REPRESENTATION THEORY OF FINITE GROUPS

PROBLEMS SET

(1) Let $p(n)$ denote the number of integer partitions of n. Prove that $p(n) \leq p(n-1) + p(n-2)$ for each $n \geq 2$. Conclude that $p(n)$ grows faster than Fibonacci numbers.

(2) Suppose that $\lambda = (n-l, l)$ and $\mu = (n-m, m)$, with $0 \leq l, m \leq n/2$. Show that there exists a unique SSYT of shape μ and type λ if and only if $m \leq l$. If $m > l$, then there is no such SSYT.

(3) For an integer partition λ, let f_λ denote the number of standard Young tableaux of shape λ. Let n be any positive integer.

   (a) For each $0 \leq k < n$, show that
$$f_{(n-k,1^k)} = \binom{n-1}{k}.$$
   (b) For each $0 \leq k \leq n/2$, show that
$$f_{(n-k,k)} = \binom{n}{k} - \binom{n}{k-1}.$$

(4) Find the least value of n for which the reverse dominance order on the set of partitions of n is not a linear order (i.e., there exist partitions λ and μ of n such that neither $\lambda \leq \mu$, nor $\mu \leq \lambda$.

(5) Let λ be a *hook*, i.e., $\lambda = (m, 1^k)$ for some positive integer m and some nonnegative integer k. Which are the partitions μ of $m+k$ that satisfy $\mu \leq \lambda$.

(6) Show that, if $\mu \leq \lambda$ in the reverse dominance order, then the number of parts of μ is at most the number of parts of λ.

(7) Exhibit a SSYT of shape $(5,2,2)$ and $(3,3,3)$. What is the Kostka number $K_{(5,2,2),(3,3,3)}$?

(8) Determine the number of 3×3 matrices with nonnegative integer entries whose rows and columns all add up to 3 (this is the number $M_{(3,3,3),(3,3,3)}$).

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