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# On an infinitesimal characterization of the discrete series

By THOMAS J. ENRIGHT and V. S. VARADARAJAN\*

## 1. Introduction and summary

The aim of this paper is to obtain an infinitesimal characterization of the discrete series of representations of a semisimple Lie group.

Let  $G$  be a semisimple Lie group, connected and having a finite center. For simplicity we assume that  $G$  is contained as a real form of a complex simply connected semisimple Lie group  $G_c$ . Let  $K \subset G$  be a maximal compact subgroup of  $G$ . We assume that  $\text{rk}(G) = \text{rk}(K)$ ,  $\text{rk}$  denoting the rank, and fix a maximal torus  $B \subset K$ . Let  $\mathfrak{g}$  be the Lie algebra of  $G$ , and  $\mathfrak{k}, \mathfrak{b}$ , the subalgebras defined by  $K, B$  respectively.  $\mathfrak{g}_c$  is the complexification of  $\mathfrak{g}$  ( $\mathfrak{g} \subset \mathfrak{g}_c$ ), and  $\mathfrak{G}$  is the universal enveloping algebra of  $\mathfrak{g}_c$ , while  $\mathfrak{R}$  is the subalgebra of  $\mathfrak{G}$  generated by  $(1, \mathfrak{k})$ .  $\Delta$  is the set of roots of  $(\mathfrak{g}, \mathfrak{b})$ , and  $\Delta_k$  those of  $(\mathfrak{k}, \mathfrak{b})$ ;  $P_k$  is a fixed positive system in  $\Delta_k$ .  $L$  is the lattice of all integral linear functions on  $\mathfrak{b}_c$ , i.e.,  $\lambda \in \mathfrak{b}_c^*$  such that  $2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbf{Z}$  for all  $\alpha \in \Delta$ .  $L'$  is the subset of all  $\lambda \in L$  such that  $\langle \lambda, \alpha \rangle \neq 0$  for all  $\alpha \in \Delta$ .  $L_t$  is the set of all  $\lambda \in \mathfrak{b}_c^*$  such that  $2\langle \lambda, \beta \rangle / \langle \beta, \beta \rangle \in \mathbf{Z}$  for all  $\beta \in \Delta_k$ ;  $C_t$  is the set of all  $\lambda \in \mathfrak{b}_c^*$  such that  $\langle \lambda, \beta \rangle$  is real and  $\geq 0$  for all  $\beta \in P_k$ . We write  $L_t^+ = L_t \cap C_t$ . For any  $\mu \in L_t^+$ ,  $\tau_\mu$  is the unique irreducible  $\mathfrak{R}$ -module with highest weight  $\mu$  (relative to  $P_k$ ). If  $\mu \in L \cap L_t^+$  (and only for such  $\mu$ ),  $\tau_\mu$  gives rise to a  $K$ -module, also denoted by  $\tau_\mu$ .  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  is the Cartan decomposition of  $\mathfrak{g}$ . Finally,  $\mathfrak{Z}$  denotes the center of  $\mathfrak{G}$  and  $\mathfrak{Q}$ , the centralizer of  $K$  in  $\mathfrak{G}$ .

The elements of  $L' \cap C_t$  parameterize the equivalence classes of the discrete series of representations of  $G$  (cf. [6]; see also Section 6). For any  $\Lambda \in L' \cap C_t$  let  $\omega(\Lambda)$  denote the corresponding equivalence class.

Fix  $\Lambda \in L' \cap C_t$ , and let  $P$  be the positive system of roots of  $(\mathfrak{g}, \mathfrak{b})$  such that  $\langle \Lambda, \alpha \rangle > 0$  for all  $\alpha \in P$ . Define  $\delta_k, \delta_n$  by (19) of Section 6. Then  $\Lambda - \delta_k + \delta_n \in L \cap L_t^+$  and one knows that at least when all the numbers  $\langle \Lambda, \alpha \rangle$  ( $\alpha \in P$ ) are sufficiently large, the discrete class  $\omega(\Lambda)$  contains  $\tau_{\Lambda - \delta_k + \delta_n}$  with multiplicity 1 ([8], [9]). By Schur's lemma the elements of  $\mathfrak{Q}$  act as scalars on the corresponding isotypical subspace, giving rise to a homomorphism  $\chi_\Lambda$  of  $\mathfrak{Q}$  into  $\mathbf{C}$ . A well-known theorem of Harish-Chandra [4] now implies that

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the pair  $(\tau_{\Lambda-\delta_{k+\delta_n}}, \chi_\Lambda)$  completely determines the infinitesimal equivalence class of  $\omega(\Lambda)$ . Consequently, the problem of infinitesimal characterization of  $\omega(\Lambda)$  may be regarded as the problem of determining  $\chi_\Lambda$ .

The algebra  $\mathfrak{Q}$  is in general not abelian. It contains  $\mathfrak{B}$  and is itself contained in the centralizer of  $B$  in  $\mathfrak{G}$ . This last circumstance enables one to define, for each positive system  $Q \subset \Delta$  and each  $\nu \in \mathfrak{h}_e^*$ , a homomorphism  $\chi_{Q,\nu}$  of  $\mathfrak{Q}$  into  $\mathbb{C}$  (cf. Section 5). If  $\Lambda \in L' \cap C_t$ , the infinitesimal character of  $\omega(\Lambda)$  is the homomorphism of  $\mathfrak{B}$  into  $\mathbb{C}$  obtained by restricting  $\chi_{-P,\Lambda+\delta}$  to  $\mathfrak{B}$ . A study of some examples suggests that  $\chi_\Lambda$  is in fact  $\chi_{-P,\Lambda+\delta}$ . We are thus led to the following two problems:

(1) To construct, for each  $\Lambda \in L' \cap C_t$ , an irreducible  $\mathfrak{k}$ -finite  $\mathfrak{G}$ -module containing  $\tau_{\Lambda-\delta_{k+\delta_n}}$  with multiplicity one and such that  $\mathfrak{Q}$  acts on the corresponding isotypical space through the homomorphism  $\chi_{-P,\Lambda+\delta}$ ; we note that, by the theorem of Harish-Chandra mentioned above, such a  $\mathfrak{G}$ -module, if it exists at all, is uniquely determined.

(2) To prove that the  $\mathfrak{G}$ -module in (1) is equivalent to the  $\mathfrak{G}$ -module determined by any representation from  $\omega(\Lambda)$  on its space of  $K$ -finite vectors.

In this paper we solve (1) completely; this is done in Sections 2 through 5. As for (2), we solve it when  $\Lambda$  is sufficiently regular; we refer the reader to Section 6 for the precise conditions on  $\Lambda$  under which this is done. This solution is an elementary consequence of recent results of Schmid [9].

We wish to point out that our infinitesimal description of  $\omega(\Lambda)$  goes far beyond the determination of  $\chi_\Lambda$ . The  $\mathfrak{G}$ -modules that we construct are determined by certain natural and canonical properties and arise naturally out of the theory of Verma modules. It is quite interesting that the infinitesimal structure of both the discrete series as well as the finite dimensional representations is thus based on the properties of Verma modules.

We would like to acknowledge our indebtedness to Peter Trombi for several interesting discussions. In particular, the suggestion that  $\chi_\Lambda$  might coincide with  $\chi_{-P,\Lambda+\delta}$  came from him, and was responsible for determining our approach to these questions.

## 2. Some lemmas for $\mathfrak{G}$ and $\mathfrak{R}$ -modules

Our notation is as explained in Section 1. In particular,  $\Delta$  is the set of roots of  $(\mathfrak{g}, \mathfrak{b})$ , and  $\Delta_k$  the subset of roots of  $(\mathfrak{f}, \mathfrak{b})$ . For each  $\alpha \in \Delta$ ,  $X_\alpha$  is a non-zero element of the corresponding root subspace of  $\mathfrak{g}_e$ , and  $s_\alpha$  is the Weyl reflexion corresponding to  $\alpha$ .  $W_k$  is the subgroup of the Weyl group of  $(\mathfrak{g}_e, \mathfrak{b}_e)$  generated by the  $s_\alpha$  ( $\alpha \in \Delta_k$ ).

If  $M$  is a  $\mathfrak{G}$  (or  $\mathfrak{R}$ )-module and  $v \in M$ ,  $v$  is said to be a *weight vector* if

for some  $\mu \in \mathfrak{b}_c^*$ ,  $Hv = \mu(H)v$  for all  $H \in \mathfrak{b}_c$ . For a given  $\mu$ , the set of all such  $v$  is a subspace of  $M$  denoted by  $M(\mu)$ , while the  $\mu$  with  $M(\mu) \neq 0$  are called the weights of  $M$ .  $M$  is said to be a *weight module* if  $M = \sum_{\mu} M(\mu)$ ; the sum is then direct. If  $M$  is a weight module for  $\mathfrak{K}$  and  $Q \subset \Delta_k$  is a positive system of roots of  $(\mathfrak{k}, \mathfrak{b})$ ,  $M$  is said to be *bounded above with respect to  $Q$*  if there are  $\nu_1, \dots, \nu_m \in \mathfrak{b}_c^*$  such that for any weight of  $M$  we can find  $i$  such that  $\nu_i - \mu$  is a sum of elements of  $Q$ . A module  $M$  for  $\mathfrak{K}$  is said to be  *$Q$ -extreme* if for some  $\nu \in \mathfrak{b}_c^*$ ,  $M(\nu) \neq 0$  and there is a non-zero  $v \in M(\nu)$  such that

- (i)  $X_{\beta}v = 0$  for all  $\beta \in Q$
- (ii)  $M = \mathfrak{K}v$ ;

$\nu$  is then uniquely determined by  $M$ ,  $\dim M(\nu) = 1$ , and for any weight  $\mu$  of  $M$ ,  $\dim M(\mu) < \infty$  while  $\nu - \mu$  is a sum of elements of  $Q$ .  $\nu$  is called the  *$Q$ -extreme weight* of  $M$  and the *non-zero* elements of  $M(\nu)$  are called the  *$Q$ -extreme vectors* of  $M$ . Let  $\nu \in \mathfrak{b}_c^*$  and let  $\mathfrak{K}_{Q,\nu}$  be the left ideal of  $\mathfrak{K}$  generated by the elements  $X_{\beta}$  ( $\beta \in Q$ ),  $H - \nu(H)$  ( $H \in \mathfrak{b}_c$ ); then  $\mathfrak{K}_{Q,\nu}$  is proper, i.e., does not contain 1. The  $\mathfrak{K}$ -module  $\mathfrak{K}/\mathfrak{K}_{Q,\nu}$ , which is  $Q$ -extreme with  $\nu$  as the  $Q$ -extreme weight, is called a *Verma module*, and we use the symbol  $V_{\mathfrak{K},Q,\nu}$  to denote any  $\mathfrak{K}$ -module isomorphic to it. We assume the reader to be familiar with the theory of extreme and Verma modules ([1], [10], [11]); since  $\mathfrak{b}_c$  is also a Cartan subalgebra of  $\mathfrak{g}_c$ , all of these notions are applicable to  $\mathfrak{G}$ -modules also. If  $Q \subset \Delta$  is a positive system of roots of  $(\mathfrak{g}, \mathfrak{b})$ , the Verma modules for  $\mathfrak{G}$  are denoted by  $V_{\mathfrak{G},Q,\nu}$  ( $\nu \in \mathfrak{b}_c^*$ ).

Let  $M$  be any  $\mathfrak{G}$ -module. If  $\mathfrak{m} \subset \mathfrak{g}_c$  is a subalgebra, then a vector  $v \in M$  is said to be  *$\mathfrak{m}$ -finite* if it belongs to a finite dimensional  $\mathfrak{m}$ -submodule of  $M$ .  $M$  is said to be  *$\mathfrak{m}$ -finite* if every vector in it is so. If  $M'$  is the set of all  $\mathfrak{m}$ -finite vectors in  $M$ , an easy argument shows that  $M'$  is a  $\mathfrak{G}$ -module of  $M$ ; in particular, if  $M = \mathfrak{G}v$  and  $v$  is  $\mathfrak{m}$ -finite, so is  $M$ . For any root  $\beta$  of  $(\mathfrak{g}, \mathfrak{b})$  we write  $\mathfrak{m}(\beta)$  for the three dimensional subalgebra of  $\mathfrak{g}$  generated by  $X_{\pm\beta}$ . If  $M$  is a weight module for  $\mathfrak{K}$  with finite dimensional weight spaces and  $\beta \in \Delta_k$ , it is a standard result that a vector  $v \in M$  is  $\mathfrak{m}(\beta)$ -finite if and only if for some integer  $m = m(v) > 0$ ,  $X_{\beta}^m v = X_{-\beta}^m v = 0$ . In particular, if  $M$  is assumed in addition to be a sum (not necessarily direct) of submodules that have finite dimensional weight spaces and are bounded above with respect to a positive system  $Q \subset \Delta_k$  that contains  $\beta$  it is enough to require only that  $X_{-\beta}^m v = 0$ . Weight modules for  $\mathfrak{K}$  which have finite dimensional weight spaces and are bounded above with respect to  $Q$  are said to be of *type  $Q$* .

LEMMA 1. *Let  $\mathfrak{R} \subset \mathfrak{K}$  be a proper left ideal. Then  $\mathfrak{G}\mathfrak{R}$  is a proper left*

ideal of  $\mathfrak{G}$  and  $\mathfrak{G}\mathfrak{N} \cap \mathfrak{R} = \mathfrak{N}$ . Let  $\bar{\mathfrak{G}} = \mathfrak{G}/\mathfrak{G}\mathfrak{N}$ ,  $a \mapsto \bar{a}$  the natural map of  $\mathfrak{G}$  onto  $\bar{\mathfrak{G}}$ , and let  $\bar{\mathfrak{G}}$  be regarded as a  $\mathfrak{G}$ -module in the usual way. Suppose  $u \in \mathfrak{R}$  and that  $\mathfrak{R}_u$  is the left ideal of  $\mathfrak{R}$  that kills  $\bar{u}$ . Then  $\mathfrak{G}\mathfrak{R}_u$  is the left ideal of  $\mathfrak{G}$  that kills  $\bar{u}$ . The  $\mathfrak{G}$ -module  $\mathfrak{G}\bar{u}$  is isomorphic to  $\mathfrak{G}/\mathfrak{G}\mathfrak{R}_u$  and  $\mathfrak{G}\bar{u} \cap \mathfrak{R}\bar{1} = \mathfrak{R}\bar{u}$ .

Let  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  be the Cartan decomposition of  $\mathfrak{g}$ . We denote by  $S(\mathfrak{g}_c)$  and  $S(\mathfrak{p}_c)$  the symmetric algebras over  $\mathfrak{g}_c$  and  $\mathfrak{p}_c$ , and identify  $S(\mathfrak{p}_c)$  with the subalgebra of  $S(\mathfrak{g}_c)$  generated by 1 and  $\mathfrak{p}_c$ . Let  $\lambda$  be the symmetrizer map of  $S(\mathfrak{g}_c)$  onto  $\mathfrak{G}$ . Let  $\Gamma$  denote the linear map of  $S(\mathfrak{p}_c) \otimes \mathfrak{R}$  into  $\mathfrak{G}$  that sends  $p \otimes u$  to  $\lambda(p)u$ . Then  $\Gamma$  is a linear bijection and

$$\Gamma(S(\mathfrak{p}_c) \otimes \mathfrak{N}) = \lambda(S(\mathfrak{p}_c))\mathfrak{N} = \lambda(S(\mathfrak{p}_c))\mathfrak{R}\mathfrak{N} = \mathfrak{G}\mathfrak{N}.$$

Hence

$$\Gamma^{-1}(\mathfrak{G}\mathfrak{N} \cap \mathfrak{R}) = (S(\mathfrak{p}_c) \otimes \mathfrak{N}) \cap (C1 \otimes \mathfrak{R}) = C1 \otimes \mathfrak{N}$$

showing that  $\mathfrak{G}\mathfrak{N} \cap \mathfrak{R} = \mathfrak{N}$ . Suppose  $u \in \mathfrak{R}$ . Let  $\mathfrak{G}_u$  be the left ideal of  $\mathfrak{G}$  that kills  $\bar{u}$ . Obviously  $\mathfrak{G}_u \supset \mathfrak{G}\mathfrak{R}_u$ . To prove the reverse inclusion, let  $x \in \mathfrak{G}_u$ . Let  $\{p_i; i \in I\}$  be a basis of  $S(\mathfrak{p}_c)$  with  $p_{i_0} = 1$ . Write  $\Gamma^{-1}(x) = \sum_i p_i \otimes u_i$ ; then  $\Gamma^{-1}(xu) = \sum_i p_i \otimes u_i u \in \Gamma^{-1}(\mathfrak{G}\mathfrak{N}) = S(\mathfrak{p}_c) \otimes \mathfrak{N}$ . This shows that  $u_i u \in \mathfrak{N}$  for all  $i$ , i.e.,  $u_i \in \mathfrak{R}_u$  for all  $i$ , proves that  $\mathfrak{G}_u = \mathfrak{G}\mathfrak{R}_u$ , and gives the isomorphism  $\mathfrak{G}\bar{u} \simeq \mathfrak{G}/\mathfrak{G}\mathfrak{R}_u$ . For the last assertion,  $\mathfrak{R}\bar{u} \subset \mathfrak{G}\bar{u} \cap \mathfrak{R}\bar{1}$  obviously. If  $x \in \mathfrak{G}$  and  $x\bar{u} = v\bar{1}$  where  $v \in \mathfrak{R}$ , then writing  $\Gamma^{-1}(x) = \sum_i p_i \otimes u_i$  we find  $\Gamma^{-1}(xu - v) = \sum_i p_i \otimes u_i u - 1 \otimes v \in S(\mathfrak{p}_c) \otimes \mathfrak{N}$ , so that  $u_i u \equiv v \pmod{\mathfrak{N}}$ . But then  $x\bar{u} = \bar{v} = u_{i_0}\bar{u} \in \mathfrak{R}\bar{u}$ .

**LEMMA 2.** *Let notation be as above. Regard  $S(\mathfrak{p}_c)$  as a  $\mathfrak{R}$ -module in the usual way. Then the linear map  $\bar{\Gamma}$  of  $S(\mathfrak{p}_c) \otimes \mathfrak{R}/\mathfrak{N}$  that sends  $p \otimes \bar{v}$  to  $\overline{\lambda(p)v}$  is a linear bijection of  $S(\mathfrak{p}_c) \otimes \mathfrak{R}/\mathfrak{N}$  onto  $\bar{\mathfrak{G}}$  that is an isomorphism of  $\mathfrak{R}$ -modules.*

For each  $X \in \mathfrak{k}_c$  we write  $\text{ad}_p X$  for the derivation of  $S(\mathfrak{p}_c)$  that extends the endomorphism  $Y \rightarrow [X, Y]$  of  $\mathfrak{p}_c$ .  $S(\mathfrak{p}_c)$  then becomes a  $\mathfrak{R}$ -module if we set  $X \cdot p = (\text{ad}_p X)p$ . From the well-known fact that  $\lambda$  commutes with the adjoint representation we find for  $X \in \mathfrak{k}_c$ ,  $p \in S(\mathfrak{p}_c)$ ,  $v \in \mathfrak{R}$ ,

$$\begin{aligned} \bar{\Gamma}(X \cdot p \otimes \bar{v}) &= \bar{\Gamma}((\text{ad}_p X)p \otimes \bar{v} + p \otimes \overline{Xv}) \\ &= (X\lambda(p) - \lambda(p)X)\bar{v} + \lambda(p)\overline{Xv} \\ &= X \cdot \bar{\Gamma}(p \otimes \bar{v}) \end{aligned}$$

so  $\bar{\Gamma}$  is a  $\mathfrak{R}$ -module homomorphism. That  $\bar{\Gamma}$  is a linear bijection is clear from the relation  $\Gamma^{-1}(\mathfrak{G}\mathfrak{N}) = S(\mathfrak{p}_c) \otimes \mathfrak{N}$ .

**LEMMA 3.** *Let notation be as above. Suppose  $\mathfrak{R}/\mathfrak{N}$  is a module of type  $Q$ ,*

$Q$  being a positive system contained in  $\Delta_k$ . Then  $\mathfrak{G}/\mathfrak{G}\mathfrak{N}$  is a weight module for  $\mathfrak{G}$ , and considered as a  $\mathfrak{R}$ -module, it is a direct sum of  $\mathfrak{R}$ -modules of type  $Q$ .

If  $M$  is a  $\mathfrak{R}$ -module of type  $Q$  then so is  $M \otimes N$  for any finite dimensional  $\mathfrak{R}$ -module  $N$ . Lemma 3 is now immediate from Lemma 2 since  $S(\mathfrak{p}_c)$  is a direct sum of finite dimensional  $\mathfrak{R}$ -modules.

We now come to the key lemma of this section. If  $M$  is a  $\mathfrak{R}$ -module and  $\beta \in \Delta_k$ , we say that  $M$  is  $X_\beta$ -free if  $X_\beta$  acts injectively on  $M$ .

LEMMA 4. Let  $V_0 = \mathfrak{G}v_0$  be a cyclic  $\mathfrak{G}$ -module with cyclic vector  $v_0$  and let  $U_0 = \mathfrak{R}v_0$ . Let  $U = \mathfrak{G}v$  be a cyclic  $\mathfrak{R}$ -module with cyclic vector  $v$ . Suppose  $\psi: U_0 \rightarrow U$  is a  $\mathfrak{R}$ -module injection. Then there exists a  $\mathfrak{G}$ -module  $V$  containing  $U$  and a  $\mathfrak{G}$ -module injection  $\psi': V_0 \rightarrow V$  such that (i)  $V = \mathfrak{G}v$ , and (ii)  $\psi'/U_0 = \psi$ . Suppose further that  $U$  is a  $\mathfrak{R}$ -module of type  $Q$ ,  $Q$  being a positive system  $\subset \Delta_k$  and that  $U$  and  $V_0$  are both  $X_{-\beta}$ -free for all  $\beta \in Q$ . Then we can choose  $V$  such that it is  $X_{-\beta}$ -free for all  $\beta \in Q$  and is the sum of  $\mathfrak{R}$ -submodules that are of type  $Q$ .

Let  $\mathfrak{N}$  be the left ideal of  $\mathfrak{R}$  annihilating  $v$ . We may assume  $v \neq 0$  so that  $\mathfrak{N}$  is proper. Let  $\mathfrak{G} = \mathfrak{G}/\mathfrak{G}\mathfrak{N}$  and use notation of Lemma 1. Since  $U \simeq \mathfrak{R}\bar{1}$ , there is no loss of generality in assuming that  $U = \mathfrak{R}\bar{1}$ , so that  $\psi$  is now a  $\mathfrak{R}$ -module injection of  $U_0$  into  $\mathfrak{R}\bar{1}$ . Write  $y = \psi(v_0)$ . By Lemma 1,  $\mathfrak{G}\mathfrak{N}_0$  is the annihilator of  $y$  in  $\mathfrak{G}$  where  $\mathfrak{N}_0$  is the annihilator of  $v_0$  in  $\mathfrak{R}$ . Let  $\mathfrak{M}_0 \supset \mathfrak{G}\mathfrak{N}_0$  be the annihilator of  $v_0$  in  $\mathfrak{G}$ , and let  $Z = \mathfrak{M}_0 y$ . Then

$$Z \cap \mathfrak{R}\bar{1} = Z \cap (\mathfrak{G}y \cap \mathfrak{R}\bar{1}) = Z \cap \mathfrak{R}y \text{ (Lemma 1)} = 0.$$

Take  $V = \mathfrak{G}/Z$  and define  $\pi$  as the natural map of  $\mathfrak{G}$  onto  $V$ . Then  $\psi': av_0 \rightarrow \pi(ay)$  is a well-defined  $\mathfrak{G}$ -module injection of  $V_0$  into  $V$ , and for  $a \in \mathfrak{R}$ ,  $\psi'(av_0) = \pi(\psi(av_0))$ . Identifying  $\mathfrak{R}\bar{1}$  with its image in  $V$  we obtain easily the first assertion. Suppose now that  $U$  is of type  $Q$ . Then by Lemma 3  $\mathfrak{G}$  is a direct sum of  $\mathfrak{R}$ -submodules of type  $Q$ , and so, in that case  $V$  will be a sum of such  $\mathfrak{R}$ -modules. In order to prove the last assertion it is enough to prove the existence of a  $\mathfrak{G}$ -submodule  $E$  of  $V$  such that (i)  $E \cap \psi'(V_0) = E \cap U = 0$ , (ii) the  $\mathfrak{G}$ -module  $V/E$  is  $X_{-\beta}$ -free for all  $\beta \in Q$ . For, we may then replace  $V$  by  $V/E$ , the condition (i) insuring that  $\psi'(V_0)$  and  $U$  are injectively imbedded in  $V/E$ .

We construct  $E$  as follows. For any  $\beta \in Q$  we define  $E_{0,\beta}$  to be the  $\mathfrak{G}$ -module of all  $m(\beta)$ -finite vectors in  $V$ . Put  $E_0 = \sum_{\beta \in Q} E_{0,\beta}$ . If  $s \geq 1$ , and  $E_i$  ( $0 \leq i < s$ ) are well-defined  $\mathfrak{G}$ -submodules of  $V$ , we define  $E_{s,\beta}$  as the  $\mathfrak{G}$ -module of all vectors that are  $m(\beta)$ -finite mod  $E_{s-1}$ , and take  $E_s = \sum_{\beta \in Q} E_{s,\beta}$ .

This defines the  $\mathfrak{G}$ -submodules  $E_s$  ( $s \geq 0$ ) inductively and  $E_0 \subset E_1 \subset \dots$ . Let  $E = \bigcup_{s \geq 0} E_s$ . We shall prove (i) and (ii) above for the  $\mathfrak{G}$ -submodule  $E$ .

If  $x \in V$ ,  $\beta \in Q$  and  $X_{-\beta}x \in E$ , then  $X_{-\beta}x \in E_s$  for some  $s \geq 0$ . Since  $x$  lies in a  $\mathfrak{K}$ -submodule that is of type  $Q$ ,  $x \in E_{s,\beta}$ , and so  $x \in E_{s+1} \subset E$ . This proves that  $V/E$  is  $X_{-\beta}$ -free for all  $\beta \in Q$ .

We shall now prove by induction on  $s \geq 0$  that  $\psi'(V_0) \cap E_s = 0$  for all  $s \geq 0$ . This will show that  $\psi'(V_0) \cap E = 0$ . Suppose first that  $x \in \psi'(V_0) \cap E_0$ . Let  $x = \sum_{\beta \in D} x_\beta$  where  $D \subset Q$ ,  $x_\beta \in E_{0,\beta}$  and  $x_\beta \neq 0$ . Let  $\beta \in D$ , and choose an integer  $m > 0$  such that  $X_{-\beta}^m x_\beta = 0$ . If  $D = \{\beta\}$ , then  $x = x_\beta$  and hence, as  $V_0$  is  $X_{-\beta}$ -free,  $x = x_\beta = 0$ . If  $D$  has more than one element,  $x' = X_{-\beta}^m x = \sum_{\beta' \in D \setminus \{\beta\}} x'_{\beta'}$  where  $x'_{\beta'} = X_{-\beta}^m x_{\beta'} \in E_{0,\beta'}$ ; here we must remember that the  $E_{0,\beta'}$  are  $\mathfrak{G}$ -modules. So, as  $x' \in \psi'(V_0) \cap E$ , we find using induction on the cardinality of  $D$ , that  $x' = 0$ . This gives  $x = 0$  since  $V_0$  is  $X_{-\beta}$ -free. Suppose  $s \geq 1$  and  $E_{s-1} \cap \psi'(V_0) = 0$ . Passing to  $V/E_{s-1}$  we are in the same situation considered previously. So we have  $E_s \cap \psi'(V_0) = 0$ . The same argument proves that  $U \cap E = 0$ .

### 3. The fundamental chain of $\mathfrak{G}$ -modules

Fix a positive system  $P_k \subset \Delta_k$ . For any weight module  $M$  for  $\mathfrak{K}$ ,  $\Gamma(M)$  is the set of all  $\mu \in \mathfrak{b}_c$  such that  $M$  has a  $P_k$ -extreme vector of weight  $\mu$ . For any submodule  $N \subset M$ ,  $\Gamma(M \setminus N)$  is the subset of all  $\mu \in \Gamma(M)$  for which there is a  $P_k$ -extreme vector of weight  $\mu$  that is non-zero mod  $N$ .

Let  $\sum_k = \{\beta_1, \dots, \beta_l\}$  be the set of simple roots of  $P_k$ . For any  $s \in W_k$ ,  $N(s)$  denotes the smallest integer  $r$  such that  $s$  is a product of  $r$  reflections  $s_\beta$ ,  $\beta \in \sum_k$  ( $N(1) = 0$ ). For the properties of  $N(\cdot)$  the reader is referred to [10]. Let  $t$  be the element of  $W_k$  such that  $tP_k = -P_k$ . Then  $N(t) = |P_k|$ . Write

$$(1) \quad t = s_1 s_2 \cdots s_n = s_n \cdots s_1 \quad (s_r = s_{\gamma_r} \text{ where } \gamma_r = \beta_{j_r}).$$

Then  $\{\gamma_n, s_n \gamma_{n-1}, \dots, s_n \cdots s_2 \gamma_1\}$  and  $\{\gamma_1, s_1 \gamma_2, s_1 s_2 \gamma_3, \dots, s_1 \cdots s_{n-1} \gamma_n\}$  are both enumerations of  $P_k$ . Put

$$(2) \quad \delta_k = \frac{1}{2} \sum_{\alpha \in P_k} \alpha.$$

For any  $s \in W_k$  let  $s'$  be the affine map of  $\mathfrak{b}_c^*$  defined by

$$(3) \quad s'\mu = s(\mu + \delta_k) - \delta_k.$$

Finally, if  $\mu_1, \mu_2 \in \mathfrak{b}_c^*$ , we write  $\mu_1 \geq \mu_2$  or  $\mu_2 \leq \mu_1$  if  $\mu_1 - \mu_2$  is either 0 or a sum of elements of  $P_k$ .

Suppose  $\lambda \in L_t^+$  so that  $2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle$  is an integer  $\geq 0$  for all  $\alpha \in P_k$ . We write

$$(4) \quad \lambda_1 = t'\lambda, \lambda_r = (s_r s_{r+1} \cdots s_n)' \lambda \quad (1 \leq r \leq n), \lambda_{n+1} = \lambda.$$

Then

$$(5) \quad \langle \lambda_r + \delta_k, \gamma_{r-1} \rangle = \langle \lambda + \delta_k, s_n s_{n-1} \cdots s_r \gamma_{r-1} \rangle > 0 \quad (2 \leq r \leq n+1)$$

while

$$(6) \quad \langle \lambda_r + \delta_k, \gamma_r \rangle = -\langle \lambda + \delta_k, s_n s_{n-1} \cdots s_{r+1} \gamma_r \rangle < 0 \quad (1 \leq r \leq n).$$

Finally define the integers  $\nu_r > 0$  by

$$(7) \quad \nu_r = 2\langle \lambda_{r+1} + \delta_k, \gamma_r \rangle / \langle \gamma_r, \gamma_r \rangle \quad (1 \leq r \leq n).$$

LEMMA 5. *Let  $P$  be a positive system of roots of  $(\mathfrak{g}, \mathfrak{b})$  such that  $P_k \subset P$ , and let  $\lambda \in L_i^+$ . Then there are  $\mathfrak{G}$ -modules  $W_r$  ( $1 \leq r \leq n+1$ ) with the following properties:*

(i) *Each  $W_r$  is a weight module,  $W_1 \subset W_2 \subset \cdots \subset W_{n+1}$ , and  $W_1$  is the Verma  $\mathfrak{G}$ -module  $V_{\mathfrak{G}, -tP, \lambda_1}$ .*

(ii)  *$W_r = \mathfrak{G}v_r$  where  $v_r \in W_r(\lambda_r)$ ,  $v_r$  is  $P_k$ -extreme, and the  $P_k$ -extreme vectors of weight  $\lambda_r$  in  $W_r$  are multiples of  $v_r$ ;  $v_r \notin W_{r-1}$  for  $r \geq 2$ ; moreover, as a  $\mathfrak{R}$ -module,  $W_r$  is the sum of  $\mathfrak{R}$ -submodules of type  $P_k$ .*

(iii)  *$v_r = X_{-\gamma_r}^{\nu_r} v_{r+1}$  ( $1 \leq r \leq n$ ).*

(iv)  *$W_{r+1}/W_r$  is  $m(\gamma_r)$ -finite ( $1 \leq r \leq n$ ).*

(v) *Each  $W_r$  is  $X_{-\beta}$ -free for all  $\beta \in P_k$ .*

(vi) *If  $s \in W_k$  and  $s'\lambda \in \Gamma(W_r)$ , then  $N(s) \geq n+1-r$  ( $1 \leq r \leq n$ ).*

We set  $W_1 = V_{\mathfrak{G}, -tP, \lambda_1}$ . Of the properties (i)-(vi), only (ii), (v) and (vi) are applicable to  $W_1$ ; (ii) and (v) are obvious. If  $s'\lambda \in \Gamma(W_1)$ , then  $s'\lambda$  is in particular a weight of  $W_1$ , and so  $t(s'\lambda - t'\lambda) = ts(\lambda + \delta_k) - (\lambda + \delta_k)$  is zero or else is a sum of roots from  $P$ . But as  $\lambda + \delta_k \in L_i^+$  and  $\langle \lambda + \delta_k, \beta \rangle > 0$  for all  $\beta \in P_k$ ,  $ts(\lambda + \delta_k) - (\lambda + \delta_k)$  is non-zero and is a sum of roots in  $-P_k$  except when  $ts = 1$ . Hence  $ts(\lambda + \delta_k) = \lambda + \delta_k$  which implies  $ts = 1$  or  $s = t$ . So  $N(s) = n$ .

Assume now that  $W_1, \dots, W_r$  have been constructed with properties (i)-(vi) (as far as they are applicable). Since  $W_r$  is  $X_{-\beta}$ -free for all  $\beta \in P_k$ ,  $\mathfrak{R}v_r \simeq V_{\mathfrak{R}, P_k, \lambda_r}$ . On the other hand,  $\lambda_r + \delta_k = s_r(\lambda_{r+1} + \delta_k)$  and  $2\langle \lambda_{r+1}, \gamma_r \rangle / \langle \gamma_r, \gamma_r \rangle$  is an integer  $\geq 0$ , so that  $V_{\mathfrak{R}, P_k, \lambda_r} \subset V_{\mathfrak{R}, P_k, \lambda_{r+1}}$  while moreover the  $P_k$ -extreme vector of  $V_{\mathfrak{R}, P_k, \lambda_r}$  is the image under  $X_{-\gamma_r}^{\nu_r}$  of the  $P_k$ -extreme vector of  $V_{\mathfrak{R}, P_k, \lambda_{r+1}}$ . Lemma 4 is thus applicable and so we can find a  $\mathfrak{G}$ -module  $W_{r+1} = \mathfrak{G}v_{r+1}$  such that (a)  $W_r \subset W_{r+1}$ , (b)  $v_r = X_{-\gamma_r}^{\nu_r} v_{r+1}$ , (c)  $v_{r+1}$  is a  $P_k$ -extreme vector of weight  $\lambda_{r+1}$ , (d)  $W_{r+1}$  is  $X_{-\beta}$ -free for all  $\beta \in P_k$ , and (e)  $W_{r+1}$  is a sum of  $\mathfrak{R}$ -submodules that are of type  $P_k$ . We wish to prove that  $W_{r+1}$  satisfies (ii), (iv) and (vi).

Let  $\tilde{W}_{r+1} = W_{r+1}/W_r$ . If  $v_{r+1} \in W_r$ ,

$$\lambda_{r+1} = (s_{r+1} \cdots s_n)' \lambda \in \Gamma(W_r) \implies N(s_{r+1} \cdots s_n) \geq n + 1 - r$$

by the hypothesis. But  $N(s_{r+1}, \dots, s_n) = n - r$  and so we have a contradiction. Thus  $v_{r+1} \notin W_r$ . Since  $X_{-r}^{v_r}$  kills the image of  $v_{r+1}$  in  $\tilde{W}_{r+1}$  this image is  $m(\gamma_r)$ -finite, proving that  $\tilde{W}_{r+1}$  is  $m(\gamma_r)$ -finite. Suppose  $u \in W_{r+1}$  and is  $P_k$ -extreme of weight  $\lambda_{r+1}$ . Let  $u' = X_{-r}^{v_r} u$ . We claim that  $u' \in W_r$ . This is trivial if  $u \in W_r$ . If  $u \notin W_r$ , the image of  $u$  in  $\tilde{W}_{r+1}$  is non-zero and  $P_k$ -extreme of weight  $\lambda_{r+1}$  so that  $X^{v_r}$  will kill it, showing  $u' \in W_r$ . Since  $u' \in W_r(\lambda_r)$  and is  $P_k$ -extreme,  $u' = c\lambda_r$  for some  $c \in \mathbb{C}$  by the induction hypothesis. As  $W_{r+1}$  is  $X_{-r}$ -free, we must have  $u = cv_{r+1}$ . Finally, let  $s \in W_k$  and  $s'\lambda \in \Gamma(W_{r+1})$ . If  $s'\lambda \in \Gamma(W_r)$ ,  $N(s) \geq n + 1 - r > n + 1 - (r + 1)$ . If  $s'\lambda \in \Gamma(W_{r+1} \setminus W_r)$ , there is a  $P_k$ -extreme vector  $v$  of weight  $s'\lambda$  which is non-zero mod  $W_r$ . If  $q = 2\langle s'\lambda, \gamma_r \rangle / \langle \gamma_r, \gamma_r \rangle$ , the  $m(\gamma_r)$ -finiteness of  $\tilde{W}_{r+1}$  implies that  $q \geq 0$  and that  $X_{-r}^{q+1}$  annihilates the image of  $v$  in  $\tilde{W}_{r+1}$ . Hence  $X_{-r}^{q+1}v$  is  $P_k$ -extreme,  $\in W_r$ , and of weight  $(s,s)\lambda$ . By the induction hypothesis,  $N(s,s) \geq n + 1 - r$  so that  $N(s) \geq n - r$ . We have thus carried forward the induction.

The argument given just now actually proves the following.

**COROLLARY 6.** *Suppose  $\mu \in \Gamma(W_{r+1} \setminus W_r)$  ( $1 \leq r \leq n$ ) and  $v$  is a  $P_k$ -extreme vector of weight  $\mu$  in  $W_{r+1}$  that is non-zero mod  $W_r$ . Then  $2\langle \mu, \gamma_r \rangle / \langle \gamma_r, \gamma_r \rangle$  is an integer  $\geq 0$ , and if  $m = 2\langle \mu + \delta_k, \gamma_r \rangle / \langle \gamma_r, \gamma_r \rangle$ , then  $X_{-r}^m v$  is a  $P_k$ -extreme vector in  $W_r$  of weight  $s'_r \mu$ . In particular  $s'_r \mu \in \Gamma(W_r)$ .*

#### 4. The submodule $\bar{W}$

**LEMMA 7.** *Let  $M$  and  $N$  be  $\mathfrak{R}$ -modules that are sums of  $\mathfrak{R}$ -submodules of type  $P_k$ . Suppose  $N$  is a quotient of  $M$ . If  $\mu \in \Gamma(N)$ , then there is  $\mu' \in \Gamma(M)$  such that  $\mu' + \delta_k \in W_k \cdot (\mu + \delta_k)$  and  $\mu' \geq \mu$ . If moreover  $\mu \in C$ , then  $\mu \in \Gamma(M)$ ; and if  $v$  is a  $P_k$ -extreme vector of weight  $\mu$  in  $N$ , we can find a  $P_k$ -extreme vector  $v'$  of  $M$  of weight  $\mu$  that lies above  $v$ .*

Let  $v$  be a  $P_k$ -extreme vector in  $N$  of weight  $\mu$ . Replacing  $N$  by  $\mathfrak{R}v$  where  $\tilde{v} \in M$  lies above  $v$ , we may assume that  $N$  is  $P_k$ -extreme with extreme weight  $\mu$  and that  $M$  is of type  $P_k$ . Let  $\mathfrak{Z}_k$  be the center of  $\mathfrak{R}$ . Elements of  $\mathfrak{Z}_k$  act as scalars on  $N$ . Let  $\chi_\mu$  be the homomorphism of  $\mathfrak{Z}_k$  into  $\mathbb{C}$  defined by this action. For any homomorphism  $\chi$  of  $\mathfrak{Z}_k$  into  $\mathbb{C}$ , let  $M_\chi$  be the subspace of all elements  $w \in M$  such that for each  $z \in \mathfrak{Z}_k$ ,  $(z - \chi(z))^m w = 0$  for sufficiently large  $m$ . The  $M_\chi$  are linearly independent  $\mathfrak{R}$ -submodules of  $M$ . Since the weight spaces of  $M$  are finite dimensional, each vector of  $M$  lies in some  $M_\chi$ . So  $M$  is their direct sum. If  $\chi \neq \chi_\mu$ , the projection of  $M$  onto  $N$  vanishes on  $M_\chi$ . We may thus assume that  $M = M_{\chi_\mu}$ . Let  $v' \in M$  be a vector of weight  $\mu$  above  $v$ . Let  $A'$  be the set of all vectors which are non-zero and of the

form  $X_{\alpha_1} \cdots X_{\alpha_q} v'$  for some choice of  $q \geq 1$ ,  $\alpha_1, \dots, \alpha_q \in P_k$ ; let  $A = A' \cup \{v'\}$ . Since  $M$  is bounded above, we can find  $w \in A$  whose weight  $\mu'$  is maximal with respect to  $\leq$  among the weights of the members of  $A$ . Clearly  $X_\beta w = 0$  for all  $\beta \in P_k$ , and so  $\mathfrak{R}w$  is a  $P_k$ -extreme module with  $w$  as a  $P_k$ -extreme vector. This implies that the elements of  $\mathfrak{B}_k$  act as scalars on  $\mathfrak{R}w$ . Therefore, as  $M = M_{\chi_\mu}$ ,  $zw = \chi_\mu(z)w$ ,  $z \in \mathfrak{B}_k$ . Since  $w$  is of weight  $\mu'$ , we must also have  $zw = \chi_{\mu'}(z)w$ ,  $z \in \mathfrak{B}_k$ , so that  $\mu' + \delta_k \in W_k \cdot (\mu + \delta_k)$ . Clearly  $\mu' \geq \mu$ . Suppose now that  $\mu \in C_i$ . Then  $\mu' + \delta_k \leq \mu + \delta_k$  and hence  $\mu' = \mu$ . But then  $v' = w$  so that  $v'$  itself is  $P_k$ -extreme.

Let  $\{W_r\}_{1 \leq r \leq n+1}$  be a fundamental chain of  $\mathfrak{G}$ -modules possessing the properties stated in Lemma 5. Fix  $r$ ,  $1 \leq r \leq n$ . We define the  $\mathfrak{G}$ -submodule  $\bar{W}_r$  as follows.  $W_{r,0}$  is the  $\mathfrak{G}$ -submodule of all vectors in  $W_{n+1}$  that are  $m(\gamma_r)$ -finite mod  $W_{r-1}$  (we define  $W_0 = 0$ ); once  $W_{r,0}, \dots, W_{r,p-1}$  are defined,  $W_{r,p}$  is the  $\mathfrak{G}$ -module of all vectors in  $W_{n+1}$  that are  $m(\gamma_{r+p})$ -finite mod  $W_{r,p-1}$ ,  $1 \leq p \leq n - r$ . The  $W_{r,p}$  are  $\mathfrak{G}$ -modules and  $W_{r,0} \subset \dots \subset W_{r,n-r}$ . We write  $\bar{W}_r = W_{r,n-r}$ . It is then clear that  $\bar{W}_r$  is the set of all vectors  $v \in W_{n+1}$  such that, for some integers  $d_r, d_{r+1}, \dots, d_n \geq 0$ ,

$$(8) \quad X_{-i_r}^{d_r} X_{-i_{r+1}}^{d_{r+1}} \cdots X_{-i_n}^{d_n} v \in W_{r-1}.$$

LEMMA 8. *Suppose  $\nu_r$  are as in (7). If  $N \subset W_{n+1}$  is a  $\mathfrak{R}$ -submodule and  $X_{-i_r}^{d_r} X_{-i_{r+1}}^{d_{r+1}} \cdots X_{-i_n}^{d_n} v_{n+1} \in N$  for some integers  $d_r, d_{r+1}, \dots, d_n \geq 0$ . Then*

$$(9) \quad X_{-i_r}^{\nu_r} X_{-i_{r+1}}^{\nu_{r+1}} \cdots X_{-i_n}^{\nu_n} v_{n+1} \in N.$$

Suppose  $\beta \in \sum_k$ ,  $L \subset W_{n+1}$  a  $\mathfrak{R}$ -submodule and  $w \in W_{n+1}$  a  $P_k$ -extreme vector of weight  $\mu$  where  $2\langle \mu, \beta \rangle / \langle \beta, \beta \rangle$  is an integer  $\geq 0$ . If  $m = 2\langle \mu + \delta_k, \beta \rangle / \langle \beta, \beta \rangle$ , we claim that  $X_\beta^m w \in L$  whenever  $X_\beta^d w \in L$  for some integer  $\geq 0$ . This is trivial if  $d \leq m$ . Suppose  $d > m$ . Then, assuming (as we may) that  $X_{\pm\beta}$  are normalized so that  $\beta(H) = 2$  where  $H = [X_\beta, X_{-\beta}]$ , we have

$$X_\beta X_\beta^d = X_\beta^d X_\beta + d X_\beta^{d-1} (H - d + 1),$$

so  $X_\beta^{d-1} w = [d(m - d)]^{-1} X_\beta X_\beta^d w \in L$ . An induction on  $d$  proves our claim. This said, we come to the proof of Lemma 8. We shall prove the lemma by downward induction on  $r$ . The case  $r = n$  is immediate from the remark made just previously. Suppose  $r < n$  and  $N'$  is the  $\mathfrak{R}$ -submodule of  $W_{n+1}$  of all vectors  $w$  that are  $m(\gamma_r)$ -finite mod  $(N)$ . Then  $X_{-i_{r+1}}^{d_{r+1}} \cdots X_{-i_n}^{d_n} v_{n+1} \in N'$ . So by the induction hypothesis,  $v_{r+1} = X_{-i_{r+1}}^{\nu_{r+1}} \cdots X_{-i_n}^{\nu_n} v_{n+1} \in N'$ . But  $v_{r+1}$  is  $P_k$ -extreme of weight  $\mu = \lambda_{r+1}$  and  $2\langle \mu, \gamma_r \rangle / \langle \gamma_r, \gamma_r \rangle = \nu_r - 1$  an integer  $\geq 0$  by (5) and (7). As  $X_{-i_r}^d v_{r+1} \in N$  for large  $d$ , our first remark shows that  $X_{-i_r}^{\nu_r} v_{r+1} \in N$ .

We now formulate the key lemma of this section. Define

$$(10) \quad \bar{W} = W_n + \sum_{1 \leq r \leq n} \bar{W}_r .$$

LEMMA 9. *Let  $\bar{W}$  be as above. Then  $v_{n+1} \notin \bar{W}$ . If  $\mu \in \Gamma(W_{n+1} \setminus \bar{W})$ , then  $\mu \in L_i^+$  and  $c_r = 2\langle \mu + \delta_k, s_n s_{n-1} \cdots s_{r+1} \gamma_r \rangle / \langle \gamma_r, \gamma_r \rangle$  are integers  $> 0$ ; moreover, for any  $P_k$ -extreme vector  $v$  of weight  $\mu$  that does not lie in  $W_n$  or any of the  $\bar{W}_r$ ,  $X_{-i_1}^{c_1} \cdots X_{-i_n}^{c_n} v$  is a  $P_k$ -extreme vector in  $W_1$  of weight  $t'\mu$ .*

Suppose  $v_{n+1} \in \bar{W}$ . Since  $\bar{W}$  may be regarded as the quotient of the abstract direct sum  $W_n \oplus \bar{W}_1 \oplus \cdots \oplus \bar{W}_n$  and since  $\lambda \in C_i$ , Lemma 7 may be applied to conclude that

$$\lambda \in \Gamma(W_n \oplus \bar{W}_1 \oplus \cdots \oplus \bar{W}_n) \subset \Gamma(W_n) \cup \Gamma(\bar{W}_1) \cup \cdots \cup \Gamma(\bar{W}_n) .$$

But  $C^x \cdot v_{n+1}$  is precisely the set of  $P_k$ -extreme vectors of weight  $\lambda$  in  $W_{n+1}$ , by Lemma 5. So  $v_{n+1}$  belongs to  $W_n$  or to some  $\bar{W}_r$ ,  $1 \leq r \leq n$ . Since  $v_{n+1} \notin W_n$ , we have  $v_{n+1} \in \bar{W}_r$  for some  $r$ . By the definition of  $\bar{W}_r$ , the relation (8), and Lemma 8, we then obtain  $v_r = X_{-i_r}^{\nu_r} X_{-i_{r+1}}^{\nu_{r+1}} \cdots X_{-i_n}^{\nu_n} v_{n+1} \in W_{r-1}$ , a contradiction.

Suppose  $v$  is as in the lemma. Since  $v \notin \bar{W}_r$  for  $1 \leq r \leq n$ , we have, for all integers  $d_r, \dots, d_n \geq 0$ ,

$$(11) \quad X_{-i_r}^{d_r} X_{-i_{r+1}}^{d_{r+1}} \cdots X_{-i_n}^{d_n} v \notin W_{r-1} .$$

Since  $v \notin W_n$ , Corollary 6 applies and we find that  $2\langle \mu, \gamma_n \rangle / \langle \gamma_n, \gamma_n \rangle = c_n - 1$  is an integer  $\geq 0$  and  $X_{-i_n}^{c_n} v \in W_n$ . By (11),  $X_{-i_n}^{c_n} v \notin W_{n-1}$  and so, as this is a  $P_k$ -extreme vector of weight  $s'_n \mu$ , Corollary 6 applies again. We find that  $c_{n-1} = 2\langle \mu + \delta_k, s_n \gamma_{n-1} \rangle / \langle s_n \gamma_{n-1}, s_n \gamma_{n-1} \rangle$  is an integer  $> 0$  and  $X_{-i_{n-1}}^{c_{n-1}} X_{-i_n}^{c_n} v$  is a  $P_k$ -extreme vector of weight  $s'_{n-1} s'_n \mu$ , and lies in  $W_{n-1}$ , but not in  $W_{n-2}$ . This argument evidently continues until we reach  $W_1$ . We note that  $\langle \mu + \delta_k, \beta \rangle$  is then  $> 0$  for all  $\beta \in P_k$  so that  $\mu \in L_i^+$ . The second assertion of the lemma is thus proved.

COROLLARY 10. *If  $s \in W_k$  and  $s \neq 1$ , then  $s'\lambda \notin \Gamma(W_{n+1} \setminus \bar{W})$ .*

Otherwise  $t's'\lambda$  is a weight of  $W_1$ . So  $s(\lambda + \delta_k) - (\lambda + \delta_k)$  is either zero or else is a sum of roots in  $P$ . As we have seen before, this is a contradiction to the assumption that  $\lambda \in \Gamma_i^+$ .

COROLLARY 11.  $\lambda - \alpha \in \Gamma(W_{n+1} \setminus \bar{W})$  for  $\alpha \in P$ .

Otherwise  $t'(\lambda - \alpha)$  is a weight of  $W_1$  and hence  $t'(\lambda - \alpha) - t'\lambda = -t\alpha$  is a sum of roots in  $tP$ , i.e.,  $-\alpha$  is a sum of roots in  $P$ , a contradiction.

### 5. The $\mathfrak{G}$ -modules $W_{P,\lambda}$ and $D_{P,\lambda}$

We are now in a position to prove our main theorems. Let  $\mathfrak{A}$  be the centralizer of  $\mathfrak{h}_c$  in  $\mathfrak{G}$  and  $\mathfrak{B} \subset \mathfrak{A}$  the subalgebra generated by 1 and  $\mathfrak{h}_c$ . If

$Q \subset \Delta$  is any positive system of roots of  $(\mathfrak{g}, \mathfrak{b})$ , then we have a homomorphism

$$(12) \quad \beta_Q: \mathfrak{X} \longrightarrow \mathfrak{B}$$

such that for any  $a \in \mathfrak{X}$

$$(13) \quad a \equiv \beta_Q(a) \pmod{\sum_{\alpha \in Q} \mathbb{G}X_\alpha}.$$

For any element  $\sigma$  of the Weyl group of  $(\mathfrak{g}_c, \mathfrak{b}_c)$ , let  $x_\sigma$  be an element of the complex adjoint group of  $\mathfrak{g}_c$  that induces  $\sigma$ , and let us write  $b \rightarrow b^\sigma$  for the induced automorphism of  $\mathbb{G}$ . If  $b \in \mathfrak{X}$ ,  $b^\sigma$  depends only on  $\sigma$  and not on the choice of  $x_\sigma$ . Obviously

$$(14) \quad \beta_{\sigma Q}(a^\sigma) = \beta_Q(a)^\sigma \quad (a \in \mathfrak{X}).$$

Since  $\mathfrak{b} \subset \mathfrak{k}$ ,  $\mathfrak{X}$  contains the centralizer  $\mathfrak{D}$  of  $\mathfrak{k}$  in  $\mathbb{G}$ . As the elements of  $W_k$  are induced from the adjoint group of  $\mathfrak{k}_c$ , it is clear that  $a^\sigma = a$  for  $a \in \mathfrak{D}$ ,  $\sigma \in W_k$ . Hence

$$(15) \quad \beta_{\sigma Q}(a) = \beta_Q(a)^\sigma \quad (a \in \mathfrak{D}, \sigma \in W_k).$$

The members of  $\mathfrak{B}$  are usually interpreted as polynomials on  $\mathfrak{b}_c^*$ . For any  $\nu \in \mathfrak{b}_c^*$  we write

$$(16) \quad \chi_{Q,\nu}(a) = \beta_Q(a)(\nu) \quad (a \in \mathfrak{D}).$$

The  $\chi_{Q,\nu}$  are homomorphisms of  $\mathfrak{D}$  into  $\mathbb{C}$ . We now have

**THEOREM 1.** *Let  $\lambda \in L_i^+$  and  $W_{P,\lambda} = W_{n+1}/\bar{W}$ . Then  $W_{P,\lambda}$  is a  $\mathfrak{k}$ -finite weight module for  $\mathbb{G}$ ;  $W_{P,\lambda} = \mathbb{G}\bar{\nu}_{n+1}$  where  $\bar{\nu}_{n+1}$  is the image of  $\nu_{n+1}$  in  $W_{P,\lambda}$ ; and  $\bar{\nu}_{n+1}$  is a  $P_k$ -extreme vector of  $W_{P,\lambda}$  of weight  $\lambda$ . If  $\tau_\mu$  is the finite dimensional  $\mathfrak{R}$ -module of highest weight  $\mu$  with respect to  $P_k$ , we have the following:*

- (i) *The multiplicity  $[W_{P,\lambda}; \tau_\lambda] = 1$ .*
- (ii) *For any  $\mu \in L_i^+$ , the multiplicity  $[W_{P,\lambda}; \tau_\mu]$  cannot exceed the maximum number (necessarily finite) of linearly independent  $P_k$ -extreme vectors of weight  $t'\mu$  in the Verma module  $V_{\mathbb{G}, -tP, t'\lambda}$ .*
- (iii) *The isotypical  $\mathfrak{R}$ -submodule of  $W_{P,\lambda}$  corresponding to  $\lambda$  is  $\mathfrak{R}\bar{\nu}_{n+1}$ ; and elements of  $\mathfrak{D}$  act as scalars there, the corresponding homomorphism of  $\mathfrak{D}$  being  $\chi_{-P, \lambda + 2\delta_k}$ .*

(iv)  $\lambda - \alpha \notin \Gamma(W_{P,\lambda})$  for  $\alpha \in P$ .

Certainly  $W_{P,\lambda} = \mathbb{G}\bar{\nu}_{n+1}$  and  $\bar{\nu}_{n+1}$  is  $P_k$ -extreme of weight  $\lambda$ . If  $\beta \in \sum_k$  and  $\lambda_\beta$  is the integer  $2\langle \lambda, \beta \rangle / \langle \beta, \beta \rangle \geq 0$ ,  $X_{-\beta}^{\lambda_\beta+1} \nu_{n+1}$  is  $P_k$ -extreme in  $W_{n+1}$  of weight  $s'_\beta \lambda$ . By Corollary 10,  $X_{-\beta}^{\lambda_\beta+1} \nu_{n+1} \in \bar{W}$ . So  $X_{-\beta}^{\lambda_\beta+1} \bar{\nu}_{n+1} = 0$ . This proves that  $\bar{\nu}_{n+1}$  is  $\mathfrak{k}$ -finite. So  $W_{P,\lambda}$  is  $\mathfrak{k}$ -finite. Let  $\mu \in \Gamma(W_{P,\lambda})$ . Then  $\mu \in L_i^+$  and so Lemma 7 applies to show that every  $P_k$ -extreme vector of weight  $\mu$  in  $W_{P,\lambda}$  lies below a  $P_k$ -extreme vector of weight  $\mu$  in  $W_{n+1}$ . Let  $v_i$  ( $1 \leq i \leq q$ ) be

linearly independent  $P_k$ -extreme vectors of weight  $\mu$  in  $W_{P,\lambda}$ , and let  $v'_i$  be a  $P_k$ -extreme vector of weight  $\mu$  in  $W_{n+1}$  lying above  $v_i$ . The  $v'_i$  are linearly independent. If the integers  $c_r$  are as in Lemma 9 and  $x = X_{-\gamma_1}^{c_1} \cdots X_{-\gamma_n}^{c_n}$ , then  $xv'_1, \dots, xv'_q$  are  $P_k$ -extreme vectors of weight  $t'\mu$  in  $W_1$ ; moreover, by (v) of Lemma 5, they are even linearly independent. This proves (ii) and, for  $\mu = \lambda$ , gives (i). Corollary 11 gives (iv) in the same manner. For (iii) it is enough to prove that  $a\bar{v}_{n+1} = \beta_{-P}(a)(\lambda + 2\delta_k)\bar{v}_{n+1}$  for  $a \in \Omega$ . Now, if  $a \in \Omega$ ,  $a$  commutes with the  $X_{-\beta}$  ( $\beta \in P_k$ ) and so

$$\begin{aligned} X_{-\gamma_1}^{v_1} \cdots X_{-\gamma_n}^{v_n}(av_{n+1}) &= a(X_{-\gamma_1}^{v_1} \cdots X_{-\gamma_n}^{v_n}v_{n+1}) \\ &= av_1 \quad (\text{cf. (iii) of Lemma 5}) \\ &= \beta_{-tP}(a)(t'\lambda)v_1 \end{aligned}$$

since  $v_1$  is  $-tP$ -extreme in the Verma module  $W_1$ . Hence

$$X_{-\gamma_1}^{v_1} \cdots X_{-\gamma_n}^{v_n}(av_{n+1}) = X_{-\gamma_1}^{v_1} \cdots X_{-\gamma_n}^{v_n}(\chi_{-tP,t'\lambda}(a)v_{n+1}),$$

showing, in view of (v) of Lemma 5, that  $av_{n+1} = \chi_{-tP,t'\lambda}(a)v_{n+1}$ . By (15),  $\chi_{-tP,t'\lambda}(a) = \chi_{-P,\lambda+2\delta_k}(a)$ . Hence

$$(17) \quad av_{n+1} = \beta_{-P}(a)(\lambda + 2\delta_k)v_{n+1} = \chi_{-P,\lambda+2\delta_k}(a)v_{n+1} \quad (a \in \Omega).$$

This gives (iii) on going over to  $W_{P,\lambda}$ .

Let  $\mathfrak{S}$  be the collection of all  $\mathfrak{G}$ -submodules of  $W_{P,\lambda}$  that do not contain  $\bar{v}_{n+1}$ . Let  $W_{P,\lambda}^z$  be the sum of all the  $\mathfrak{R}$ -submodules of  $W_{P,\lambda}$  that are irreducible and have  $P_k$ -extreme weights  $\neq \lambda$ . If  $M \in \mathfrak{S}$ , then  $M \subset W_{P,\lambda}^z$ . Let

$$(18) \quad M_{P,\lambda} = \sum_{M \in \mathfrak{S}} M.$$

Then  $M_{P,\lambda} \subset W_{P,\lambda}^z$  and so  $M_{P,\lambda} \in \mathfrak{S}$  also. It is obvious that the  $\mathfrak{G}$ -module  $D_{P,\lambda} = W_{P,\lambda}/M_{P,\lambda}$  is irreducible, and  $\mathfrak{k}$ -finite. We have  $[D_{P,\lambda}; \tau_\lambda] = 1$  and the action of  $\Omega$  on the corresponding isotypical  $\mathfrak{R}$ -submodule is given by the homomorphism  $\chi_{-P,\lambda+2\delta_k}$ . These two properties already determine  $D_{P,\lambda}$  up to equivalence in view of the well-known theorem of Harish-Chandra [4]. We thus have

**THEOREM 2.** *Let  $\lambda \in L_i^+$ . Then, given any positive system  $P$  of roots of (g, b) that contains  $P_k$ , there exists a unique (up to equivalence) irreducible  $\mathfrak{k}$ -finite  $\mathfrak{G}$ -module  $D_{P,\lambda}$  having the following property: the irreducible  $\mathfrak{R}$ -module  $\tau_\lambda$  with  $P_k$ -extreme vector  $\lambda$  occurs with multiplicity 1 in  $D_{P,\lambda}$ , and the action of  $\Omega$  on the corresponding isotypical  $\mathfrak{R}$ -submodule is given by the homomorphism  $\chi_{-P,\lambda+2\delta_k}$ . If  $\mu \in L_i^+$ , the multiplicity of  $\tau_\mu$  in  $D_{P,\lambda}$  cannot exceed the (finite) maximum number of linearly independent  $P_k$ -extreme vectors in the Verma  $\mathfrak{G}$ -module  $V_{\mathfrak{a},-tP,t'\lambda}$ . Moreover, if  $\alpha \in P$  and  $\lambda - \alpha \in L_i^+$ ,*

the irreducible  $\mathfrak{K}$ -module  $\tau_{\lambda-\alpha}$  does not occur in  $D_{P,\lambda}$ .

### 6. Identification with the discrete series

Let  $G_c$  be a complex simply connected group with Lie algebra  $\mathfrak{g}_c$  and  $G \subset G_c$  the real analytic subgroup determined by  $\mathfrak{g}$ . As in Section 1,  $B$  and  $K$  are the analytic subgroups of  $G$  defined by  $\mathfrak{b}$  and  $\mathfrak{k}$  respectively. Let  $L, L'$  be as in Section 1. As usual, fix a positive system  $P_k$  of roots of  $(\mathfrak{k}, \mathfrak{b})$ ; let  $P$  be a positive system of roots of  $(\mathfrak{g}, \mathfrak{b})$  with  $P_k \subset P$ ; and let

$$(19) \quad \delta = \frac{1}{2} \sum_{\alpha \in P} \alpha, \quad \delta_k = \frac{1}{2} \sum_{\alpha \in P_k} \alpha, \quad \delta_n = \frac{1}{2} \sum_{\alpha \in P_n} \alpha$$

where  $P_n = P \setminus P_k$ . If  $\Lambda \in L'$  and  $\langle \Lambda, \alpha \rangle > 0$  for all  $\alpha \in P$ , we denote by  $\omega(\Lambda)$  the equivalence class of discrete series representations of  $G$  whose character  $\Theta_{\omega(\Lambda)}$  is given on the regular subset  $B'$  of  $B$  by (cf. [6])

$$(20) \quad \Theta_{\omega(\Lambda)}(\exp H) = (-1)^q \sum_{s \in W_k} \varepsilon(s) e^{s\Lambda(H)} / \Delta(\exp H) \quad (H \in \mathfrak{b}, \exp H \in B')$$

where

$$q = \frac{1}{2} \dim(G/K) \quad \text{and}$$

$$\Delta(\exp H) = \prod_{\alpha \in P} \left\{ \exp\left(\frac{1}{2} \alpha(H)\right) - \exp\left(-\frac{1}{2} \alpha(H)\right) \right\} \quad (H \in \mathfrak{b}).$$

In order to relate the  $\mathfrak{G}$ -modules  $D_{P,\lambda}$  constructed earlier with the discrete series of  $G$  we make use of the results of Schmid [9] (see also Hotta and Parthasarathy [7]). Let  $\Lambda$  be as above and assume that  $\Lambda$  satisfies the following additional conditions ("sufficiently regular"):

$$(21) \quad \begin{cases} \text{(i)} & \langle \Lambda - \delta, \alpha \rangle > 0 & \text{for all } \alpha \in P_n \\ \text{(ii)} & \langle \Lambda - \delta, \beta \rangle \geq \langle \delta_n - \langle Q \rangle, \beta \rangle & \text{for all } \beta \in P_k, \quad Q \subset P_n \\ & \text{where } \langle Q \rangle \text{ denotes the sum } \sum_{\alpha \in Q} \alpha. \end{cases}$$

The results in [7] that are relevant for our purposes may then be summarized as follows:

1.  $\Lambda - \delta_k + \delta_n, \Lambda - \delta_k + \delta_n - \alpha \in L \cap L_t^+, \alpha \in P_n$  being arbitrary. Moreover,  $[\omega(\Lambda): \tau_{\Lambda-\delta_k+\delta_n}] = 1$  and  $[\omega(\Lambda): \tau_{\Lambda-\delta_k+\delta_n-\alpha}] = 0$  for all  $\alpha \in P_n$ .

2. Let  $V$  be a unitary  $K$ -module belonging to  $\tau_{\Lambda-\delta_k+\delta_n}$ . Let  $\pi_\Lambda$  be an irreducible unitary representation belonging to  $\omega(\Lambda)$ . Let  $\mathfrak{H}$  be the Hilbert space of  $\pi_\Lambda$ ;  $H_0$  the isotypical subspace of  $H$  corresponding to  $\tau_{\Lambda-\delta_k+\delta_n}$ ;  $E$  the orthogonal projection  $H \rightarrow H_0$ ; and  $F(x) = \eta^{-1}(E\pi(x)E|H_0)\eta$  ( $x \in G$ ) where  $\eta: V \rightarrow H_0$  is a unitary  $K$ -module isomorphism. Let  $\mathfrak{D}$  be the set of all  $C^\infty$  maps  $F'$  of  $G$  into the algebra  $A$  of endomorphisms of  $V$  such that

- (i)  $F'(1) = 1$ ;
- (ii)  $F'(k_1 x k_2) = k_1 F'(x) k_2$  ( $x \in G$ ,  $k_1, k_2 \in K$ );
- (iii) If  $\alpha \in P_n$  and  $\zeta_\alpha$  is the character (on  $K$ ) of

$\tau_{\Lambda - \delta_k + \delta_n - \alpha}$ , then, for all  $X \in \mathfrak{p}$ ,  $x \in G$ ,

$$\int_K \zeta_\alpha(k)^{\text{conj}}(\tilde{X} F')(xk) dk = 0 ;$$

here  $\tilde{X}$  is the left invariant vector field on  $G$  defined by  $X$ . Then  $F \in \mathfrak{D}$  and  $\mathfrak{D}$  contains no member other than  $F$ .

Consider now the irreducible  $\mathfrak{G}$ -module  $D_{P, \Lambda - \delta_k + \delta_n}$ . By Harish-Chandra's subquotient theorem [4] there exists a Hilbert space  $\tilde{H}$  and an irreducible representation  $\tilde{\pi}$  of  $G$  in  $\tilde{H}$ , quasi-simple in the sense of Harish-Chandra [3] such that  $\tilde{\pi}/K$  is unitary and the  $\mathfrak{G}$ -module defined by  $\tilde{\pi}$  on the  $K$ -finite vectors of  $\tilde{H}$  is equivalent to  $D_{P, \Lambda - \delta_k + \delta_n}$ . Let  $\tilde{H}_0$  be the isotypical subspace of  $\tilde{H}$  corresponding to  $\tau_{\Lambda - \delta_k + \delta_n}$ ;  $\tilde{E}$ , the orthogonal projection  $\tilde{H} \rightarrow \tilde{H}_0$ ; and

$$\tilde{F}(x) = \tilde{\eta}^{-1}(\tilde{E}\tilde{\pi}(x)\tilde{E}|_{\tilde{H}_0})\tilde{\eta} \quad (x \in G)$$

where  $\tilde{\eta}$  is a unitary  $K$ -module isomorphism of  $V$  with  $\tilde{H}_0$ . The fact that the  $\mathfrak{K}$ -modules  $\tau_{\Lambda - \delta_k + \delta_n - \alpha}$  ( $\alpha \in P_n$ ) do not occur in  $D_{P, \Lambda - \delta_k + \delta_n}$  then shows that  $\tilde{F}$  satisfies condition (iii) of 2. above. So  $\tilde{F} \in \mathfrak{D}$  and hence  $\tilde{F} = F$ . Standard results in representation theory of semisimple groups [6] now imply that  $\tilde{\pi}$  and  $\pi$  are infinitesimally equivalent. We thus have

**THEOREM 3.** *Let  $\Lambda \in L'$  and let  $P$  be the positive system of roots of  $(\mathfrak{g}, \mathfrak{b})$  such that  $\langle \Lambda, \alpha \rangle > 0$  for all  $\alpha \in P$ . Suppose  $\Lambda$  satisfies the additional restrictions (21). Then  $\Lambda - \delta_k + \delta_n \in L \cap L_i^+$  and the  $\mathfrak{G}$ -module  $D_{P, \Lambda - \delta_k + \delta_n}$  is equivalent to the  $\mathfrak{G}$ -module defined by any representation of the discrete class  $\omega(\Lambda)$  on its  $K$ -finite vectors. In particular,  $[\omega(\Lambda): \tau_{\Lambda - \delta_k + \delta_n}] = 1$ , and the action of  $\mathfrak{D}$  on the isotypical subspace of  $\omega(\Lambda)$  corresponding to  $\tau_{\Lambda - \delta_k + \delta_n}$  is given by the homomorphism  $\chi_{-P, \Lambda + \delta}$ .*

*Remark.* For the cases in which  $G$  equals  $SU(n, 1)$  or the simply connected double covering of the connected component of the identity in  $SO(2k, 1)$ , the conclusions of Theorem 3 are valid when the restrictions (i) and (ii) of (21) are replaced by restriction (i) only (see [7]).

It appears likely that Theorem 3 remains true for all  $\Lambda \in L'$ . This has been shown to be the case by Enright [2] for the groups  $SU(m, 1)$ ,  $SO(2k, 1)$ . In [2] this infinitesimal description of the discrete series is used to determine the exact rate of decay at infinity on  $G$  of the matrix coefficients of the discrete series representations of  $G$ .

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