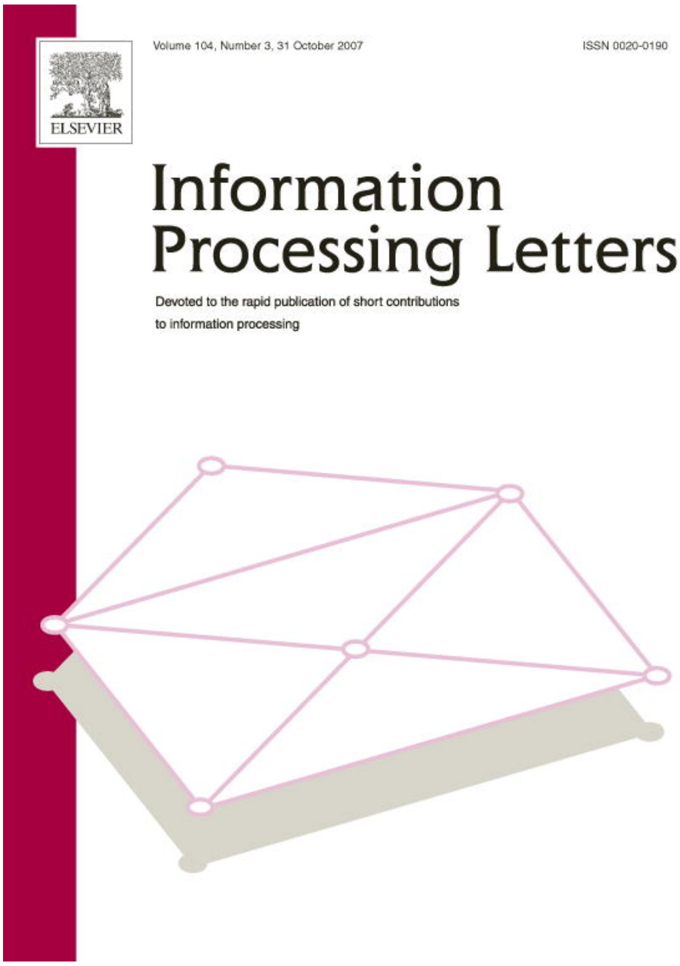
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Parameterized complexity of the induced subgraph problem in directed graphs

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Abstract

In this Letter, we consider the parameterized complexity of the following problem: Given a hereditary property $\mathcal{P}$ on digraphs, an input digraph D and a positive integer k, does D have an induced subdigraph on k vertices with property $\mathcal{P}$? We completely characterize hereditary properties for which this induced subgraph problem is $W[1]$-complete for two classes of directed graphs: general directed graphs and oriented graphs. We also characterize those properties for which the induced subgraph problem is $W[1]$-complete for general directed graphs but fixed parameter tractable for oriented graphs. These results are among the very few parameterized complexity results on directed graphs.

Keywords: Computational complexity; Combinatorial problems; Parameterized complexity; Directed graphs; Induced subgraph; Hereditary properties

1. Introduction

Parameterized complexity is an approach developed by Downey and Fellows for dealing with computationally hard problems where small parameter values cover many practical applications. Consider, for instance, the NP-complete VERTEX COVER and DOMINATING SET problems. These problems are defined as follows: Given a graph G and a positive integer parameter k, decide whether G has a vertex cover (respectively, dominating set) of size at most k. Both problems can be solved in time $O(n^{k+2})$, where n is the number of vertices of G. What is interesting is that for the VERTEX COVER problem there exists an algorithm with run time $O(c^k \cdot n)$, where c is a constant, whereas for DOMINATING SET there is reason to believe that no such algorithm exists.

Parameterized complexity is mainly concerned with obtaining algorithms for parameterized problems with run time $O(f(k) \cdot n^{O(1)})$, where f is a function of k alone, as against a run time of $O(n^{O(k)})$. Here k is the parameter for the problem. A parameterized problem which admits an algorithm with run time $O(f(k) \cdot n^{O(1)})$ is called *fixed parameter tractable* (FPT). For a comprehensive introduction to parameterized complexity see the classic monograph by Downey and Fellows [3] or the recent texts by Niedermeier [10] and Flum and Grohe [4].

In this Letter, we consider the parameterized complexity of a class of problems in (directed) graphs that are loosely termed as the INDUCED SUBGRAPH prob-

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lem. This problem is defined as follows: Given a graph G and a positive integer k, does G have a vertex induced subgraph of size k satisfying some prespecified property? Lewis and Yannakakis [9] proved that this problem is NP-complete when the property is nontrivial and hereditary. Khot and Raman [8] studied the parameterized complexity of this problem in undirected graphs and completely characterized for which properties the problem is FPT and for which ones the problem is $W[1]$-complete. We extend their result for hereditary properties on directed graphs. As a corollary of our results, we show, for example, that the problem of deciding whether an input digraph D has a transitive induced subdigraph of size k is fixed parameter tractable while the problem of deciding whether D has a planar induced subdigraph of size k is $W[1]$-complete.

There have been very few results on parameterized problems on directed graphs since, in general, many problems which can be formulated for both directed and undirected graphs are significantly more difficult for directed graphs [7]. For instance, the FEEDBACK EDGE SET problem is polynomial-time solvable in undirected graphs but NP-complete in directed graphs [5]. From the parameterized complexity point of view, the UNDIRECTED FEEDBACK VERTEX SET problem is known to be fixed parameter tractable but the DIRECTED FEEDBACK VERTEX (EDGE) SET is a celebrated open problem. Our results in this paper add to the growing literature on parameterized complexity results on directed graphs.

This paper is organized as follows. In Section 2, we define the problem formally and briefly survey some previous work. In Section 3, we give a complete specification of when the INDUCED SUBGRAPH problem is fixed parameter tractable and when it is not, for hereditary properties on general directed graphs. In Section 4, we consider the problem for hereditary properties on oriented graphs. An oriented graph is a directed graph which has at most one arc between any pair of vertices. In Section 5, we characterize those hereditary properties for which the INDUCED SUBGRAPH problem is hard on general digraphs but FPT on oriented graphs. We end with some concluding remarks in Section 6.

2. Problem definition and previous work

A *graph property* $\mathcal{P}$ is an isomorphism-closed set of graphs. A graph property $\mathcal{P}$ is *nontrivial* if there exists an infinite family of graphs satisfying $\mathcal{P}$ and an infinite family not satisfying $\mathcal{P}$. A graph property $\mathcal{P}$ is *hereditary* if $G \in \mathcal{P}$ implies that every induced subgraph of G is also in $\mathcal{P}$ (see [9]). Examples of hereditary properties (for undirected graphs) include the class of planar, outerplanar, bipartite, interval, comparability, acyclic, bounded-degree, chordal, complete, independent set and line invertible graphs [9]. Similarly for digraphs, the following graph classes are hereditary: acyclic, transitive, symmetric, anti-symmetric, line-digraph, maximum outdegree r, maximum indegree r, without cycles of length l, without cycles of length $\leqslant l$ [9].

A property $\mathcal{P}$ has a *forbidden set characterization* if there exists a set $\mathcal{F}$ of graphs such that G has property $\mathcal{P}$ if and only if no element of $\mathcal{F}$ is an induced subgraph of G. The set $\mathcal{F}$ is called the *forbidden set* of $\mathcal{P}$. It is well known that a property $\mathcal{P}$ is hereditary if and only if it has a forbidden set characterization [2]. For if a property $\mathcal{P}$ has a forbidden set characterization, it is clearly hereditary. Conversely suppose $\mathcal{P}$ is hereditary and consider the set $\mathcal{S}$ of graphs *not* in $\mathcal{P}$. The induced subgraph relation defines a partial order among the elements of $\mathcal{S}$ and the minimal elements of this partial order form the forbidden set of $\mathcal{P}$.

For a property $\mathcal{P}$ on (directed) graphs, the INDUCED SUBGRAPH problem is defined as follows: Given a (directed) graph G find a vertex subset of maximum size that induces a subgraph with property $\mathcal{P}$. Lewis and Yannakakis [9] proved this problem to be NP-hard when the property $\mathcal{P}$ is nontrivial and hereditary. If, in addition, the given property can be tested in polynomial time, their results show that the INDUCED SUBGRAPH problem is NP-complete. The parameterized version of this problem for a given property $\mathcal{P}$ is defined as follows.

$P(G, k, \mathcal{P})$

Input: A graph $G = (V, E)$ with vertex set V and edge set E.
Parameter: A positive integer $k \leqslant |V|$.
Question: Does G have an induced subgraph on at least k vertices with property $\mathcal{P}$?

Call the directed graphs version of this problem $P(D, k, \mathcal{P})$.

Khot and Raman [8] resolved the problem $P(G, k, \mathcal{P})$ when $\mathcal{P}$ is a nontrivial hereditary property on undirected graphs. They show that if the property $\mathcal{P}$ either contains all independent sets and all cliques or excludes an independent set and a clique then the problem $P(G, k, \mathcal{P})$ is fixed parameter tractable and $W[1]$-complete otherwise. The proof techniques employed by them make heavy use of Ramsey theory. In particular, they make use of the fact that any "sufficiently large" undirected graph either contains an independent set or a clique.

In this Letter, we consider the problem $P(D,k,\mathcal{P})$ when $\mathcal{P}$ is a nontrivial hereditary property on directed graphs. We give a complete specification of when the problem $P(D,k,\mathcal{P})$ is fixed parameter tractable and when it is not.

3. The induced subgraph problem for general directed graphs

We begin with by examining a specialization of Ramsey's theorem applicable to directed graphs.

Fact 1. *(See [6].) Suppose that for every set S with n elements, the 2-element subsets of S are partitioned into m disjoint families $\mathcal{F}_1,\dots,\mathcal{F}_m$. Let $p_1,\dots,p_m$ be any positive integers with $p_i \geqslant 2$, $1 \leqslant i \leqslant m$. Then there is a number $r(p_1,\dots,p_m)$, such that for every set S with $n \geqslant r(p_1,\dots,p_m)$ elements there exists an i, $1 \leqslant i \leqslant m$, and a subset A_i of S with p_i elements all of whose 2-subsets are in the family $\mathcal{F}_i$.*

If $D=(V,A)$ is a digraph with $V=\{u_1,\dots,u_n\}$, partition the 2-subsets of V into four classes, as follows:

$$\mathcal{F}_1 = \bigl\{\{u_i,u_j\colon (u_i,u_j),(u_j,u_i)\notin A\}\bigr\},$$
$$\mathcal{F}_2 = \bigl\{\{u_i,u_j\colon (u_i,u_j),(u_j,u_i)\in A\}\bigr\},$$
$$\mathcal{F}_3 = \bigl\{\{u_i,u_j\colon (u_i,u_j)\in A,(u_j,u_i)\notin A, i<j\}\bigr\},$$
$$\mathcal{F}_4 = \bigl\{\{u_i,u_j\colon (u_i,u_j)\notin A,(u_j,u_i)\in A, i<j\}\bigr\}.$$

From Ramsey's theorem we have

Corollary 1. *Let p_1,p_2,p_3 be any positive natural numbers $\geqslant 2$. Then there exists a positive number $r(p_1,p_2,p_3)$ such that any directed graph D on at least $r(p_1,p_2,p_3)$ vertices contains either an independent set of size p_1, or a complete symmetric digraph of size p_2, or an acyclic tournament of size p_3.*

Thus if $\mathcal{P}$ is a nontrivial hereditary property on digraphs, then it must contain either all independent sets, all complete symmetric digraphs and all acyclic tournaments or exactly two of these graph types of all sizes or exactly one of these graph types of all sizes.

Theorem 1. *If $\mathcal{P}$ is a hereditary property on digraphs that either contains all independent sets (i.s.), all complete symmetric (c.s.) digraphs and all acyclic tournaments (a.t.) or excludes a graph of each of these three types, then the problem $P(D,k,\mathcal{P})$ is fixed parameter tractable.*

Proof. Suppose $\mathcal{P}$ excludes an independent set of size c_1, a c.s. digraph of size c_2 and an acyclic tournament of size c_3. Then $\mathcal{P}$ cannot contain any digraph D such that $|V(D)| \geqslant r(c_1,c_2,c_3)$ and is therefore finite. The problem $P(D,k,\mathcal{P})$ can then be decided in polynomial time.

Therefore assume that $\mathcal{P}$ contains all independent sets, all c.s. digraphs and all acyclic tournaments. If $|V(D)| \geqslant r(k,k,k)$[1] then, by Ramsey's theorem, D has either an independent set, or a c.s. digraph or an acyclic tournament of size k as an induced subgraph. Thus the given instance is a YES-instance. Otherwise, $|V(D)| < r(k,k,k)$ and we check all subsets $S \subseteq V(D)$ of size k to see whether $D[S]$ has property $\mathcal{P}$. This takes time $\binom{r(k,k,k)}{k} \cdot f(k)$, where $f(k)$ is the time taken to decide whether a digraph on k vertices has property $\mathcal{P}$. This proves that the problem $P(D,k,\mathcal{P})$ is fixed parameter tractable. □

Corollary 2. *Given any directed graph D and an integer k, it is fixed parameter tractable to decide whether D has an induced subdigraph on k vertices that is* (1) *a kernel perfect digraph,* (2) *an intersection digraph,* (3) *a chordal digraph,* (4) *a transitive digraph, or* (5) *a quasi-transitive digraph.* (*See* [1] *for the definitions of these graphs.*)

3.1. *W*[1]-completeness results

We show that if the property $\mathcal{P}$ contains exactly two of the graph types of all sizes or exactly one of the graph types of all sizes then the problem $P(D,k,\mathcal{P})$ is $W[1]$-complete. To do this, we first show that the problem $P(D,k,\mathcal{P})$ is in $W[1]$ for any nontrivial decidable hereditary property $\mathcal{P}$. We next show that the problem is $W[1]$-hard by exhibiting a parametric reduction from a $W[1]$-hard problem.

Lemma 1. *Let $\mathcal{P}$ be a nontrivial decidable hereditary property on digraphs. Then the problem $P(D,k,\mathcal{P})$ is in $W[1]$.*

Proof. We reduce the $P(D,k,\mathcal{P})$ problem to the SHORT TURING MACHINE ACCEPTANCE problem (defined below) which is complete for the class $W[1]$ [3].

Input: A nondeterministic Turing machine M and a string x.
Parameter: A positive integer k.

[1] We do not need to know the number $r(k,k,k)$ exactly. An upper bound on $r(k,k,k)$ will serve our purpose.

Question: Does M have a computation path accepting x in at most k steps?

Let $(D=(V,E),k)$ be an instance of the $P(D,k,\mathcal{P})$ problem, with $|V|=n$. We will show that we can construct an instance (M_D,x,k') of the SHORT TURING MACHINE ACCEPTANCE problem in time $\mathrm{O}(f(k)\cdot n^{\mathrm{O}(1)})$ such that D has an induced subgraph of size k satisfying property $\mathcal{P}$ if and only if M_D accepts x within k' steps, where k' depends only on k.

First note that since we assumed $\mathcal{P}$ to be decidable, there exists a DTM M' that takes a digraph D as input and in time $t(|V(D)|)$ decides whether D satisfies $\mathcal{P}$. The input alphabet of M_D consists of the $n+1$ symbols $1,2,3,\ldots,n,\sharp$. The NTM M_D performs the following steps.

- (i) M_D nondeterministically writes a sequence of k numbers on its tape out of its tape alphabet $\{1,2,\ldots,n\}$.
- (ii) It then verifies whether the k numbers it has picked are distinct.
- (iii) It then constructs the subgraph D' of D represented by these k vertices.
- (iv) M_D passes control to M' which then verifies whether D' satisfies $\mathcal{P}$. If yes, M_D accepts.

The time taken in Steps 1, 2 and 4 are, respectively, $\mathrm{O}(k)$, $\mathrm{O}(k^2)$ and $t(k)$. Assuming that the graph D is hardwired in M_D as an adjacency matrix, Step 3 takes time $\mathrm{O}(k^2)$.

It is easy to see that (D,k) is a YES-instance of the problem $P(D,k,\mathcal{P})$ if and only if the nondeterministic Turing machine M_D accepts the empty string in $k'=\mathrm{O}(k+k^2+t(k))$ steps. □

To prove $W[1]$-hardness, we consider the following four cases:

- (i) The property $\mathcal{P}$ contains all c.s. digraphs but not all independent sets.
- (ii) The property $\mathcal{P}$ contains all independent sets but not all c.s. digraphs.
- (iii) $\mathcal{P}$ contains all acyclic tournaments but not all independent sets.
- (iv) $\mathcal{P}$ contains all independent sets but not all acyclic tournaments.

Note that (i)–(iv), though not mutually exclusive, are exhaustive.

We first show that the problem $P(D,k,\mathcal{P})$ is $W[1]$-hard in cases (i) and (ii).

Theorem 2. *Let $\mathcal{P}$ be a hereditary property on digraphs that contains all c.s. digraphs but not all independent sets or vice versa. Then the problem $P(D,k,\mathcal{P})$ is $W[1]$-complete.*

Proof. Membership in $W[1]$ was shown in Lemma 1. We therefore need only establish $W[1]$-hardness.

Let $\mathcal{P}$ be a property on digraphs. Define $\mathcal{P}_1$ as follows. An undirected graph $G\in\mathcal{P}_1$ if and only if the directed graph D obtained from G by replacing every edge $\{u,v\}\in E(G)$ by the arcs (u,v) and (v,u) is in $\mathcal{P}$. Note that G contains a clique of size k if and only if D contains a c.s. digraph of size k and G contains an independent set of size k if and only if D contains an independent set of size k. Also note that

- (i) $\mathcal{P}_1$ is nontrivial and hereditary if and only if $\mathcal{P}$ is nontrivial and hereditary,
- (ii) $\mathcal{P}_1$ and contains all cliques but not all independent sets if and only if $\mathcal{P}$ contains all c.s. digraphs but not all independent sets, and
- (iii) $\mathcal{P}_1$ and contains all independent sets but not all cliques if and only if $\mathcal{P}$ contains all independent sets but not all c.s. digraphs.

By Khot and Raman [8], the problem $P_1(G,k,\mathcal{P}_1)$ is $W[1]$-hard when $\mathcal{P}_1$ contains all cliques but not all independent sets or vice versa.

We now exhibit a parametric reduction from $P_1(G,k,\mathcal{P}_1)$ to $P(D,k,\mathcal{P})$. Let (G,k) be an instance of P_1. Construct a directed graph D as follows: $V(D)=V(G)$ and for all $u,v\in V(G)$, if $\{u,v\}\in E(G)$ add the arcs (u,v) and (v,u) in $A(D)$. D has no other arcs. From the manner in which property $\mathcal{P}_1$ was defined, it is clear that G has an induced subgraph on k vertices satisfying $\mathcal{P}_1$ if and only if D has an induced subdigraph on k vertices satisfying $\mathcal{P}$. This completes the proof. □

We next show that the problem $P(D,k,\mathcal{P})$ is $W[1]$-hard in cases (iii) and (iv).

Theorem 3. *Let $\mathcal{P}$ be a hereditary property on digraphs that contains all acyclic tournaments but not all independent sets or vice versa. Then the problem $P(D,k,\mathcal{P})$ is $W[1]$-complete.*

Proof. As before, we show only $W[1]$-hardness. Let $\mathcal{P}$ be a property on digraphs. Define $\mathcal{P}_1$ to be a set of undirected graphs with the following property: An undirected graph $G \in \mathcal{P}_1$ if and only if the directed graph $D \in \mathcal{P}$, where $V(D) = V(G)$ and

$$A(D) = \{(u, v)\colon u < v \text{ and } \{u, v\} \in E(G)\}.$$

Clearly G has an independent set of size k if and only if D has an independent set of size k and G has a clique of size k if and only if D has an acyclic tournament of size k. Also

(i) $\mathcal{P}_1$ is nontrivial and hereditary if and only if $\mathcal{P}$ is nontrivial and hereditary,
(ii) $\mathcal{P}_1$ contains all independent sets but not all cliques if and only if $\mathcal{P}$ contains all independent sets but not all acyclic tournaments, and
(iii) $\mathcal{P}_1$ contains all cliques but not all independent sets if and only if $\mathcal{P}$ contains all acyclic tournaments but not all independent sets.

The problem $P_1(G, k, \mathcal{P}_1)$ is $W[1]$-hard by [8] when $\mathcal{P}_1$ contains all independent sets but not all cliques or vice versa.

We now exhibit a parametric reduction from the problem $P_1(G, k, \mathcal{P}_1)$ to the problem $P(D, k, \mathcal{P})$. Let (G, k) be an instance of the problem $P_1(G, k, \mathcal{P}_1)$. Let D be the directed graph obtained by orienting the edges of G from lower ordered vertices to higher ordered vertices. From the manner in which we constructed $\mathcal{P}_1$, it is easy to see that G has an induced subgraph on k vertices satisfying $\mathcal{P}_1$ if and only if D has an induced subdigraph on k vertices satisfying $\mathcal{P}$. This proves the theorem. □

We now look at some applications. For definitions of digraph properties introduced in the remainder of this section, one may consult Bang-Jensen and Gutin [1].

The set of symmetric digraphs contains all independent sets and all c.s. digraphs but no acyclic tournament. The following hereditary properties contain all independent sets and acyclic tournaments but not all c.s. digraphs:

(1) acyclic digraphs,
(2) antisymmetric digraphs,
(3) digraphs without dicycles of length l, and
(4) digraphs without dicycles of length $\leqslant l$.

Hence the following corollary is immediate from Theorems 2 and 3.

Corollary 3. *Given a digraph D and a positive integer k, it is $W[1]$-complete to decide whether D has an induced subdigraph of size k that is*

(1) *a symmetric digraph,*
(2) *acyclic,*
(3) *an antisymmetric digraph,*
(4) *without dicycles of length l, or*
(5) *without dicycles of length $\leqslant l$.*

The following digraph properties contain all independent sets but not all c.s. digraphs and acyclic tournaments:

(1) with maximum indegree r,
(2) with maximum outdegree r,
(3) bipartite,
(4) colorable with c colors, for some constant $c \geqslant 1$,
(5) planar,
(6) a line digraph.

Hence the following corollary is immediate from Theorem 3.

Corollary 4. *Given a digraph D and a positive integer k, it is $W[1]$-complete to decide whether D has an induced subdigraph of size k that is*

(1) *of maximum indegree r,*
(2) *of maximum outdegree r,*
(3) *bipartite,*
(4) *colorable with c colors, for some constant $c \geqslant 1$,*
(5) *planar, or*
(6) *a line digraph.*

4. The induced subgraph problem for oriented graphs

Though Corollary 3 says that finding an acyclic subdigraph is hard in general digraphs, Raman and Saurabh [11] have shown that the problem is FPT in oriented graphs. In this section, we look at the general INDUCED SUBGRAPH problem in oriented graphs.

An oriented graph is a directed graph in which every pair of vertices has at most one arc between them. Thus oriented graphs are precisely those digraphs with no 2-cycle. For oriented graphs, Ramsey's theorem says: For positive integers p and q there exists an integer $r(p, q) \in \mathbb{N}$ such that any oriented graph on at least $r(p, q)$ vertices either has an independent set of size p or an acyclic tournament of size q.

Any nontrivial hereditary property $\mathcal{P}$ on oriented graphs can therefore be classified into one of the three types:

(1) $\mathcal{P}$ contains all independent sets and all acyclic tournaments;
(2) $\mathcal{P}$ contains all independent sets but not all acyclic tournaments;
(3) $\mathcal{P}$ contains all acyclic tournaments but not all independent sets.

As one might suspect, the problem $P(D,k,\mathcal{P})$ is fixed parameter tractable for case (1) and $W[1]$-complete for cases (2) and (3). Membership in $W[1]$ can be easily proved by a parametric reduction to the SHORT TURING MACHINE ACCEPTANCE PROBLEM similar to the proof of Lemma 1.

Theorem 4. *Let $\mathcal{P}$ be a hereditary property on oriented graphs that either contains all independent sets and all acyclic tournaments or excludes an independent set and an acyclic tournament. Then the problem $P(D,k,\mathcal{P})$ is fixed parameter tractable.*

Proof. Suppose $\mathcal{P}$ excludes an independent set of size c_1 and an acyclic tournament of size c_2. Then $\mathcal{P}$ cannot contain any oriented graph D such that $|V(D)| \geqslant r(c_1, c_2)$ and is therefore finite. The problem $P(D,k,\mathcal{P})$ can then be decided in polynomial time.

Therefore assume that $\mathcal{P}$ contains all independent sets and all acyclic tournaments. If $|V(D)| \geqslant r(k,k)$ then, by Ramsey's theorem, D has either an independent set or an acyclic tournament of size k as an induced subgraph. Thus the given instance is a YES-instance. Otherwise, $|V(D)| < r(k,k)$ and we check all subsets $S \subseteq V(D)$ of size k to see whether $D[S]$ has property $\mathcal{P}$. This takes time $\binom{r(k,k)}{k} \cdot f(k)$, where $f(k)$ is the time taken to decide whether an oriented graph on k vertices has property $\mathcal{P}$. This proves that the problem $P(D,k,\mathcal{P})$ is fixed parameter tractable. □

Theorem 5. *Let $\mathcal{P}$ be a hereditary property on oriented graphs that contains all independent sets but not all acyclic tournaments or vice versa. Then the problem $P(D,k,\mathcal{P})$ is $W[1]$-complete.*

Proof. Let $\mathcal{P}$ be a property on oriented graphs. Define a property $\mathcal{P}'$ on undirected graphs as follows: An undirected graph G satisfies $\mathcal{P}'$ if and only if the directed graph D satisfies $\mathcal{P}$, where $V(D) = V(G)$ and $A(D) = \{(u,v)\colon u < v, \{u,v\} \in E(G)\}$. Clearly G has an independent set of size k if and only if D has an independent set of size k and G has a clique of size k if and only if D has an acyclic tournament of size k. Also

(i) $\mathcal{P}'$ is nontrivial and hereditary if and only if $\mathcal{P}$ is nontrivial and hereditary,
(ii) $\mathcal{P}'$ contains all independent sets but not all cliques if and only if $\mathcal{P}$ contains all independent sets but not all acyclic tournaments, and
(iii) $\mathcal{P}'$ contains all cliques but not all independent sets if and only if $\mathcal{P}$ contains all acyclic tournaments but not all independent sets.

The problem $P(G,k,\mathcal{P}')$ is $W[1]$-hard by Khot and Raman [8] and it is easy to see that $P(G,k,\mathcal{P}') \leqslant_{\mathrm{FPT}} P(D,k,\mathcal{P})$. □

5. General digraphs vs. oriented graphs

In this section, we characterize general digraph properties for which the problem $P(D,k,\mathcal{P})$, when restricted to oriented graphs, becomes fixed parameter tractable. In what follows, if $\mathcal{P}$ is a property on general directed graphs then its restriction $\mathcal{P}'$ to oriented graphs is defined to be the set of all oriented graphs satisfying $\mathcal{P}$.

Corollary 5. *Let $\mathcal{P}$ be a nontrivial hereditary property on digraphs such that the induced subgraph problem, $P(D,k,\mathcal{P})$, is $W[1]$-complete. If $\mathcal{P}'$ is its restriction to oriented graphs then the problem $P(D,k,\mathcal{P}')$, restricted to oriented graphs, is fixed parameter tractable if and only if either one of the following conditions are satisfied*:

(1) *$\mathcal{P}$ satisfies all independent sets and all acyclic tournaments but not all c.s. digraphs, or*
(2) *$\mathcal{P}$ satisfies all c.s. digraphs but not all independent sets and acyclic tournaments.*

Proof. ($\Leftarrow$) If $\mathcal{P}$ satisfies (1), then $\mathcal{P}'$ contains all independent sets and all acyclic tournaments. If $\mathcal{P}$ satisfies (2), then $\mathcal{P}'$ is finite. The FPT result then follows from Theorem 4.

($\Rightarrow$) If $\mathcal{P}$ does not satisfy either conditions (1) or (2) of the theorem and, if the problem $P(D,k,\mathcal{P})$ is $W[1]$-complete, then $\mathcal{P}'$ is a nontrivial hereditary property on oriented graphs that satisfies either all independent sets but not all acyclic tournaments or vice versa. The hardness proof then follows from Theorem 5. □

Acyclic digraphs form an example of a hereditary property that contains all independent sets and acyclic tournaments but no c.s. digraphs. Consequently, the INDUCED ACYCLIC SUBGRAPH problem is $W[1]$-

complete on general directed graphs but FPT on oriented graphs.

6. Conclusion

In this Letter, we have characterized hereditary properties on digraphs for which finding an induced subdigraph with k vertices in a given digraph is $W[1]$-complete. We first did this for general directed graphs and then for oriented graphs. We also characterized hereditary properties for which the induced subgraph problem is $W[1]$-complete on general directed graphs but FPT for oriented graphs.

A related problem is the GRAPH MODIFICATION problem $\mathcal{P}(i, j, k)$ which asks whether a given input graph G can be 'modified' by deleting at most i vertices, j edges and adding at most k edges so that the resulting graph satisfies property $\mathcal{P}$. More formally, this problem is defined as follows: Given an undirected graph $G = (V, E)$ does there exist $V' \subseteq V$, $E' \subseteq E$ and $E'' \subseteq E^c$ (the edge set of the complement graph) with $|V'| \leqslant i$, $|E'| \leqslant j$ and $|E^c| \leqslant k$, such that $G - V' - E' \cup E''$ satisfies $\mathcal{P}$?

Cai [2] has shown that if a hereditary property has a finite forbidden set the graph modification problem $\mathcal{P}(i, j, k)$ is fixed parameter tractable with parameters i, j, k. The case when the forbidden set is infinite is open. The graph modification problem can be framed for directed graphs as well (for directed graphs, E^c can be viewed as the set of all arcs not in input digraph D). The well-known DIRECTED FEEDBACK VERTEX SET problem can then be cast as the problem $\mathcal{P}(k, 0, 0)$, where $\mathcal{P}$ is the set of all acyclic digraphs. Hence it would be interesting to investigate the parameterized complexity of the GRAPH MODIFICATION problem in directed graphs.

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