KERALA SCHOOL OF MATHEMATICS
ALGEBRA II 2021 MAY–AUG, MID-TERM TEST

Instructions: The time allowed to complete the test is three hours. Answer all questions. To get full credit, you must justify your answers.

(1) Let X be the set of subsets of cardinality 4 of $[6] := \{1,2,3,4,5,6\}$. Consider the action of the symmetric group $\mathfrak{S}_6$ on X induced from its defining action on $[6]$, and the diagonal action on $X \times X$. Describe the decomposition of $X \times X$ as a disjoint union of $\mathfrak{S}_6$-orbits. You must in particular determine the following:
- The number of orbits in $X \times X$.
- Cardinality of each orbit.
- A representative point in each orbit and the isotropy subgroup there.

SOLUTION: Two elements (S,T) and (S',T') of $X \times X$ belong to the same $\mathfrak{S}_6$-orbit if and only if $|S \cap T| = |S' \cap T'|$. The possibilities for $|S \cap T|$ are: 2, 3, 4. Thus there are three orbits. As respective representative points of these, we could choose:

$$(\{1,2,3,4\},\{3,4,5,6\}), \quad (\{1,2,3,4\},\{2,3,4,5\}), \quad (\{1,2,3,4\},\{1,2,3,4\}).$$

Denoting by $\mathfrak{S}_S$ the subgroup of bijections from S to itself (for any subset S of $[6]$), the isotropy subgroups at the three points above are respectively:

$$\mathfrak{S}_{\{1,2\}} \times \mathfrak{S}_{\{3,4\}} \times \mathfrak{S}_{\{5,6\}}, \quad \mathfrak{S}_{\{1\}} \times \mathfrak{S}_{\{2,3,4\}} \times \mathfrak{S}_{\{5\}} \times \mathfrak{S}_{\{6\}}, \quad \mathfrak{S}_{\{1,2,3,4\}} \times \mathfrak{S}_{\{5,6\}}$$

The cardinality of an orbit is the index of the isotropy subgroup at any of its points. The orders of the isotropy subgroups above being 8, 6, and 48, the respective cardinalities of the three orbits are 90, 120, 15. □

(2) A circular merry-go-round has twenty identical horses mounted at equally spaced intervals along its outer rim. Its owner wants to paint half of the horses red and half of them blue. In how many different ways can this be done?

SOLUTION: Imagine twenty labelled points—let us call them "vertices"—placed at equally spaced intervals along the circumference of a fixed circle. Let X be the set of colourings of these vertices, with ten of the vertices being coloured blue and the other ten being coloured red. Evidently $|X| = \binom{20}{10}$.

The cyclic group G of order 20 acts on X and the desired answer is the number of orbits for this action. Burnside's Orbit Number Lemma (ONL) says:

$$\text{the number of orbits} = \frac{1}{|G|} \sum_{g \in G} \text{the number of points in } X \text{ fixed by } g$$

Letting c be a generator of c, it is easy to see that the number of fixed points for every element c^n of the group are as in the following table:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\lvert X^{c^n}\rvert$	0	2	0	6	0	2	0	6	0	$\binom{10}{5}$	0	6	0	2	0	6	0	2	0	$\lvert X\rvert$

Plugging the above values into the right hand side of the ONL, we get the answer:

$$\frac{1}{20}\left\{\binom{20}{10} + 32 + \binom{10}{5}\right\} = 9252 \qquad \square$$

(3) State as much as you can about Sylow 3-subgroups in the symmetric group $\mathfrak{S}_9$. You could in particular determine:

- the cardinality of a Sylow 3-subgroup
- the number of Sylow 3-subgroups
- the structure of a Sylow 3-subgroup: what is its centre? nilpotency class?
- cardinality and structure of the normaliser of a Sylow 3-subgroup.

SOLUTION: Sylow 3-subgroups of a finite group are those of cardinality 3^n with n being maximal such that 3^n divides the order of the group. In the present case, the cardinality of such a subgroup is $3^4 = 81$. To construct such a subgroup, consider the subgroup Q generated by the three 3-cycles $(1,2,3)$, $(4,5,6)$, and $(7,8,9)$. Evidently Q has order 27 and $Q \simeq \frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{3\mathbb{Z}}$. Observe that the element $g := (1,4,7)(2,5,8)(3,6,9)$, which has order 3 normalizes Q. The semi-direct product $H := \langle g\rangle \ltimes Q$ is clearly of order 81. It is thus a Sylow 3-subgroup.

We may think of g as acting on $Q \simeq \frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{3\mathbb{Z}}$ by permuting the factors cyclically, and thus identify H with the resulting semi-direct product. The centre Z of H is then the diagonal subgroup ($\simeq \frac{\mathbb{Z}}{3\mathbb{Z}}$) of $\frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{3\mathbb{Z}}$. The subgroup of H that maps to the centre modulo Z is the kernel K of the homomorphism $\frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{3\mathbb{Z}} \to \frac{\mathbb{Z}}{3\mathbb{Z}}$ given by $(a,b,c) \mapsto a+b+c$. Given that H/K has cardinality 9, it follows that it is abelian. Considering the sequence of subgroups $\{1\} \subseteq Z \subseteq K \subseteq H$ we see that the nilpotency class of H is 3.

The number of Sylow 3-subgroups turns out to be equal the number of ways we can partition $\{1,2,\ldots,9\}$ into three sets each of cardinality three (why? proof?), which is $\frac{9!}{3!\,3!\,3!} \times \frac{1}{6}$. $\square$

(4) Describe the set of all polynomials $f(x)$ with real coefficients such that

$$f(1) = -1, \quad f(-1) = 1, \quad f'(1) = 0, \quad \text{and} \quad f'(-1) = 1$$

Here $f'(x)$ denotes the first derivative of $f(x)$.

SOLUTION: The given conditions on $f(x)$ are equivalent to the following:

$$f(x) \equiv -1 \bmod (x-1)^2 \qquad f(x) \equiv 1 + (x+1) \bmod (x+1)^2$$

We have $1 = \frac{1}{2}(x+1) - \frac{1}{2}(x-1)$, cubing which we get $1 = p(x) + q(x)$ where

$$p(x) = \frac{1}{8}(x+1)^3 - \frac{3}{8}(x+1)^2(x-1), \qquad q(x) = \frac{3}{8}(x+1)(x-1)^2 - \frac{1}{8}(x-1)^3$$

Observe the following:

$$p(x) \equiv 0 \bmod (x+1)^2, \quad q(x) \equiv 0 \bmod (x-1)^2,$$
$$\text{and so} \quad p(x) \equiv 1 \bmod (x-1)^2, \quad q(x) \equiv 1 \bmod (x+1)^2$$

Thus the polynomial $f_0(x) = -p(x) + (x+2)q(x)$ has the desired properties. It follows from the Chinese Remainder Theorem (since $(x-1)^2$ and $(x+1)^2$ generate comaximal ideals in $\mathbb{R}[x]$), that the set S of all polynomials with the desired properties is given by:

$$S = f_0 + I \quad \text{where } I \text{ is the ideal generated by } (x-1)^2(x+1)^2$$

As a routine calculation shows, the unique polynomial in S of degree less than 4 is:

$$\frac{1}{4}\left(3x^3 - x^2 - 7x + 1\right)$$ □

(5) Let S be the multiplicative set $\{300^n \mid n \text{ non-negative integer}\}$ in the ring $\mathbb{Z}$ of integers. Identify all the ideals of $\mathbb{Z}$ maximal with respect to not meeting S. Let U be their union. What is the cardinality of the following set?:

$$\{n \mid n \text{ integer}, 1 \leqslant n \leqslant 50, n \notin U\}$$

SOLUTION: The ideals maximal with respect to not meeting a given multiplicative set are prime (result proved in class). The primes dividing 300 (or 300^n for some $n \geqslant 0$) are only 2, 3, and 5. Thus the ideals maximal with respect to not meeting S are $p\mathbb{Z}$ where p is a prime other than 2, 3, 5.

There are thus 24 integers n, $1 \leqslant n \leqslant 50$, such that $n \notin U$, namely:

$$1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, 30, 32, 36, 40, 45, 48, 50$$

No prime other than 2, 3, and 5 is involved in the factorisation of any of these. □

(6) Let R be the ring $\mathbb{Z}/240\mathbb{Z}$ of integers modulo 240, and $R[x]$ the ring of polynomials in the variable x with coefficients from the ring R. Identify the nil-radical $\mathfrak{n}$ of $R[x]$. Does there exist a positive integer k such that $\mathfrak{n}^k = 0$? If so find the smallest such k.

SOLUTION: A polynomial is nilpotent if and only if its coefficients are all nilpotent (HW exercise). Since $2^4 \times 3 \times 5$ is the prime factorisation of 240, the nilradical of R is generated by $2 \times 3 \times 5 = 30$. The nilradical $\mathfrak{n}$ of $R[x]$ consists of all polynomials whose coefficients are all multiples of 30. Since $30^4 = 0$ in R, it follows that $\mathfrak{n}^4 = 0$ (and $\mathfrak{n}^3 \neq 0$ since $30^3 \neq 0$ in R). □