Exercise set 5

The character of a finite dimensional (complex) representation V is denoted by $\mathscr{T}_V$.

(1) (Difficulty level 2) Let G be a finite group and V a finite dimensional complex linear representation of G. Observe the following:
   (a) Each element g of G is diagonalizable as an operator on V.
   (b) The value at g of the character of V is the sum of some n^{th} roots of unity where n is the order of g.
   (c) Any character value that is rational is actually integral.

(2) (Difficulty level 2, 3) Compute the primitive central idempotents in the group ring $\mathbb{C}G$, for G a cyclic group; for G the symmetric group $\mathfrak{S}_n$, $n = 3$, $n = 4$, $n = 5$, ...

(3) (Difficulty level 1) True or false?: every finite dimensional representation over the real numbers of a finite group is orthogonal.

(4)
- (Difficulty level 1) True or false?: If two group elements of a finite group G act the same way on all irreducible representations over the rational numbers of the finite group, then they are equal.
- (Difficulty level 2) True or false?: If two elements of the complex group ring $\mathbb{C}G$ of a finite group G act the same way on all irreducible representations over the complex numbers of the finite group, then they are equal.
- (Difficulty level 2) True or false?: If two elements of a ring act the same way on all simple modules, then they are equal.

(5) (Difficulty level 2) An element g in a finite group G belongs to the centre of G if and only if $|\mathscr{T}_V(g)| = \dim V$ for every complex irreducible representation of G.

(6) (Difficulty level 3) If a finite group G admits a faithful irreducible complex linear representation, then its centre is cyclic. (A representation V is called *faithful* if the group homomorphism $G \to GL(V)$ defining it is injective.)

(7) (Difficulty level 3) Count the number of k-dimensional vector subspaces in a fixed n-dimensional vector space over the finite field $\mathbb{F}_q$ of q elements.

(8) (Difficulty level 3) The cycle type of a permutation of 25 elements is $4, 4, 4, 3, 3, 2, 2, 2, 1$. Compute the order of its centralizer.

Additional problems (Optional)

(1) (Difficulty level 2) Check that the isomorphism $\mathrm{End}_R({}_RR) \simeq R^{\text{opp}}$ in natural (where R denotes a ring with identity).

(2) (Difficulty level 3) Construct a ring (with identity) which is not isomorphic to its opposite.