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Definition

$$b(n) = \text{number of chiral partitions of } n.$$

n	$p(n)$	$b(n)$	$b(n)/p(n)$
1	1	1	
2	2	1	
3	3	2	
4	5	3	
5	7	5	
6	11	**4**	
7	15	8	
8	22	12	
9	30	20	
10	42	**8**	
11	56	16	
12	77	24	
13	101	40	
14	135	32	
15	176	64	
16	231	88	
17	297	152	
18	385	**16**	
19	490	32	
20	627	48	
21	792	80	
22	1002	64	
23	1255	128	
24	1575	192	
25	1958	320	
26	2436	128	
27	3010	256	
28	3718	384	
29	4565	640	
30	5604	512	
31	6842	1024	
32	8349	1360	
33	10143	2384	
34	12310	**32**	

n	$p(n)$	$b(n)$	$b(n)/p(n)$
100	190569292	6144	$3.2240241518 \times 10^{-05}$
200	3972999029388	98304	$2.47430213984 \times 10^{-08}$
300	9253082936723602	196608	$2.12478372176 \times 10^{-11}$
400	6727090051741041926	2883584	$4.28652504697 \times 10^{-13}$
500	2300165032574323995027	3221225472	$1.40043232828 \times 10^{-12}$
600	458004788008144308553622	6291456	$1.37366598881 \times 10^{-17}$
700	60378285202834474611028659	805306368	$1.33376820043 \times 10^{-17}$
800	5733052172321422504456911979	178257920	$3.10930224673 \times 10^{-20}$
900	415873681190459054784114365430	50331648	$1.21026288213 \times 10^{-22}$
1000	24061467864032622473692149727991	412316860416	$1.71359811773 \times 10^{-20}$
1100	1147240591519695580043346988281283	1572864	$1.37099751493 \times 10^{-27}$
1200	46240102378152881298913555099661657	369098752	$7.98222177325 \times 10^{-27}$
1300	1607818855017534550841511230454411672	12582912	$7.82607565568 \times 10^{-30}$
1400	49032194652550394774839040691532998261	103079215104	$2.10227618475 \times 10^{-27}$
1500	1329461690763193888825263136701886891117	824633720832	$6.20276407031 \times 10^{-28}$
1600	32417690376154241824102577250721959572183	22699573248	$7.0022179201 \times 10^{-31}$
1700	717802041964941442478681516751205185010007	6442450944	$8.97524744617 \times 10^{-33}$
1800	14552716211005418005132948684850541312590849	1610612736	$1.10674372581 \times 10^{-34}$
1900	272089289788583262011466359201428623427767364	6597069766656	$2.42459737088 \times 10^{-32}$

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A045923 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **a045923**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A045923 Number of irreducible representations of symmetric group S_n for which every matrix has determinant 1. +30 0

1, 1, 1, 2, 2, 7, 7, 10, 10, 34, 40, 53, 61, 103, 112, 143, 145, 369, 458, 579, 712, 938, 1127, 1383, 1638, 2308, 2754, 3334, 3925, 5092 (list; graph; refs; listen; history; text; internal format)

OFFSET	1,4
COMMENTS	Irreducible representations of S_n contained in the special linear group were first considered by L. Solomon (unpublished).
REFERENCES	R. P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, Cambridge and New York, 1999, Exercise 7.55.
LINKS	Table of n, a(n) for n=1..30.
EXAMPLE	a(5)=2, since only the irreducible representations indexed by the partitions (5) and (3,2) are contained in the special linear group.
KEYWORD	nonn,nice
AUTHOR	Richard Stanley
STATUS	approved

page 1

Search completed in 0.768 seconds

Closed Formula for number of representations of S_n with non-trivial determinant

Suppose n has binary expansion:

$$n = \epsilon + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}, \text{ with } 0 < k_1 < \cdots < k_r, \epsilon \in \{0, 1\},$$

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Then the number of partitions λ of n for which $w \mapsto \det(\rho_\lambda(w))$ is the sign character is

$$2^{k_2+\cdots+k_r}\left(2^{k_1-1} + \sum_{v=1}^{k_1-1} 2^{(v+1)(k_1-2)-\binom{v}{2}} + \epsilon 2^{\binom{k_1}{2}}\right).$$

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Conclusion

$\det \circ \rho_\lambda$ is the sign character if and only if g_λ is odd.

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(Macdonald, *Symmetric functions and Hall polynomials*, p. 118, using the theory of skew-Schur functions)

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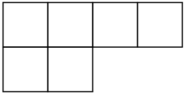

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Character Formula:

$$\chi_\lambda(2, 1^{n-2}) = \frac{f_\lambda C(\lambda)}{\binom{n}{2}}.$$

Formula for g_λ

$$\begin{aligned} g_\lambda &= (f_\lambda - \chi_\lambda(2, 1^{n-2}))/2 \\ &= f_\lambda \left(\frac{\binom{n}{2} - C(\lambda)}{\binom{n}{2}} \right) \end{aligned}$$

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$$= f_\lambda \left(\frac{\binom{n}{2} - C(\lambda)}{\binom{n}{2}} \right)$$

So ρ_λ is chiral if and only if:

$$v_2(f_\lambda) + v_2\left(\binom{n}{2} - C(\lambda) \right) = v_2 \binom{n}{2}$$

To understand $v_2(f_\lambda)$

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2	1		

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so

$$f_{(4,2)} = \frac{6!}{5 \times 4 \times 2 \times 1 \times 2 \times 1} = 9.$$

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$$|\lambda| = |\mathrm{core}_p\lambda| + p(|\lambda_0| + \cdots + |\lambda_{p-1}|).$$

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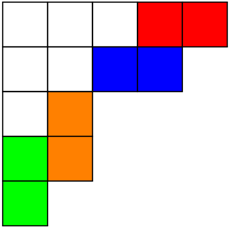
$$\lambda \leftrightarrow (\mathrm{core}_p\lambda, \mathrm{quo}_p\lambda).$$

The partition $\mathrm{core}_p\lambda$ is what remains of Young diagram of λ after successively removing the rims of as many p-hooks as possible. The p-quotient $\mathrm{quo}_p\lambda$ is a p-tuple $(\lambda_0, \dots, \lambda_{p-1})$ of partitions. The total number of cells in $\mathrm{quo}_p\lambda$ is the number of p-hooks whose rims were removed from λ to obtain $\mathrm{core}_p\lambda$.

$$|\lambda| = |\mathrm{core}_p\lambda| + p(|\lambda_0| + \cdots + |\lambda_{p-1}|).$$

The size of the partition λ_k in the p-quotient is the number of nodes in the Young diagram of λ whose hook-lengths are multiples of p, and whose hand-nodes have content congruent to k modulo p (by definition, the content of the node (i, j) is $j - i$). The partition λ can be recovered uniquely from $\mathrm{core}_p\lambda$ and $\mathrm{quo}_p\lambda$.

Example of core

The 2-core of $(5, 4, 2, 2, 1)$ is $(3, 2, 1)$: 

Example of quotient

The hook-lengths of $(5, 4, 2, 2, 1, 1)$ are:

10	7	4	3	1
8	5	2	1	
5	2			
4	1			
2				
1				

Example of quotient

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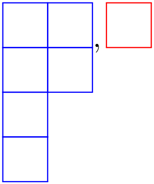
And its 2-quotient is given by



A result from Frame-Robinson-Thrall

Lemma

There exists a bijection from the set of cells in $\mathrm{quo}_p\lambda$ onto the set of cells in λ whose hook-lengths are divisible by p under which a cell of hook-length h in $\mathrm{quo}_p\lambda$ is mapped to a cell of hook-length ph in λ.

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In our example:

10	7	4	3	1
8	5	2	1	
5	2			
4	1			
2				
1				

$\rightarrow$

5	2
4	1
2	
1	

,

1

Recursive Criterion for odd Dimensionality

If n has binary expansion

$$n = \epsilon + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}, \text{ with } 0 < k_1 < \cdots < k_r, \epsilon \in \{0, 1\},$$

and λ is a partition of n with $\mathrm{core}_2\lambda$ of size a, and $\mathrm{quo}_2\lambda$ having partitions μ_0 and μ_1 of sizes m_0 and m_1 (so $n = a + 2m_0 + 2m_1$),

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Theorem (Macdonald)

f_λ is odd if and only if

- $a = \epsilon$,
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Remark

The binomial coefficient $\frac{n!}{k!(n-k)!}$ is odd if and only if the binary digits of k and $n-k$ are in disjoint positions.

Recursion of core and quotient construction: the core tower

If λ is a partition with core α and quotient μ_0 and μ_1, then its 2-core tower $T(\lambda)$ is a binary tree, defined recursively as follows:

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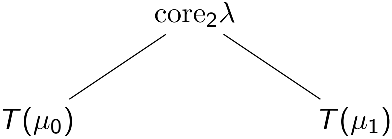

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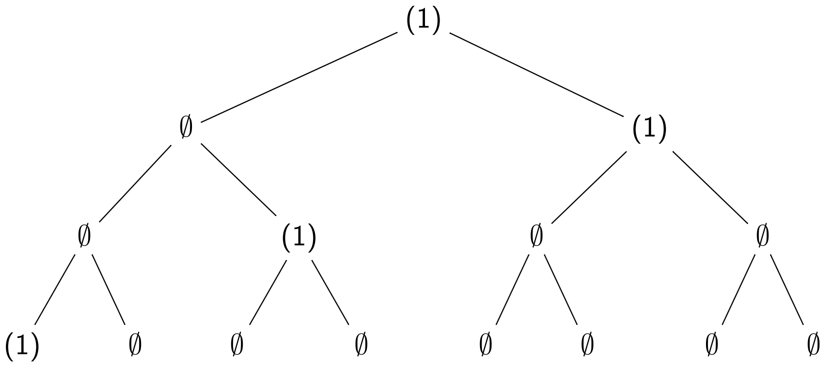

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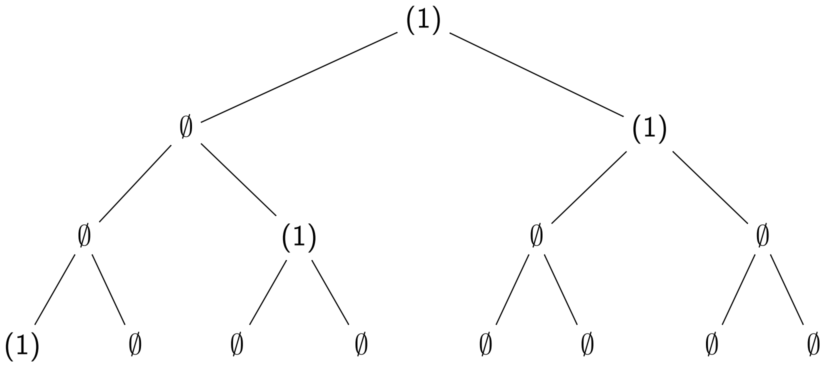

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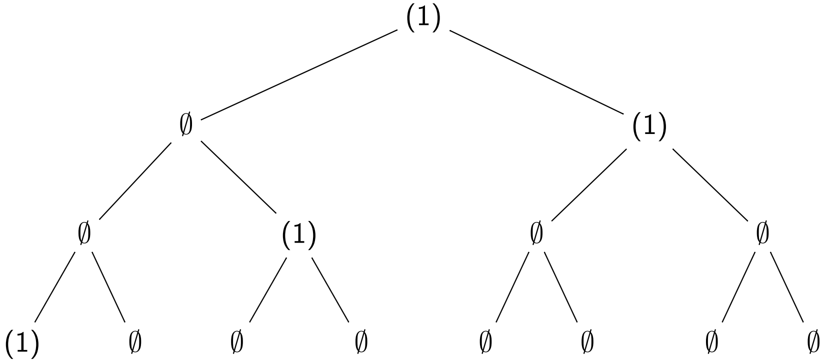

Let $w_i(\lambda) =$ sum of sizes of entries in ith row.
Here: $w_i(\lambda) = 1$ for $i = 0, 1, 2, 3$, and 0 otherwise.

Olsson's criterion for f_λ being odd

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Theorem

Let λ be a partition of n. Then f_λ is odd if and only if

$$\sum w_i(\lambda) = \sum \nu_i(n).$$

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The 2-core tower of $\lambda = (5, 4, 2, 2, 1, 1)$ (a partition of 15) is:

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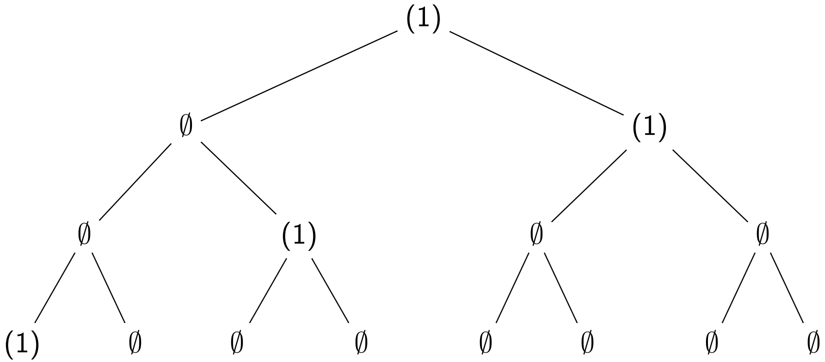

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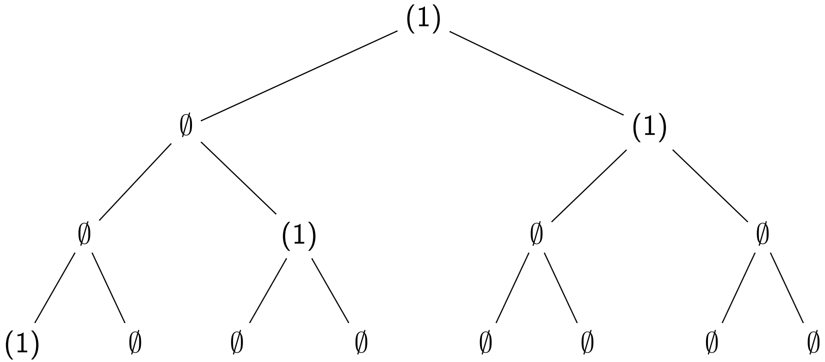

f_λ is odd.

Counting odd dimensional representations

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Theorem (Macdonald, bijective proof via Olsson)

The number of partitions λ of n such that f_λ is odd is $2^{\sum_i i\nu_i(n)}$.

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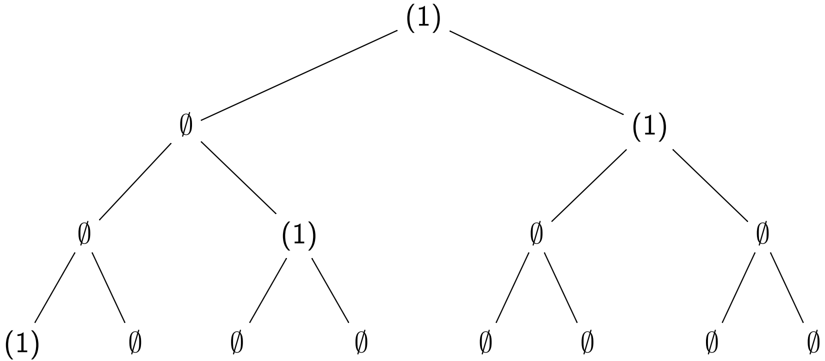
To prove, count the possible 2-core towers.

$n = 15$



2-core tower and 2-power hooks

The 2-core tower of λ tells us how the core of λ can be obtained by removing a sequence of maximal 2-power rim hooks.

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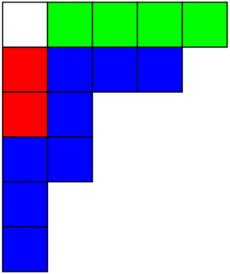
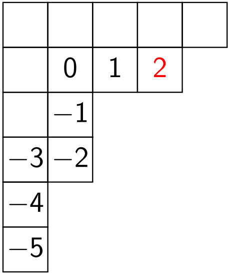


Contents in a rim-hook

	0	1	2	
	−1			
−3	−2			
−4				
−5				

$$C(h_3) = {\color{red}2} \times 2^3 - \binom{2^3}{2}$$

Contents in a rim-hook



$$C(h_3) = 2 \times 2^3 - \binom{2^3}{2}$$

The head node contribution is even on the left side of the tree and odd on the right side.

2-core towers of chiral partitions

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Chiral partitions λ of n are partitions for which

$$v_2(f_\lambda) + v_2\left(\binom{n}{2} - C(\lambda)\right) = v_2\binom{n}{2}$$

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By carefully keeping track of the contributions of different rim-hooks, we were able to characterize the 2-core towers of chiral partitions.

If $n = \epsilon + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}$, with $0 < k_1 < \cdots < k_r, \epsilon \in \{0, 1\}$. Then a partition λ of n is chiral if and only if one of the following happens:

1. The partition λ satisfies

$$w_i(\lambda) = \begin{cases} 1 & \text{if } i \in \{k_1, \ldots, k_r\}, \text{ or if } \epsilon = 1 \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

   and the unique non-trivial partition in the k_1th row of the 2-core tower of λ is α_x, where the binary sequence x of length k begins with ϵ. In this case f_λ is odd.

2. For some $0 < v < k_1$,

$$w_i(\lambda) = \begin{cases} 2 & \text{if } i = k_1 - v, \\ 1 & \text{if } k_1 - v + 1 \leq i \leq k_1 - 1 \text{ or } i \in \{k_2, \ldots, k_r\}, \\ & \text{or if } \epsilon = 1 \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

   and the two non-trivial partitions in the $(k - v)$th row of the 2-core tower of λ are α_x and α_y, for binary sequences x and y such that x begins with 0 and y begins with 1. In this case $v_2(f_\lambda) = v$.

3. We have $\epsilon = 1$ and the partition λ satisfies

$$w_i(\lambda) = \begin{cases} 3 & \text{if } i = 0, \\ 1 & \text{if } i \in \{1, \ldots, k_1 - 1, k_2, \ldots, k_r\}. \end{cases}$$

   In this case, $v_2(f_\lambda) = k_1$.

Counting such towers gives:

If $n = \epsilon + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}$, with $0 < k_1 < \cdots < k_r, \epsilon \in \{0, 1\}$, then the number $b_v(n)$ of chiral partitions λ of n for which $v_2(f_\lambda) = v$ is given by

$$b_v(n) = 2^{k_2+\cdots+k_r} \times \begin{cases} 2^{k_1-1} & \text{if } v = 0, \\ 2^{(v+1)(k_1-2)-\binom{v}{2}} & \text{if } 0 < v < k_1, \\ \epsilon 2^{\binom{k_1}{2}} & \text{if } v = k, \\ 0 & \text{if } v > k_1. \end{cases}$$

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$$b(n) = 2^{k_2+\cdots+k_r}\left(2^{k_1-1} + \sum_{v=1}^{k_1-1} 2^{(v+1)(k_1-2)-\binom{v}{2}} + \epsilon 2^{\binom{k_1}{2}}\right).$$

Growth

Let $a(n)$ be the number of partitions of n for which f_λ is odd.
Recall $b(n)$ is the number of chiral partitions of n.

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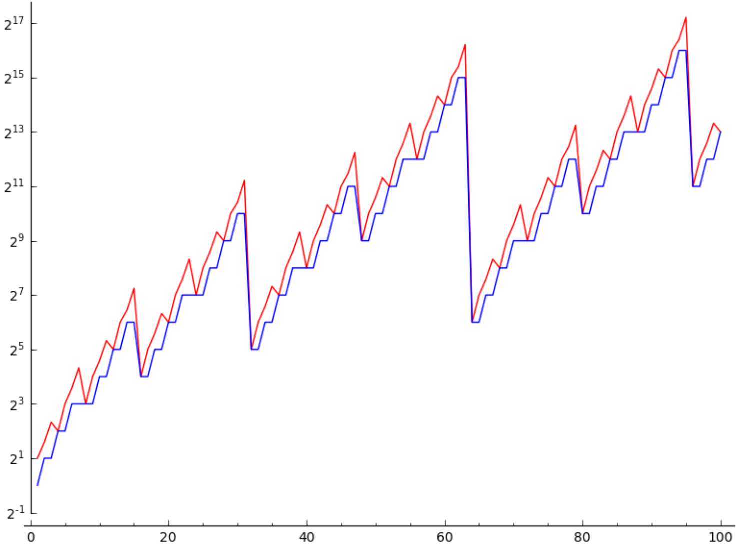
$$a(n) = 2^{k_1+\cdots+k_r},$$

$$b(n) = 2^{k_2+\cdots+k_r}\left(2^{k_1-1} + \sum_{v=1}^{k_1-1} 2^{(v+1)(k_1-2)-\binom{v}{2}} + \epsilon 2^{\binom{k_1}{2}}\right).$$

Comparison of $a(n)$ and $b(n+2)$



$2/5 \leq a(n)/b(n+2) \leq 1.$

Growth

For all n:

$$n \leq a(n) \leq 2^{\log_2(n+1)(\log_2(n+1)-1)/2}.$$

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Hardy-Ramanujan formula:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3}) \text{ as } n \to \infty.$$

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So $a(n)/p(n) \to 0$, and $b(n)/p(n) \to 0$.

n	$p(n)$	$b(n)$	$b(n)/p(n)$
100	190569292	6144	$3.2240241518 \times 10^{-05}$
200	3972999029388	98304	$2.47430213984 \times 10^{-08}$
300	9253082936723602	196608	$2.12478372176 \times 10^{-11}$
400	6727090051741041926	2883584	$4.28652504697 \times 10^{-13}$
500	2300165032574323995027	3221225472	$1.40043232828 \times 10^{-12}$
600	458004788008144308553622	6291456	$1.37366598881 \times 10^{-17}$
700	60378285202834474611028659	805306368	$1.33376820043 \times 10^{-17}$
800	5733052172321422504456911979	178257920	$3.10930224673 \times 10^{-20}$
900	415873681190459054784114365430	50331648	$1.21026288213 \times 10^{-22}$
1000	24061467864032622473692149727991	412316860416	$1.71359811773 \times 10^{-20}$
1100	1147240591519695580043346988281283	1572864	$1.37099751493 \times 10^{-27}$
1200	46240102378152881298913555099661657	369098752	$7.98222177325 \times 10^{-27}$
1300	1607818855017534550841511230454411672	12582912	$7.82607565568 \times 10^{-30}$
1400	49032194652550394774839040691532998261	103079215104	$2.10227618475 \times 10^{-27}$
1500	1329461690763193888825263136701886891117	824633720832	$6.20276407031 \times 10^{-28}$
1600	32417690376154241824102577250721959572183	22699573248	$7.0022179201 \times 10^{-31}$
1700	717802041964941442478681516751205185010007	6442450944	$8.97524744617 \times 10^{-33}$
1800	14552716211005418005132948684850541312590849	1610612736	$1.10674372581 \times 10^{-34}$
1900	272089289788583262011466359201428623427767364	6597069766656	$2.42459737088 \times 10^{-32}$

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Mathematics > Representation Theory

Representations of symmetric groups with non-trivial determinant

Arvind Ayyer, Amritanshu Prasad, Steven Spallone

(Submitted on 29 Apr 2016)

We give a closed formula for the number of partitions λ of n such that the corresponding irreducible representation V_λ of S_n has non-trivial determinant. We determine how many of these partitions are self-conjugate and how many are hooks. This is achieved by characterizing the 2-core towers of such partitions. We also obtain a formula for the number of partitions of n such that the associated permutation representation of S_n has non-trivial determinant.

Comments:	22 pages, 4 figures
Subjects:	**Representation Theory (math.RT)**; Combinatorics (math.CO)
MSC classes:	05E10, 20C30, 05A17, 05A15
Cite as:	arXiv:1604.08837 [math.RT]
	(or arXiv:1604.08837v1 [math.RT] for this version)

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