Problem Sheet - Bilinear forms

Problem 1. *Let A and B be real $n \times n$ matrices. Prove that if $x^T Ay = x^T By$ for all vectors $x, y \in \mathbb{R}^n$, then $A = B$.*

Problem 2. *Prove directly that the bilinear form represented by the matrix $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$ is positive definite if and only if $a > 0$ and $ad - b^2 > 0$.*

Problem 3. *Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Find an orthonormal basis for $\mathbb{R}^2$ with respect to the form $x^T Ay$.*

Problem 4. *Prove that AA^T and $A^T A$ are positive semidefinite matrices for any $m \times n$ real matrix A.*

Problem 5. *Prove that if the columns of an $n \times n$ matrix A forms an orthonormal basis, then the rows do too.*

Problem 6. *Let A, B be symmetric matrices such that $A = PBP^T$, where $P \in GL_n(\mathbb{F})$. Is it true that the ranks of A and of B are equal?*

Problem 7. *Let A be the matrix of a symmetric bilinear form $\langle , \rangle$ with respect to some basis. Prove or disprove: The eigenvalues of A are independent of basis.*

Problem 8. *(a) Prove that every real square matrix is the sum of a symmetric and a skew-symmetric matrix in exactly one way.*

(b) Let $\langle \ , \rangle$ be a bilinear form on a real vector spave V. Show that there is a symmetric form $(\ ,)$ and a skew-symmetric form $[\ ,]$ so that $\langle \ , \rangle = (\ ,) + [\ ,]$.

Problem 9. *Let $\langle , \rangle$ be a symmetric bilinear form on a vector space over a field F. Consider the function $q : V \to F$ defined by $q(v) = \langle v, v \rangle$. Show how to recover the bilinear form from q, if the characterisitic of the field F is not 2, by expanding $q(v + w)$.*

Problem 10. *Let x, y be vectors in $\mathbb{C}^n$, and assume that $x \neq 0$. Prove that there is a symmetric matrix B such that $Bx = y$.*

Problem 11. *The vector space P of all real polynomials of degree atmost 2 has a bilinear form defined by*

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx.$$

Find an orthonormal basis for P.

Problem 12. *Let V denote the vector space of real $n \times n$ matrices. Prove that $\langle A, B\rangle = trace(A^T B)$ is a positive definite form on V. Find an orthonormal basis for this form.*

Problem 13. *Let W be a subspace of V, and consider the restriction of a symmetric form $\langle , \rangle$ to W. If the form is nondegenerate on W, then $V = W \oplus W^{\perp}$.*

Problem 14. *Prove that every symmetric nonsingular complex matrix A has the form $A = P^T P$.*

Problem 15. *Prove that if $x^* Ax$ is real for all complex vectors x, then A is hermitian.*

Problem 16. *Prove that a hermitian matrix A is positive definite if and only if $A = P^* P$ for some invertible matrix P.*

Problem 17. *Prove that the eigenvalues of hermitian matrices are real numbers, and the eigenvalules of real symmetric matrices are real numbers*

Problem 18. *Prove that the eigenvectors associated to distinct eigenvalues of a hermitian matrix A are orthogonal.*

Problem 19. *Prove or disprove: dim$V$=dim U+ dim $U^{\perp}$, for any subspace $U \subset V$.*

Problem 20. *Let W_1 and W_2 be subspaces of a vector space V with a symmetric bilinear form. Prove:*

1. $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2 \perp$
2. $W \subset W^{\perp\perp}$
3. *If $W_1 \subset W_2$, then $W_2^{\perp} \subset W_1^{\perp}$.*

Problem 21. *Prove that AB and BA has same eigenvalues for any two matricse A and B.*

Problem 22. *Let A(symmetric) and ABA be positive definite matrices, then prove that B is positive definite.*

Problem 23. *Prove that if A is normal, then A^2 is normal. Is the converse true?*

Problem 24. *Prove that if A is normal and $AB = BA$, then $A^T B = BA^T$.*

Problem 25. *Let A and B be two matrices with PAP^{-1} and PBP^{-1} are diagonal matrices for some matrix P, then $AB = BA$.*

Problem 26. *Prove that for any square matrix A, $KerA = (ImA^*)^\perp$*

Problem 27. *Show that, if A is a real normal matrix with real eigenvalues, then A is symmetric.*

Problem 28. *Let A be a normal matrix. Prove that A is hermitian if and only if all the eigenvalues of A are real, and that the eigenvalues of A have absolute value 1 if and only if A is unitary.*