained in $N+P$. This means $S \cap (N+P) = 0$, by the simplicity of S, so $N \cap (S+P) = 0$. Since $S + P \supsetneq P$, the maximality of P is violated. Thus $N \oplus P = M$, and the third conditon holds.

Suppose now that the third condition holds. We will show that the second holds too. Choose, by Zorn, a maximal collection $\mathfrak{C}$ of simple submodules whose sum is their direct sum. Let N be the sum of submodules in such a collection, and suppose that $N \subsetneq M$. Choose $y \in M \subseteq N$. Choose, by Zorn, a maximal submodule P of Ay. Let S be complement to P in Ay (it exists by the observation we made before beginning the proof). Being isomorphic to Ay/P, it is simple. And its existence violates the maximality of the collection $\mathfrak{C}$, which finishes the proof.

[ss:ssmodfacts]

7.2.1. Facts about semisimple modules.

(1) A simple module is semisimple. Vector spaces (over division rings) are semisimple. The ring $\mathbb{Z}$ is not a semisimple module over itself.
(2) Let M be a sum of simple submodules N_i, $i \in I$. For any submodule N, there exists a subset J of I such that N is isomorphic to the direct sum of N_j, $j \in J$; and there exists a subset K of I such that the direct sum of N_k, $k \in K$, is a complement of N. In particular, $M/N \simeq \oplus_{k \in K} N_k$.
(3) Subquotients of semisimple modules are semisimple.

[ss:isossmod]

7.3. **Isotypic components of semisimple modules.** For an isomorphism class λ of simple modules, we denote by M_λ the sum of submodules of M that are isomorphic to a representative in the class λ. We call M_λ the *λ-isotypic component*.

- The isotypic components are semisimple (by definition); their sum is direct.
- $N = \oplus_\lambda (N \cap M_\lambda)$ for any submodule N of a semisimple module M.
- The λ-isotypic is mapped to the λ-isotypic under homomorphisms.

- The only submodules that are preserved by all endomorphisms of a semisimple module are the isotpyic components and their sums.

[ss:lssmod]

7.4. **Length of a semisimple module.** Let M be a semisimple module. If $\oplus_{i\in I} M_i$ and $\oplus_{j\in J} M_j$ are two expressions for M as a direct sum of simple submodules, then I and J have the same cardinality, which we then call the *length* of M and denote by $\ell_A M$. If S is a simple module, we denote by $[M : S]$ the length of the S-isotypic component of M.

- When $\ell_A M$ is finite, M has a composition series, and $\ell_A M$ coincides with Jordan-Hölder length of M.
- Two semisimple modules are isomorphic if and only if their S-isotypic lengths are equal for every simple module S.
- A semisimple module has finite length if and only if it is finitely generated.
- The length of a vector space equals the cardinality of a base.