Graph Modification Problems

A Modern Perspective

Setting the stage: Definitions and Preliminary Observations

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Brief excursions into specific examples

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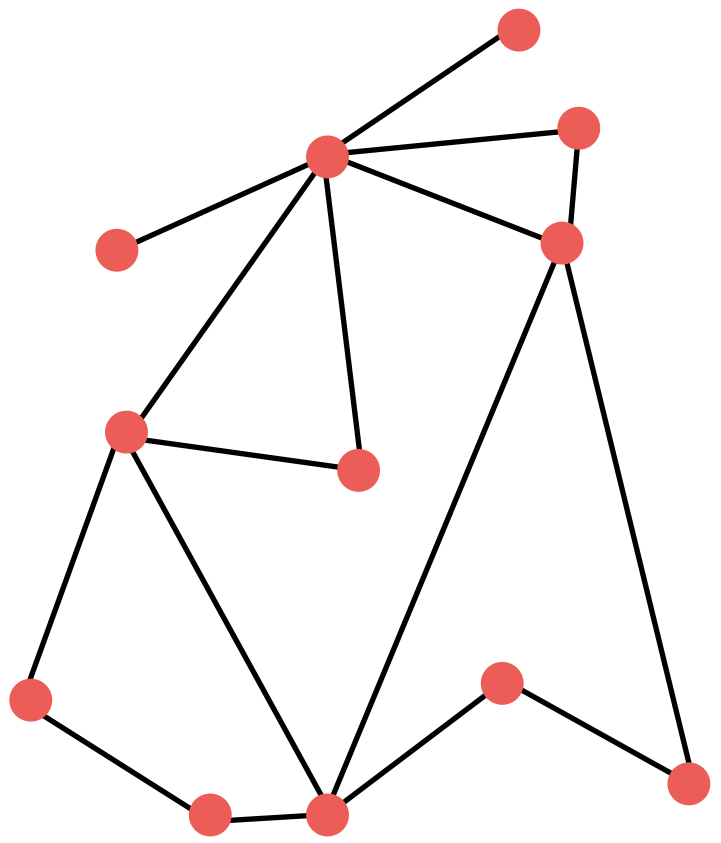
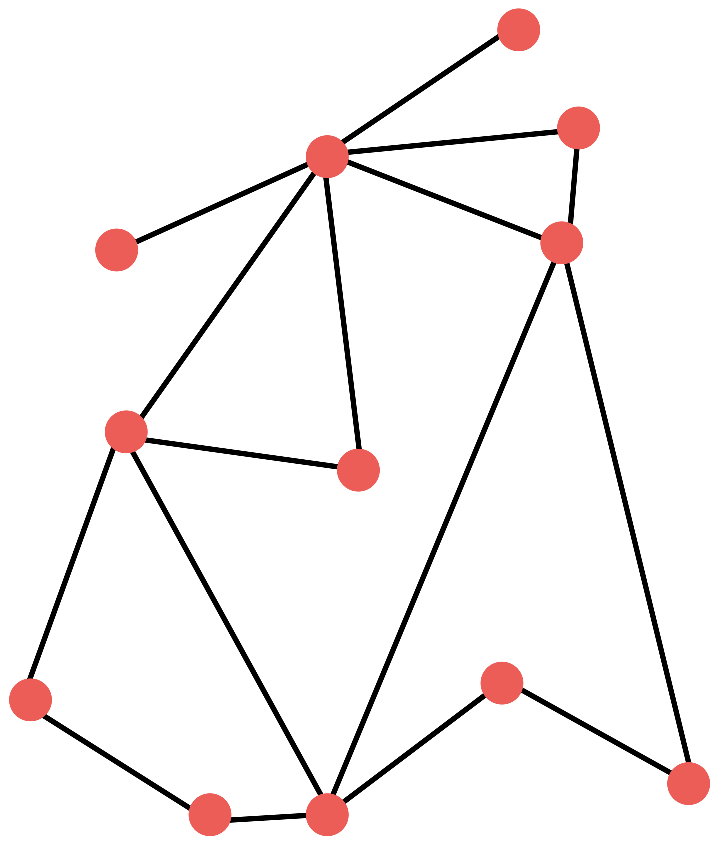
Brief excursions into specific examples

Current Trends & Future Directions

Ingredients of a typical Graph Modification Problem



Input Graph, G



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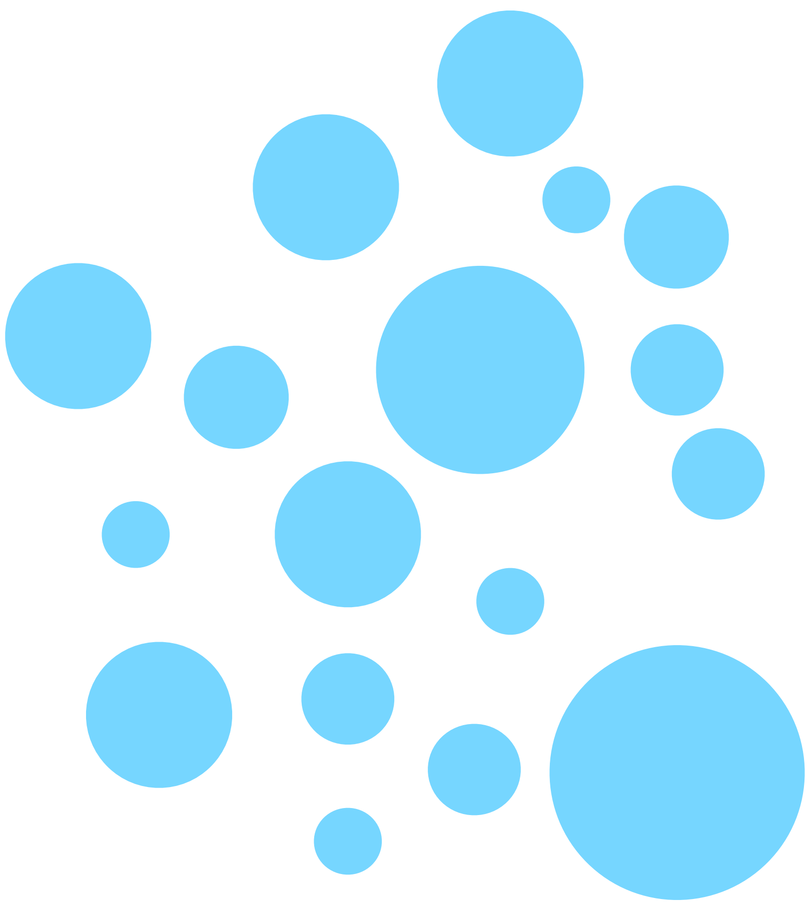
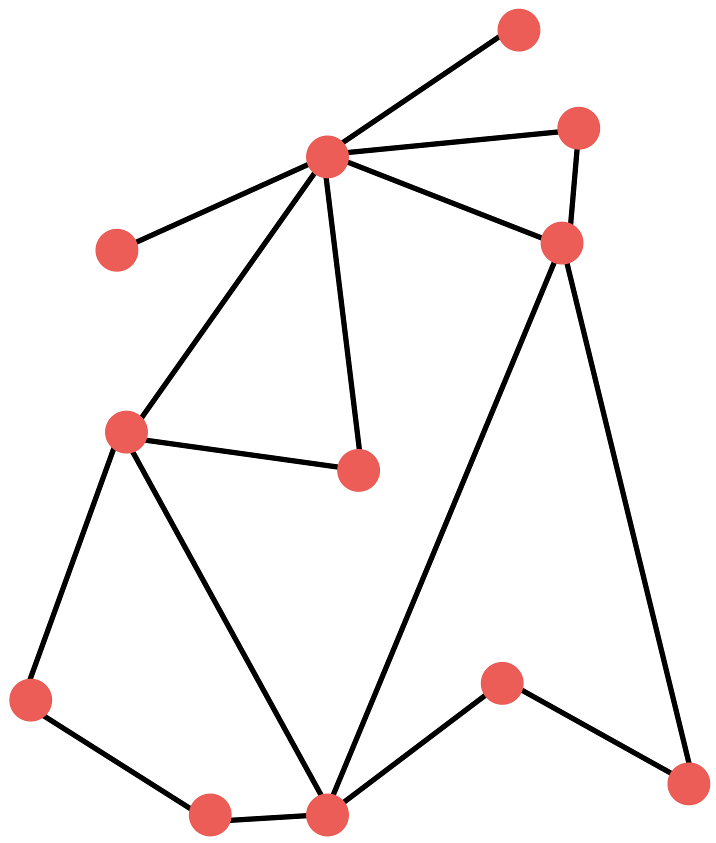


Graph Class or “property”, π

Goal: Transform G so that it belongs to **π** with "minimum damages".



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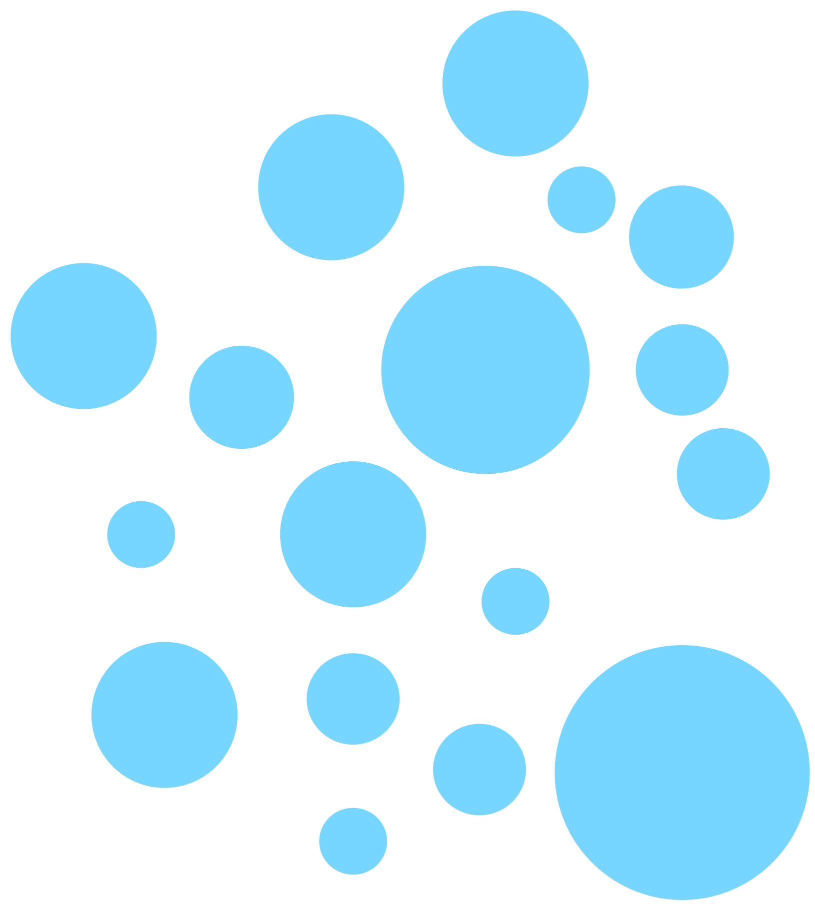

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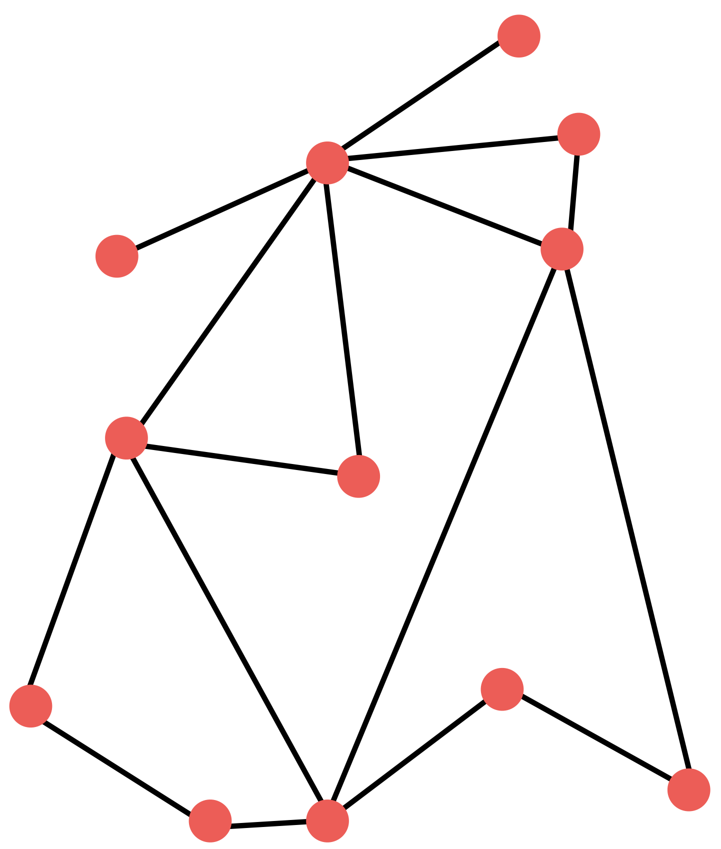

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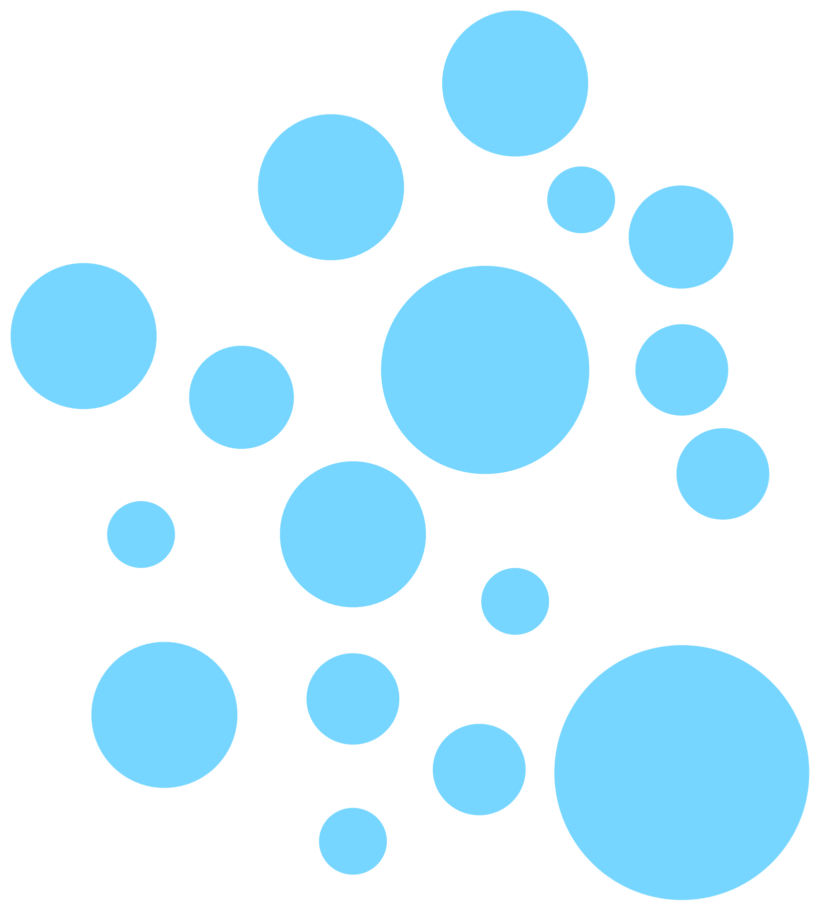

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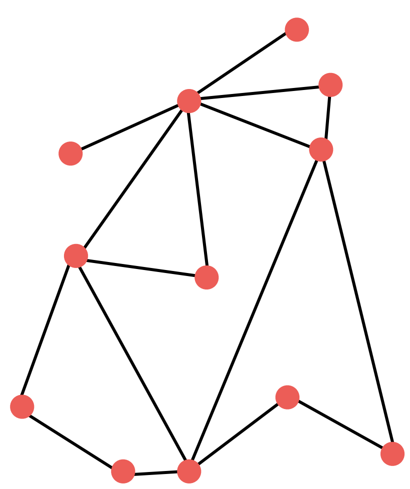

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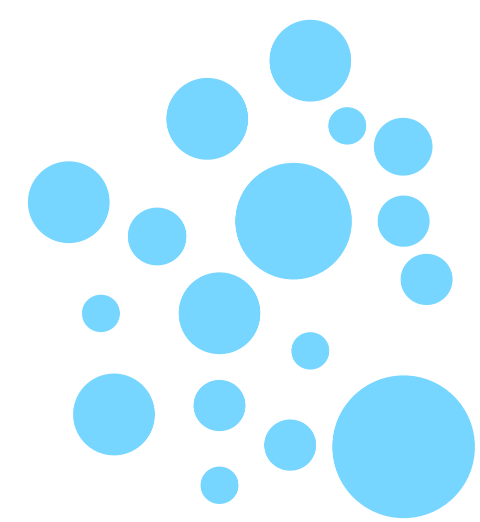

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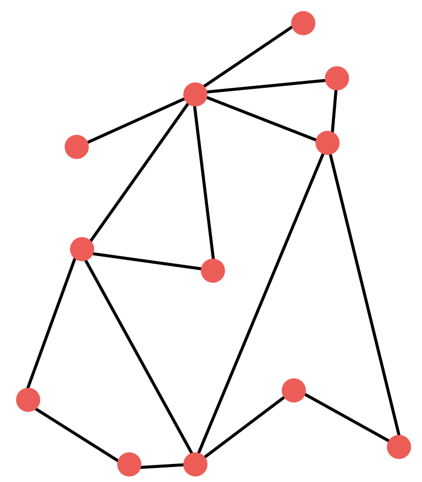
vertex deletions

Input Graph, G



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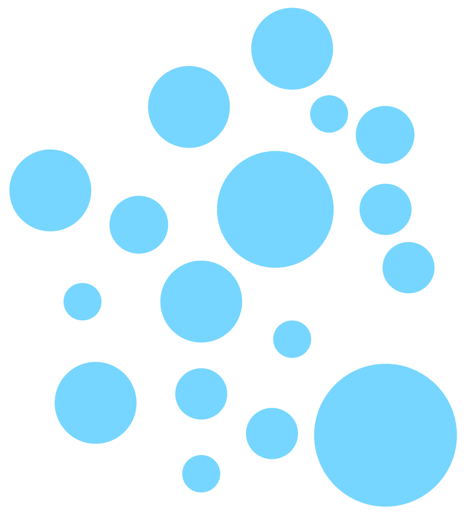

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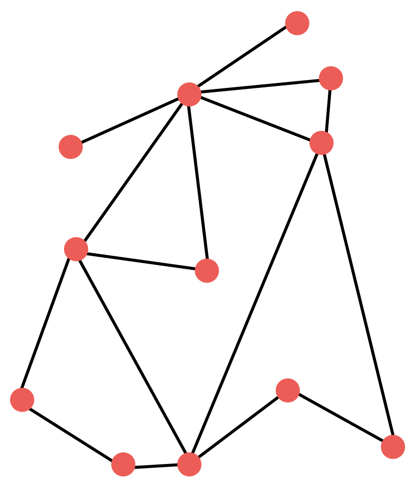
vertex deletions   edge deletions

Input Graph, G



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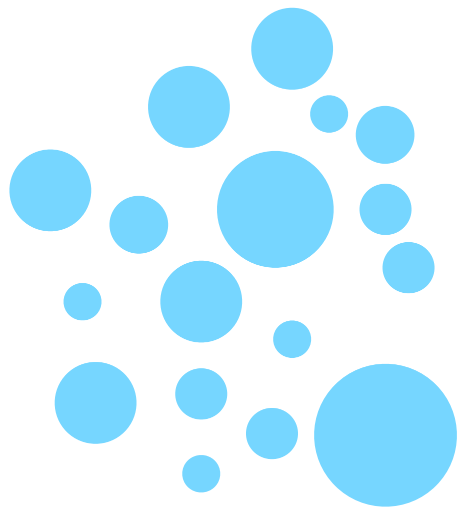

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vertex deletions edge deletions edge additions

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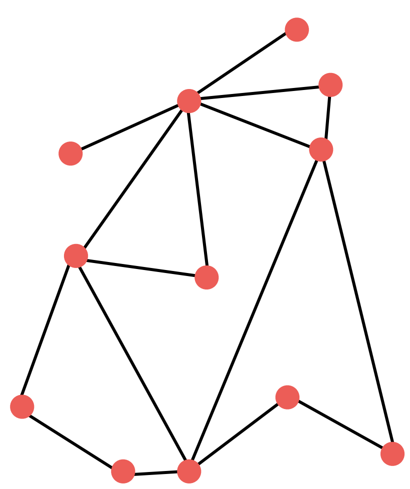

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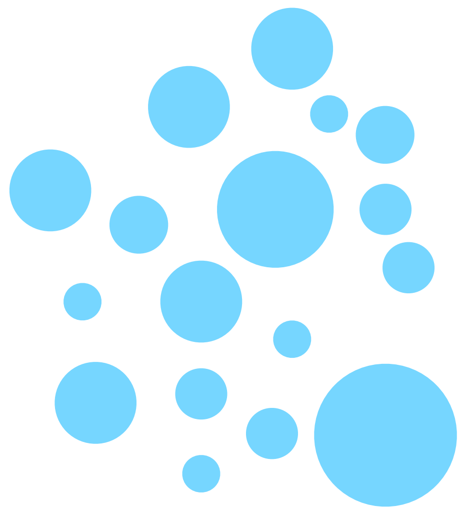

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vertex deletions edge deletions edge additions edge contractions

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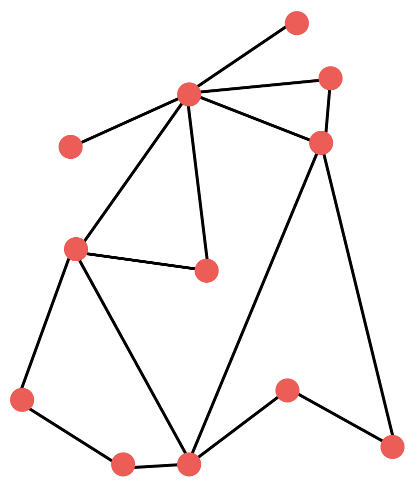

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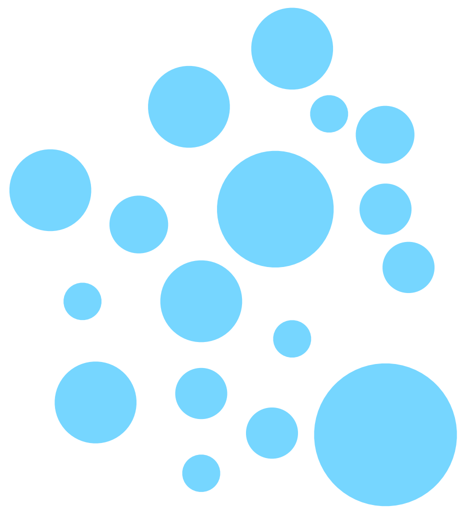

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vertex deletions edge deletions edge additions edge contractions edge editing

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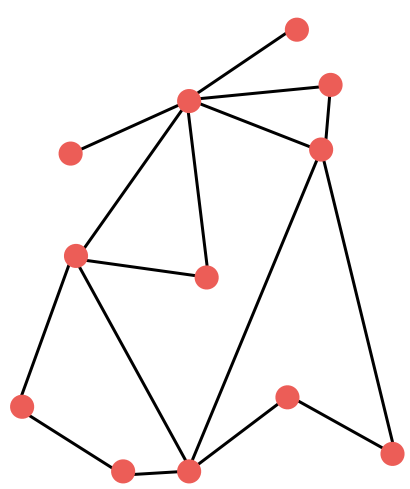

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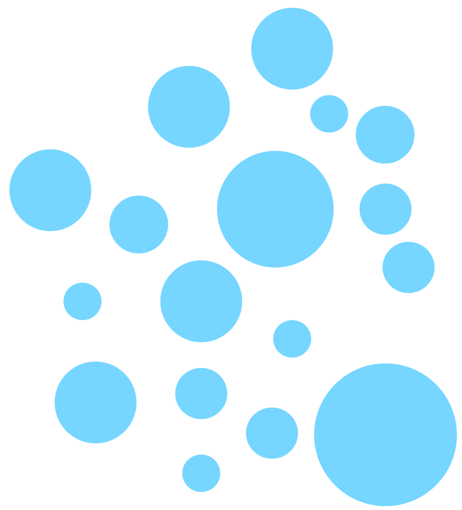

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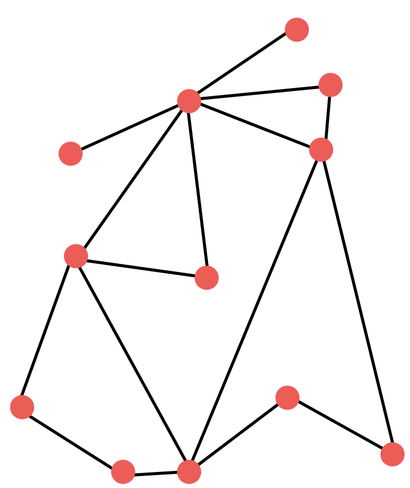
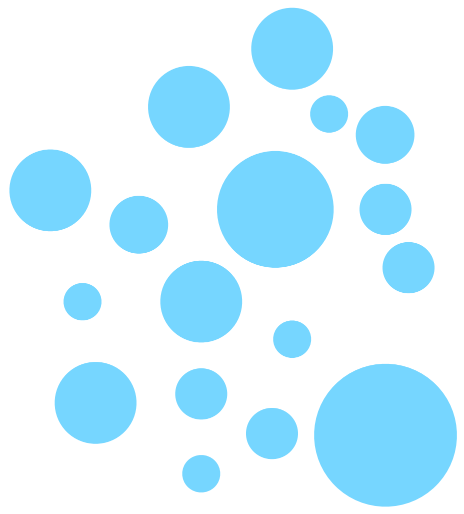
VERTEX COVER

Input Graph, G



Edgeless Graphs



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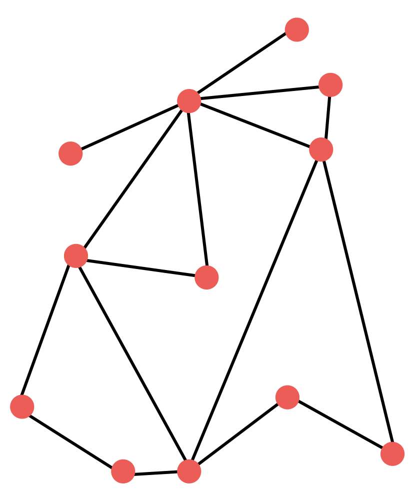
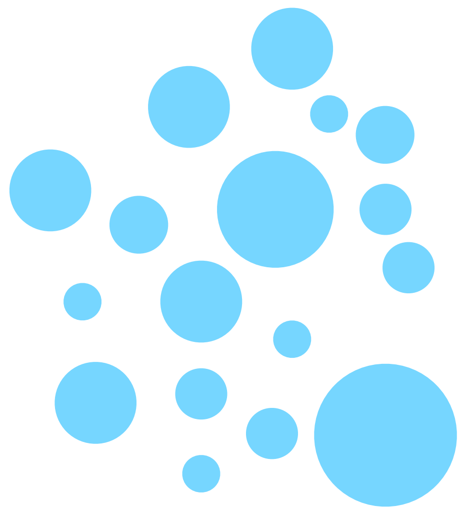
vertex deletions   edge deletions   edge additions   edge contractions   edge editing

Feedback Vertex Set

Input Graph, G



Acyclic Graphs



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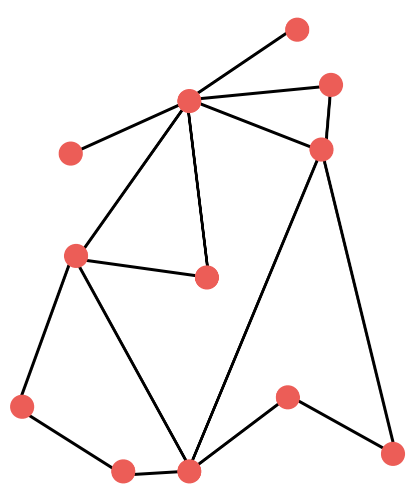
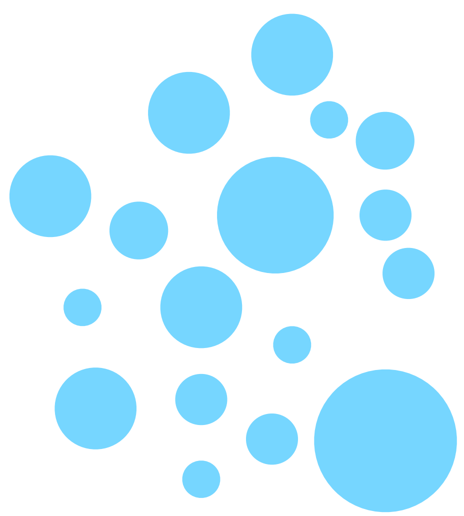
vertex deletions edge deletions **edge additions** edge contractions edge editing

MINIMUM FILL-IN

Input Graph, G



Chordal Graphs



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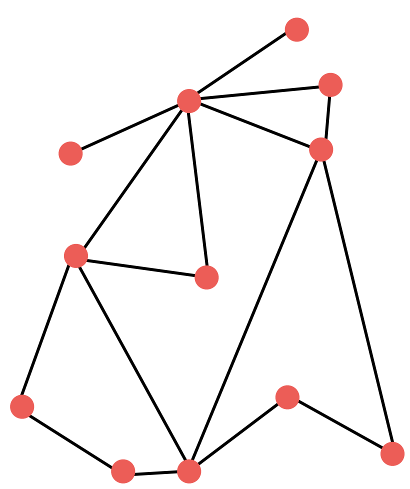
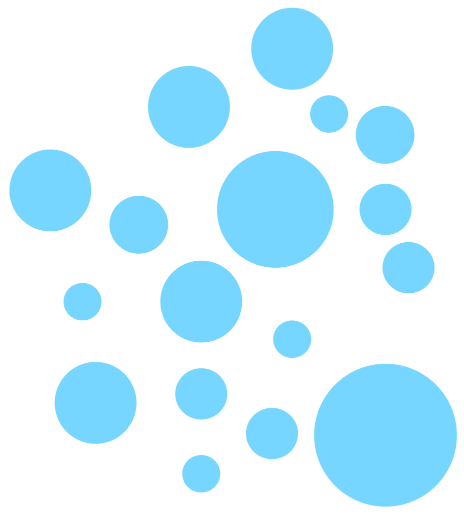
vertex deletions edge deletions edge additions edge contractions edge editing

Cluster Editing

Input Graph, G



Cluster Graphs



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Minimum **π**-Completion

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Minimum **π**-Completion

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Closest **π**-Graph

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Goal: Transform G so that it belongs to **π** with "minimum damages".

Minimum **π**-Completion	Minimum **π**-Supergraph
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Minimum **π**-Vertex Deletion	Maximum **π**-Induced Subgraph

Goal: Transform G so that it belongs to π with "minimum damages".

Minimum π-Completion

Minimum π-Deletion

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Minimum π-Supergraph

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Closest π-Graph

Maximum π-Induced Subgraph

Completion to Minimum Max-Clique

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Minimum π-Deletion

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Modification with Restrictions, eg, the Sandwich problem

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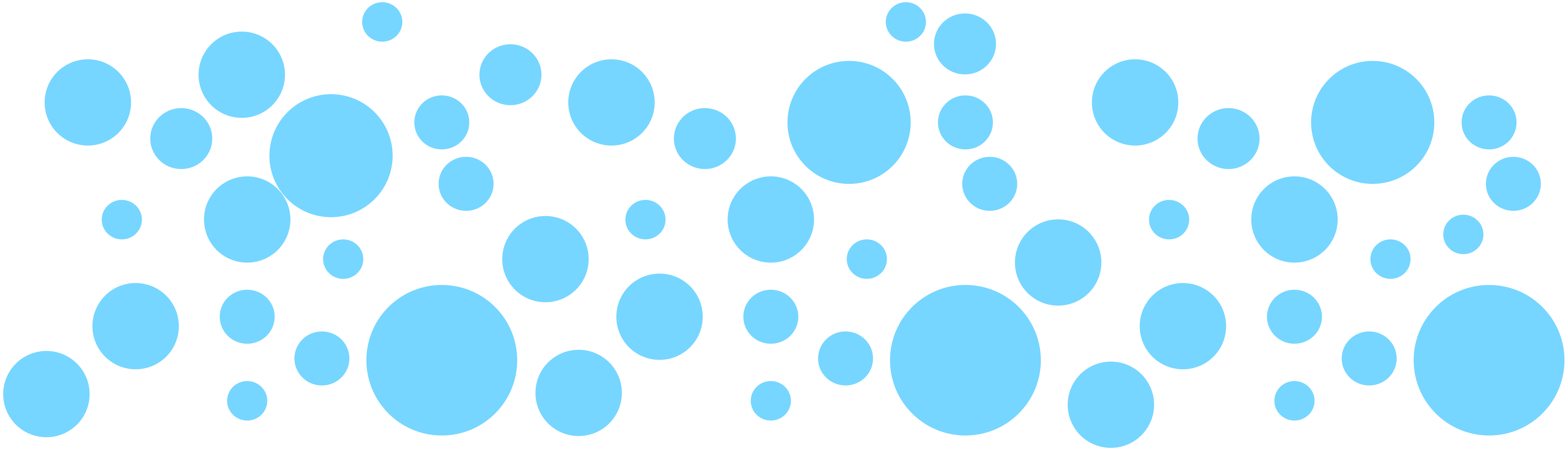
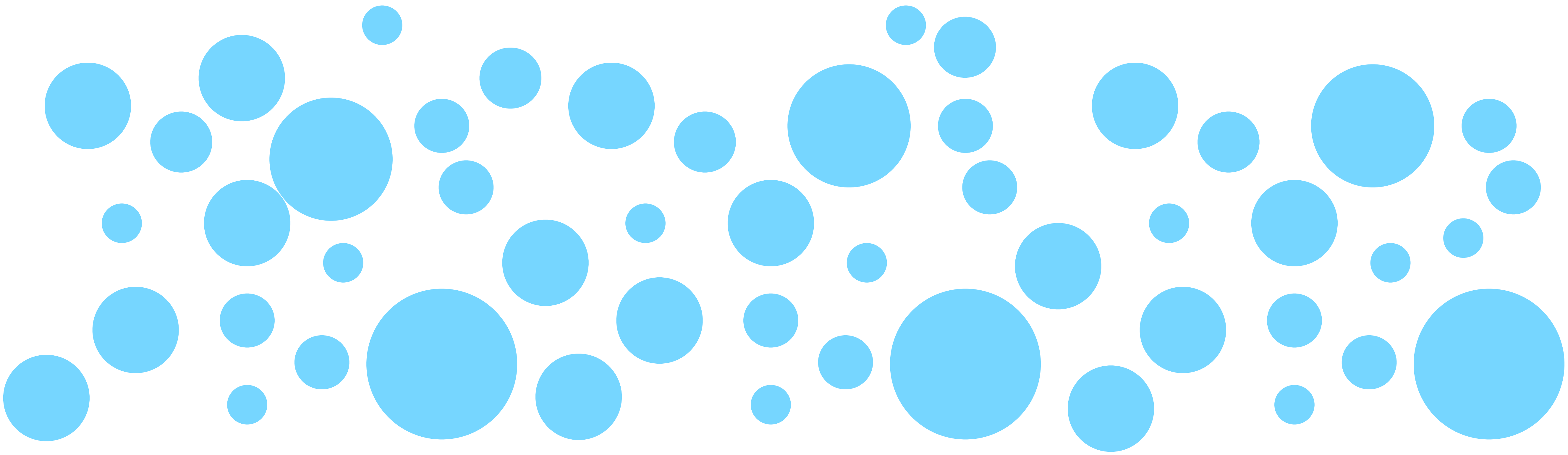
Modification with Restrictions, eg, the Sandwich problem

Restricted Classes of Input



Graph Class or “property”, π

<u>Non-Trivial</u> **π** admits infinitely many graphs, and also excludes infinitely many graphs.



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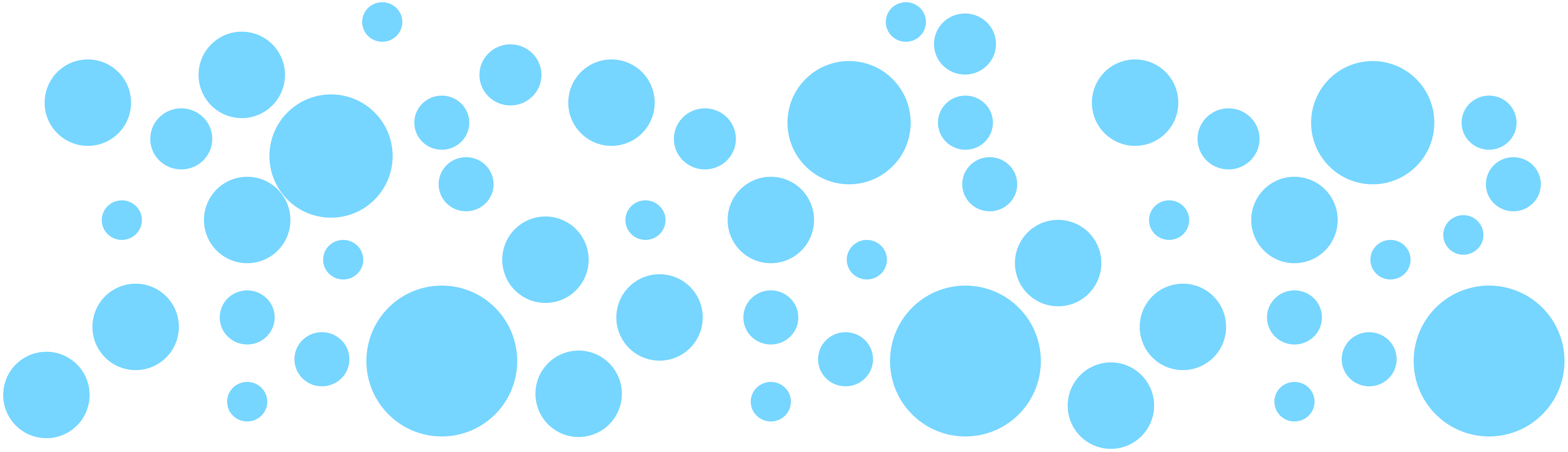

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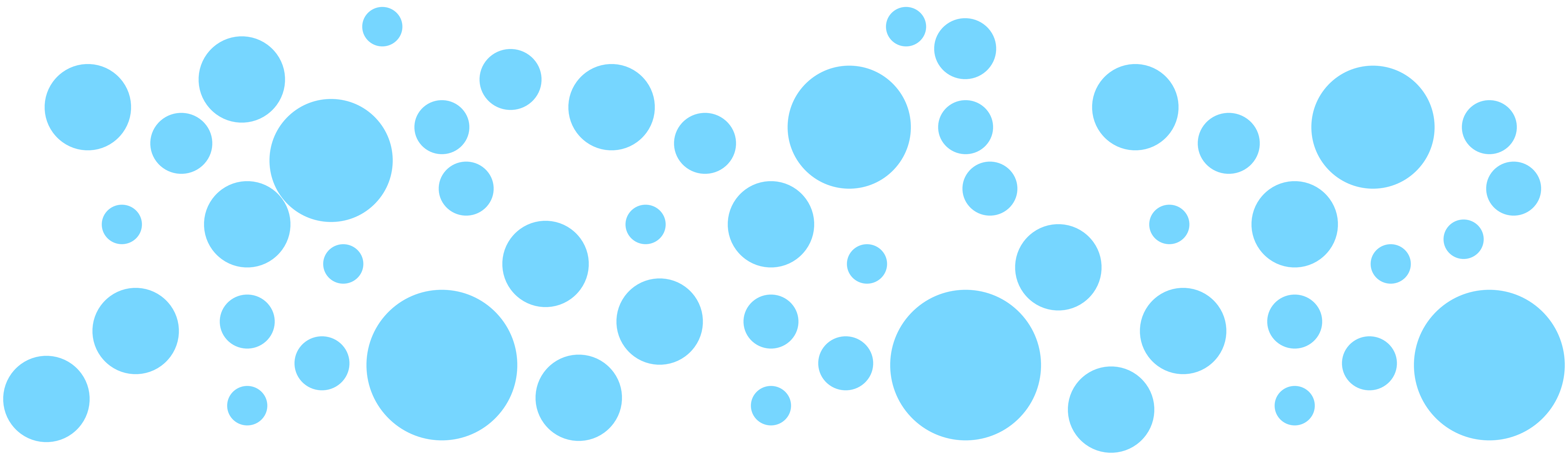

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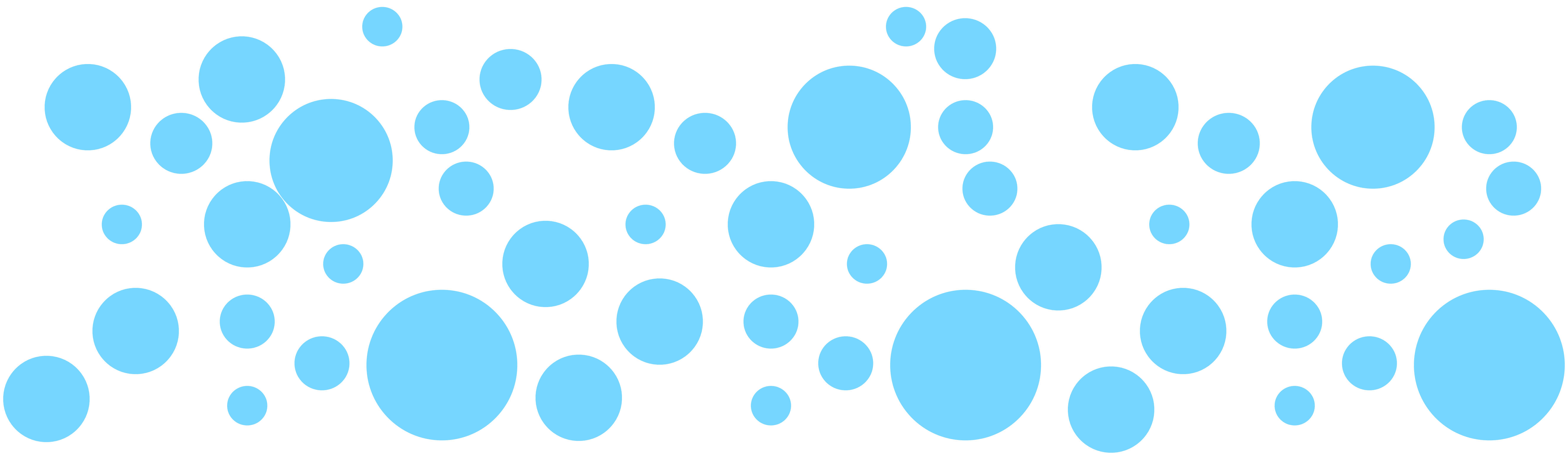

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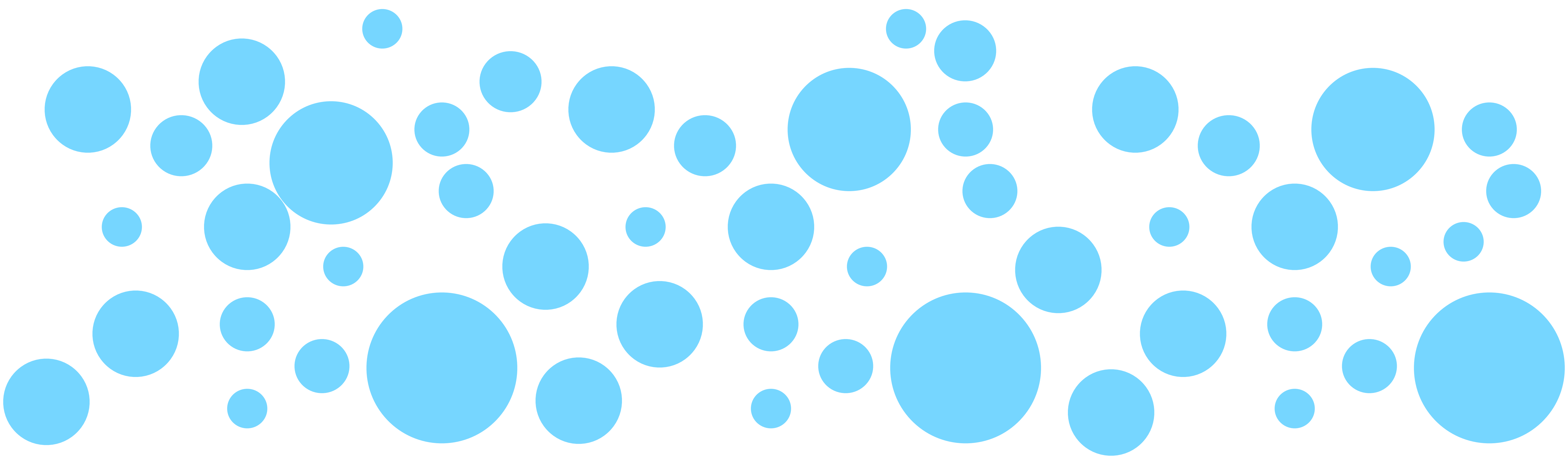

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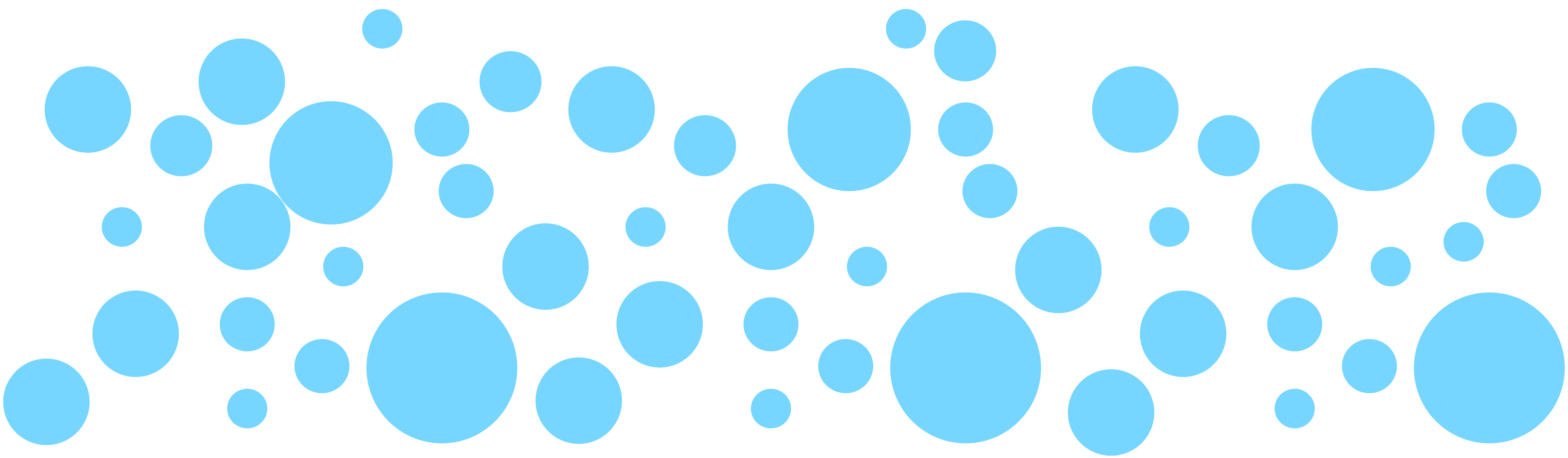
Connectivity, Biconnectivity, Trees, Stars, Eulerian, etc.



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Remove edges to make the input graph bipartite.

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<u>Colbourn & El-Mallah</u> [1988]

Making a graph P_k-free by deleting the minimum number of edges is NP-hard for every $k > 2$.

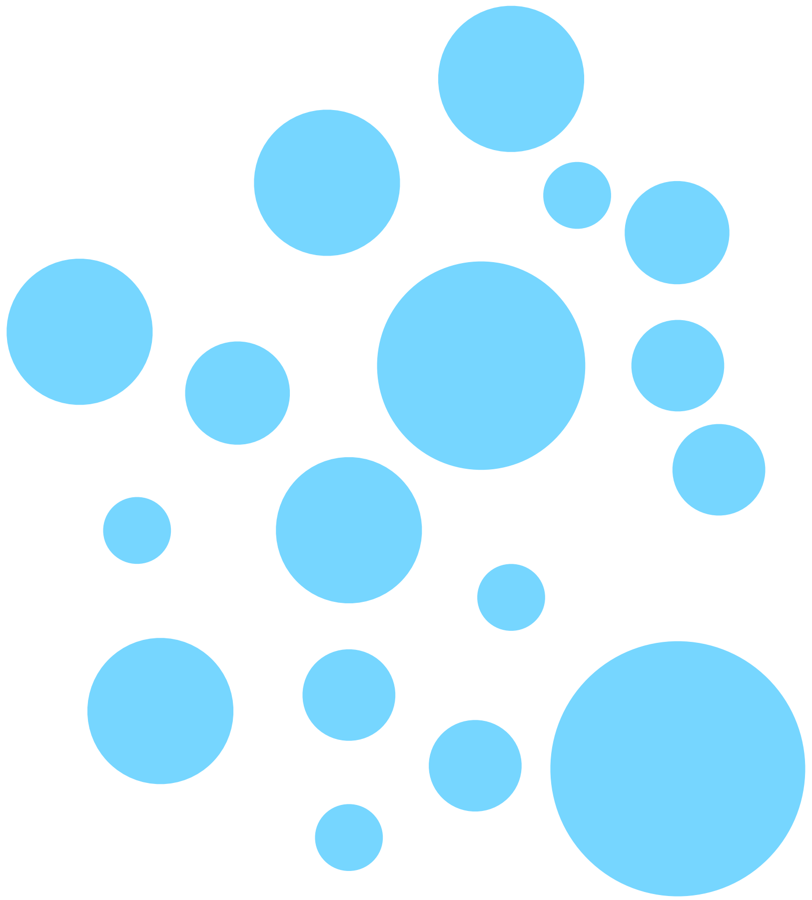
edge editing

edge contractions

vertex deletions

edge additions

edge deletions



Graph Class or “property”, π

Perfect
Trivially Perfect
Cographs
Permutation Graphs
Bipartite
Trees
Weakly Chordal
Chordal
Strongly Chordal
Split
Interval
Proper Interval
Unit Interval
Forests
Caterpillars
Chain
Chordal Bipartite
Distance Hereditary
Comparability
Trapezoid
CoChordal
Circle
Planar

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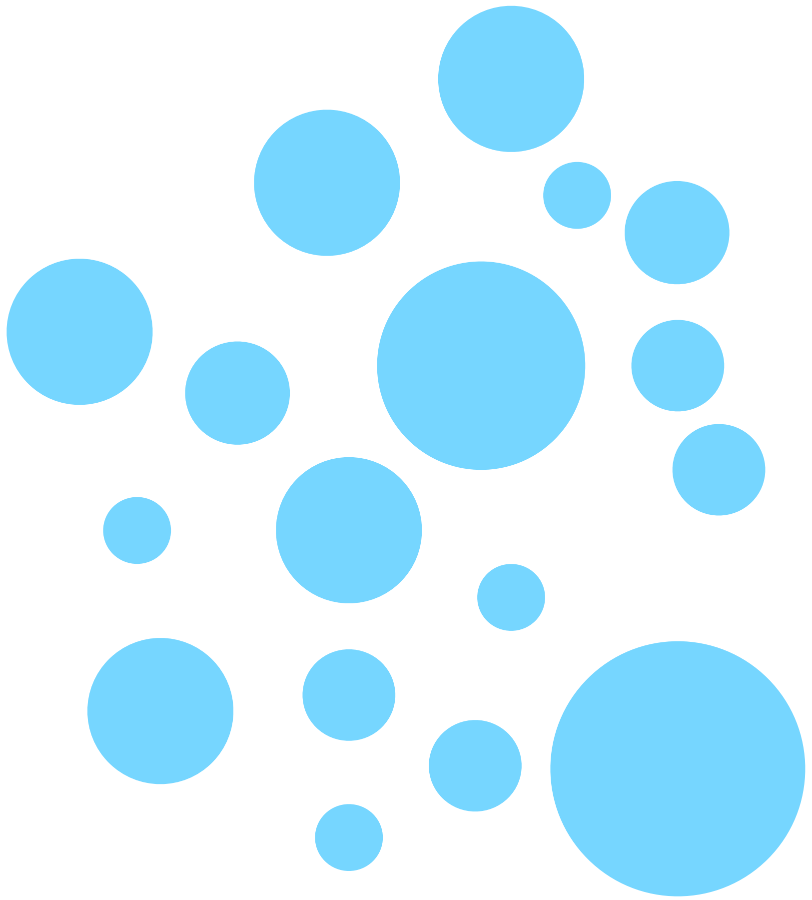

Graph Class or “property”, π

Graph Modification Problems

A Modern Perspective

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Why bother?

Special Graph Classes – correspond to some desirable property in real-world data.

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Planar Graphs

Interval Graphs

Cluster Graphs

k-Connected Graphs

Directed Acyclic Graphs

Chordal Graphs

Special Graph Classes – correspond to some desirable property in real-world data.

Planar Graphs	Graph Drawing
Interval Graphs	Physical Mapping of DNA
Cluster Graphs	Correlation Clustering
k-Connected Graphs	Reliability of Networks
Directed Acyclic Graphs	Deadlock Recovery, Operating Systems
Chordal Graphs	Gaussian Elimination over Sparse Matrices

Special Graph Classes – correspond to some desirable property in real-world data.

We would like to "achieve" these properties with minimum cost, which naturally leads us to the graph modification framework.

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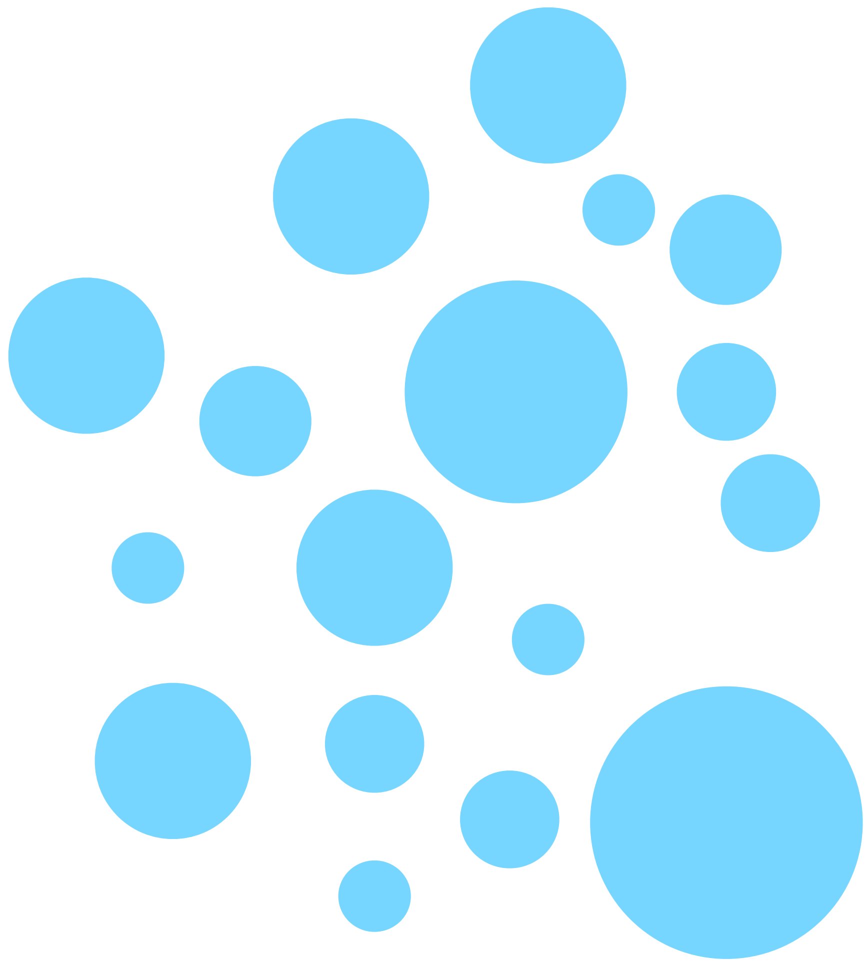
Graph Modification Problems

Algorithms



Graph Class or "property", π

Forbidden Subgraph Characterization

A graph G belongs to **π** if, and only if,
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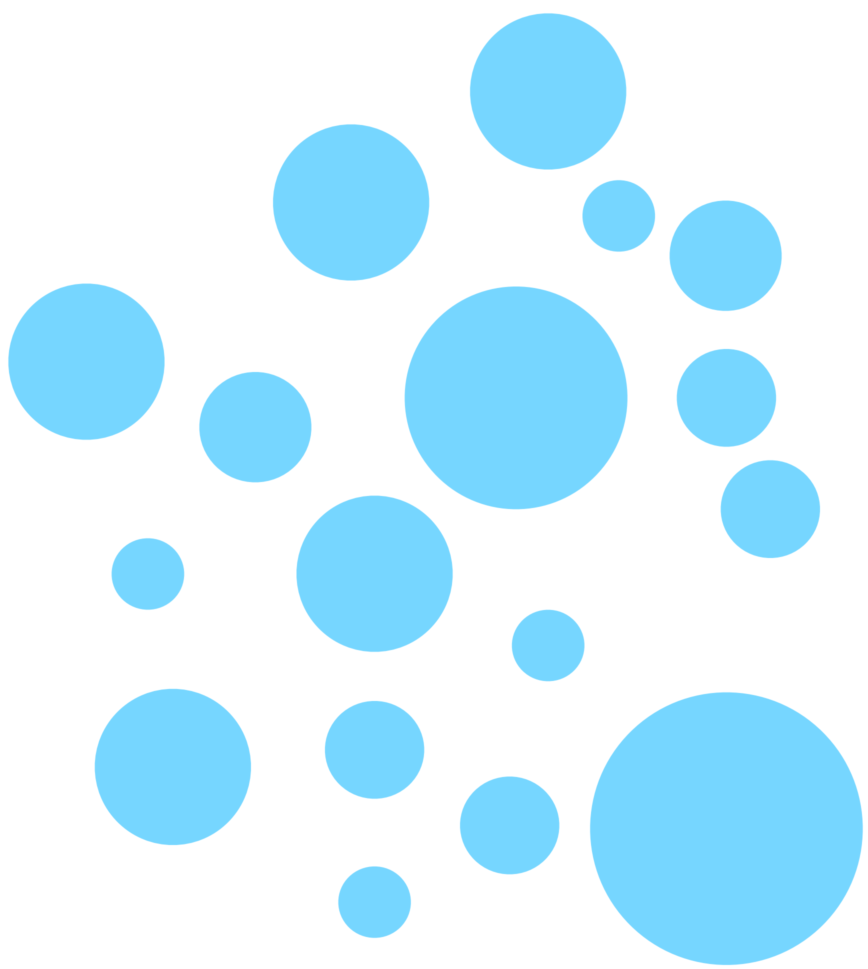

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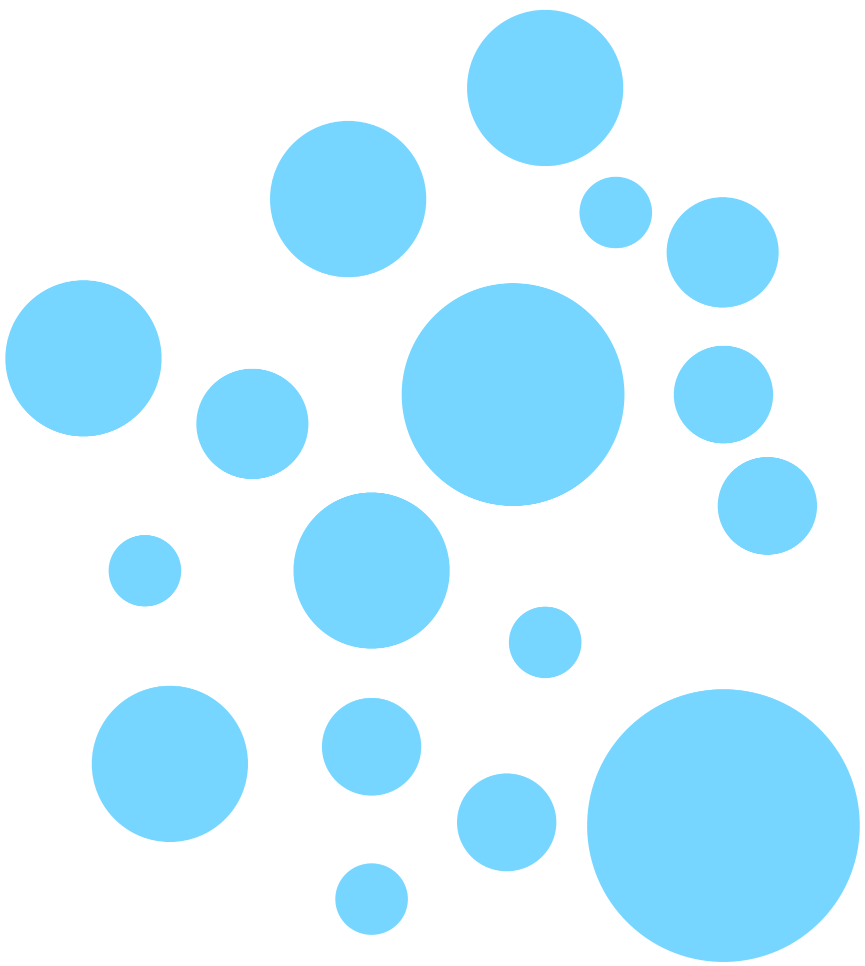

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Forbidden Subgraph Characterization

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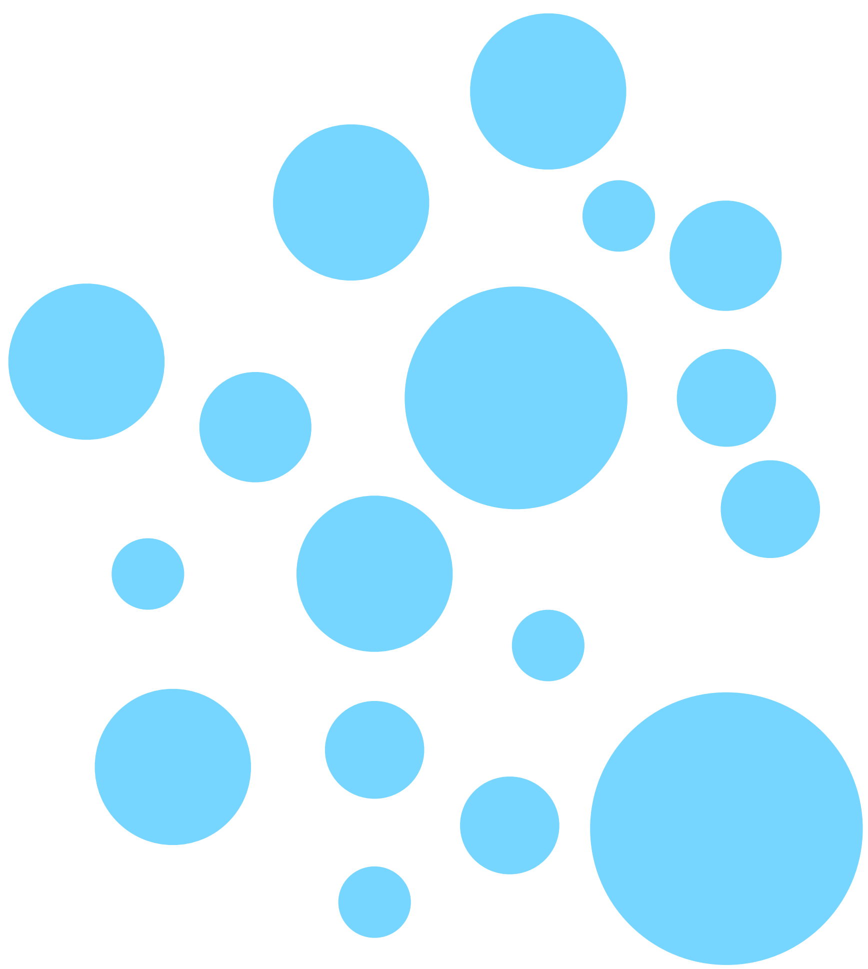

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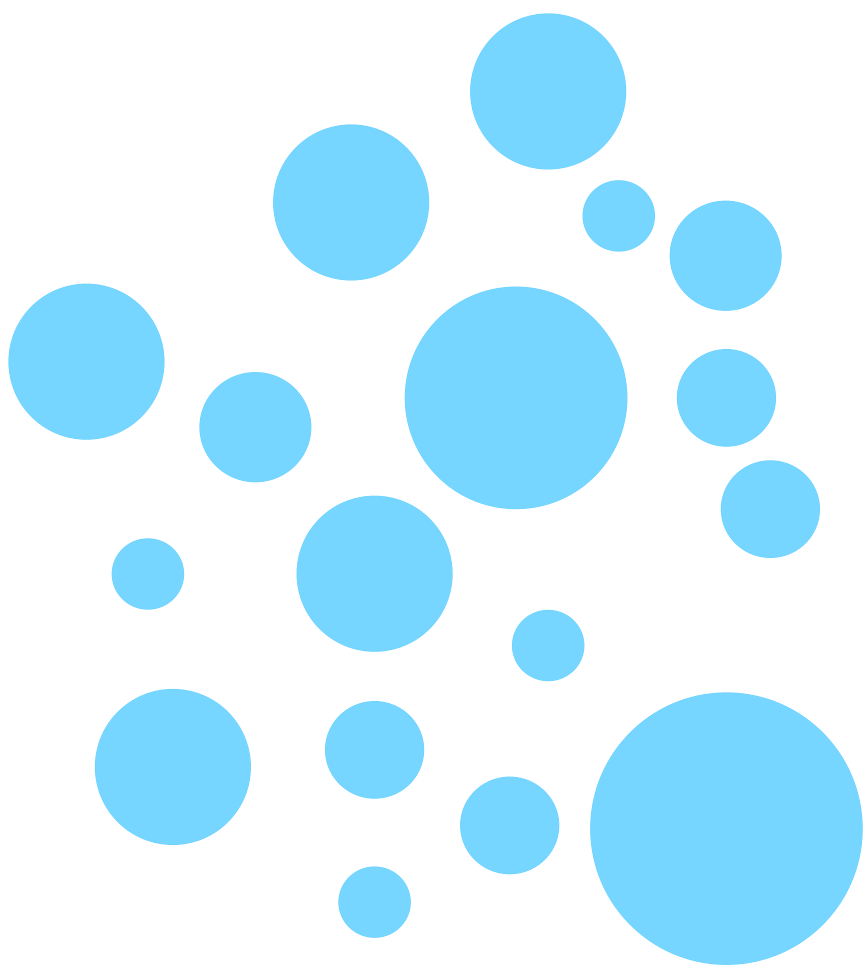

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Forbidden Minor Characterization

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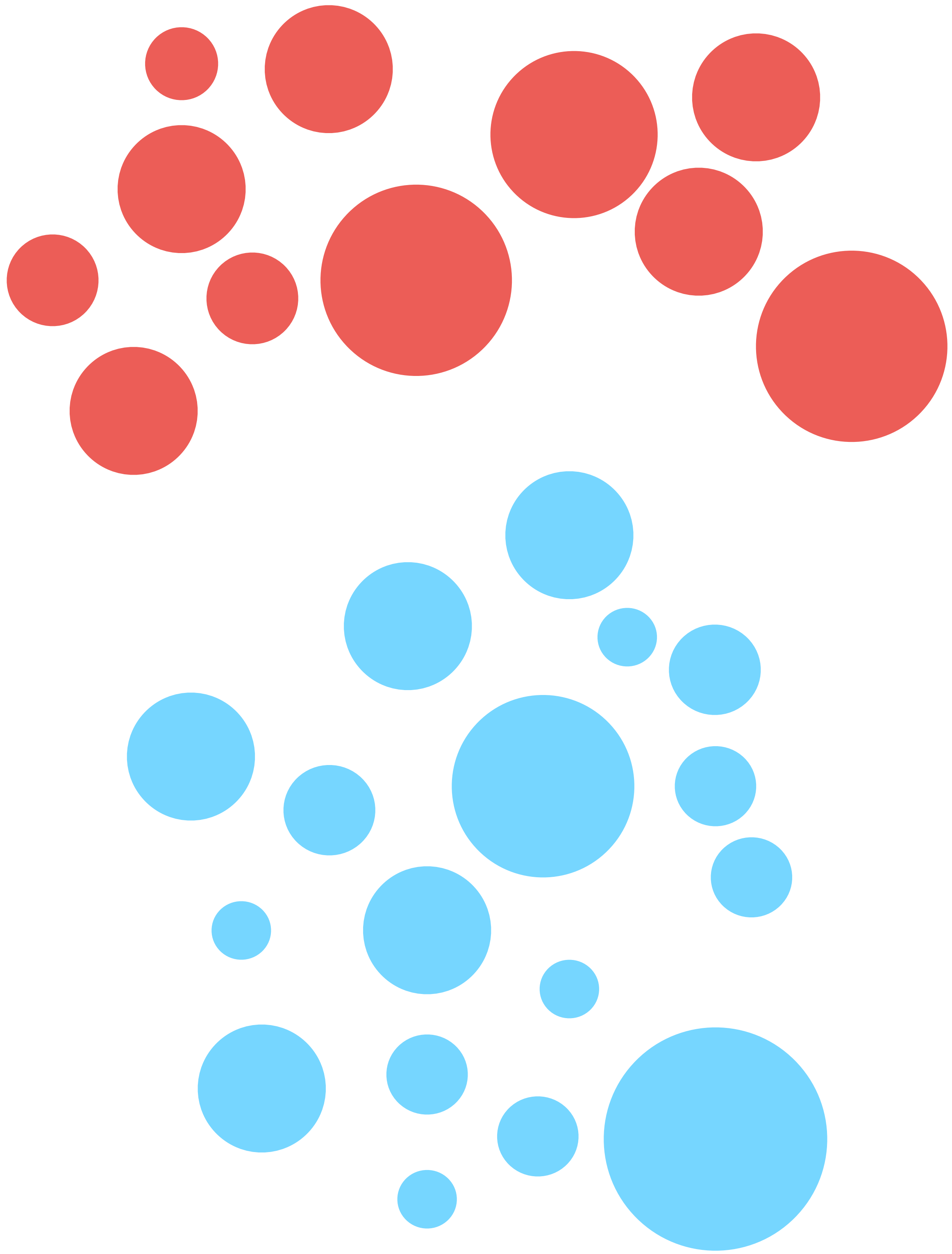
edge deletions and edge contractions.



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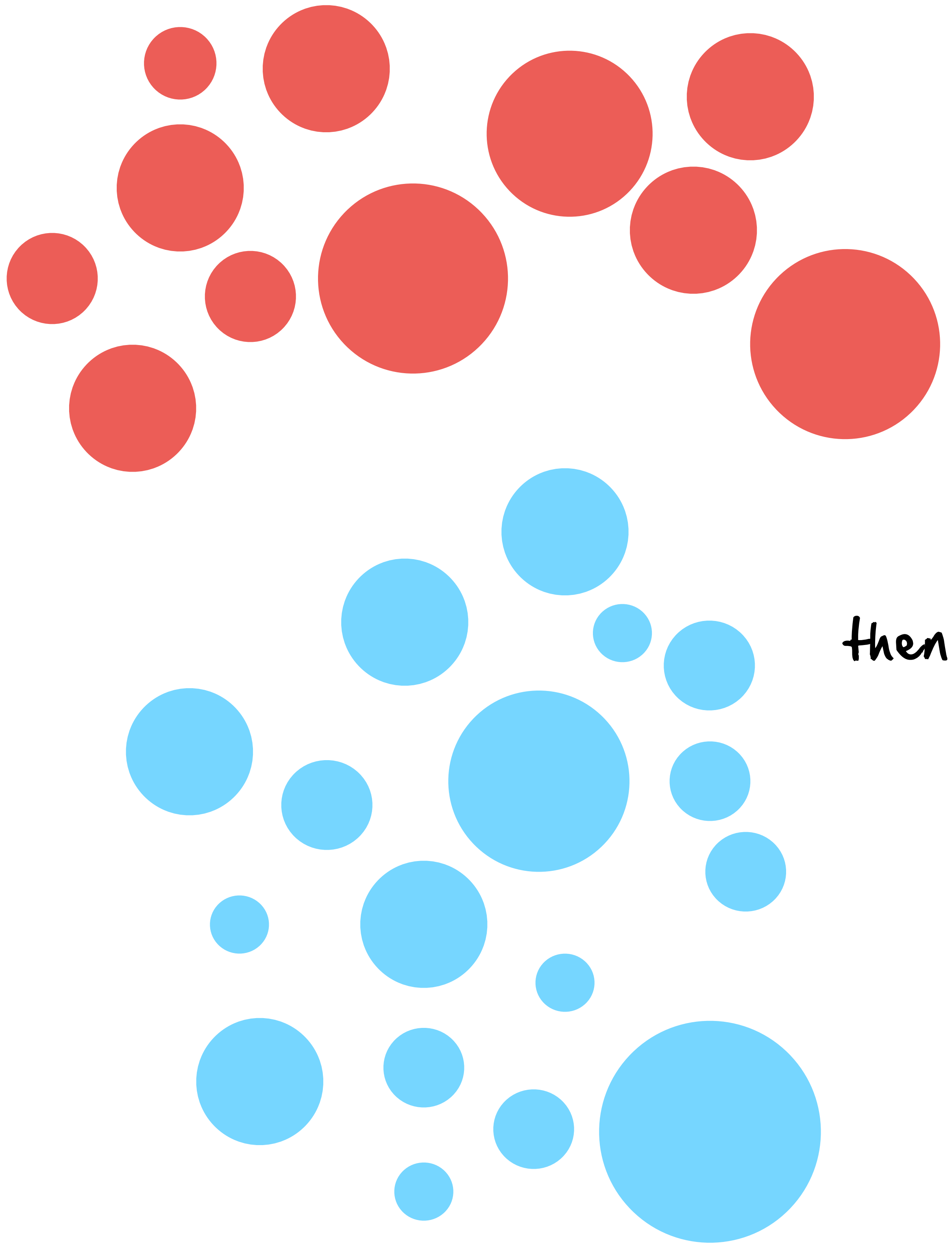

Graph Class or "property", π

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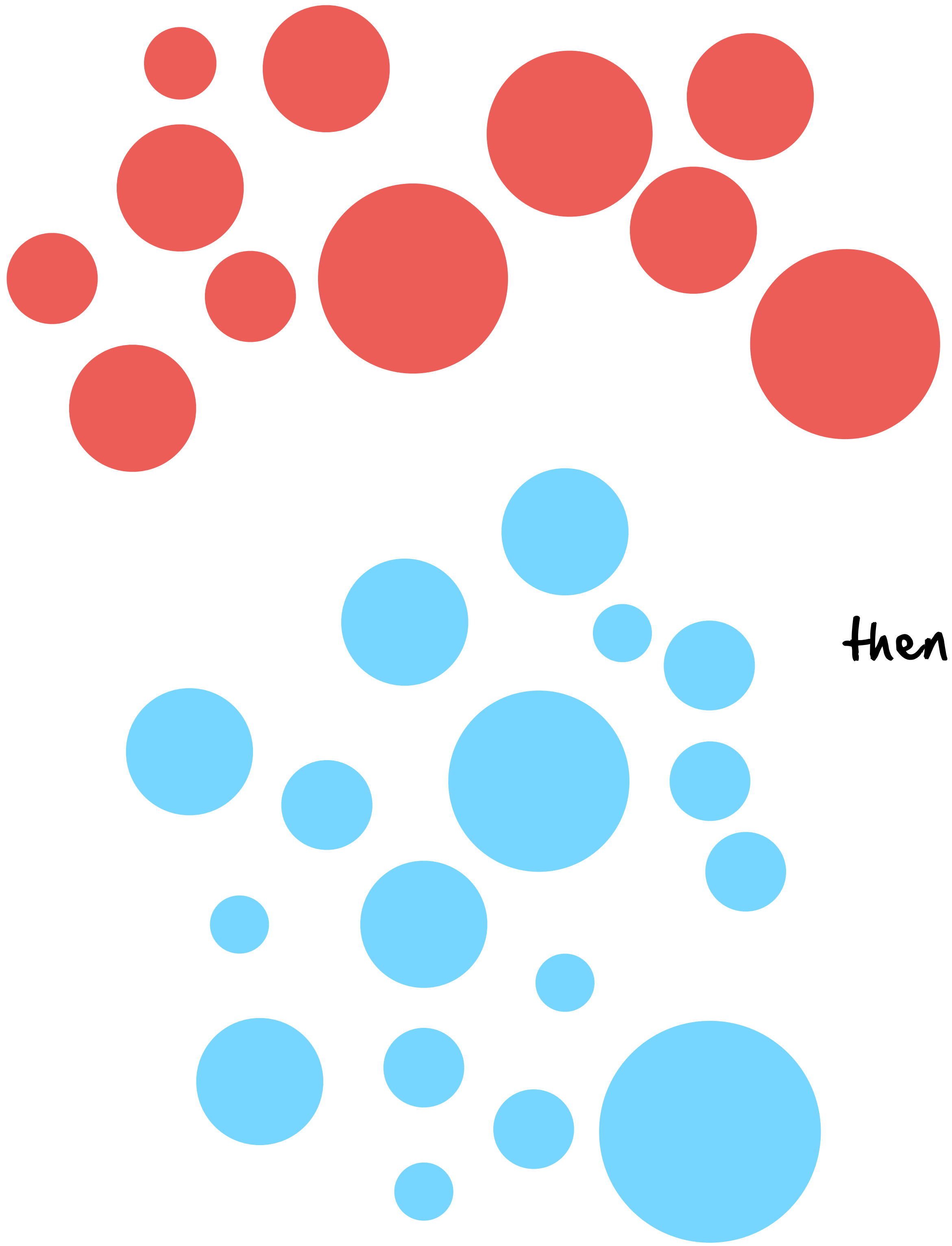

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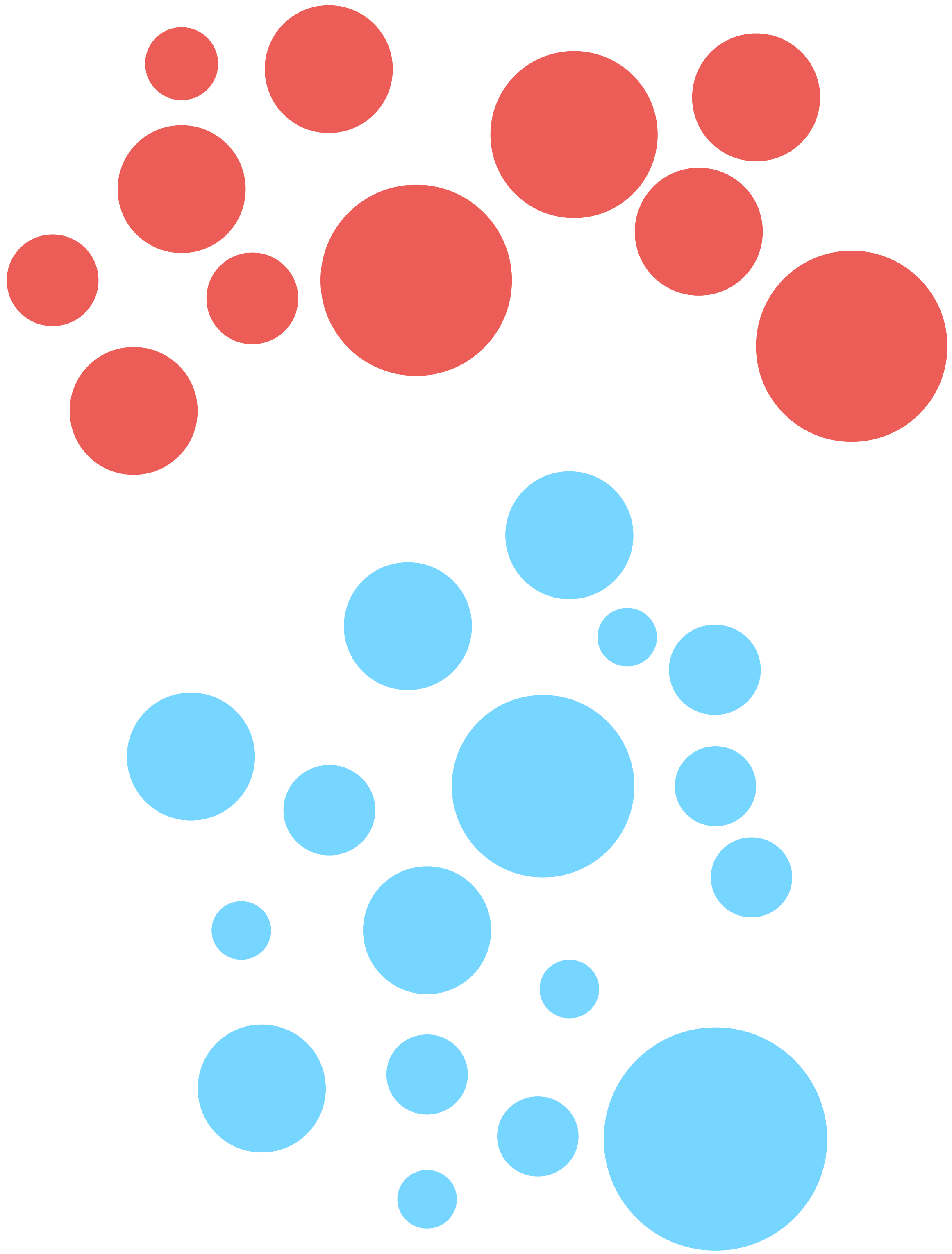

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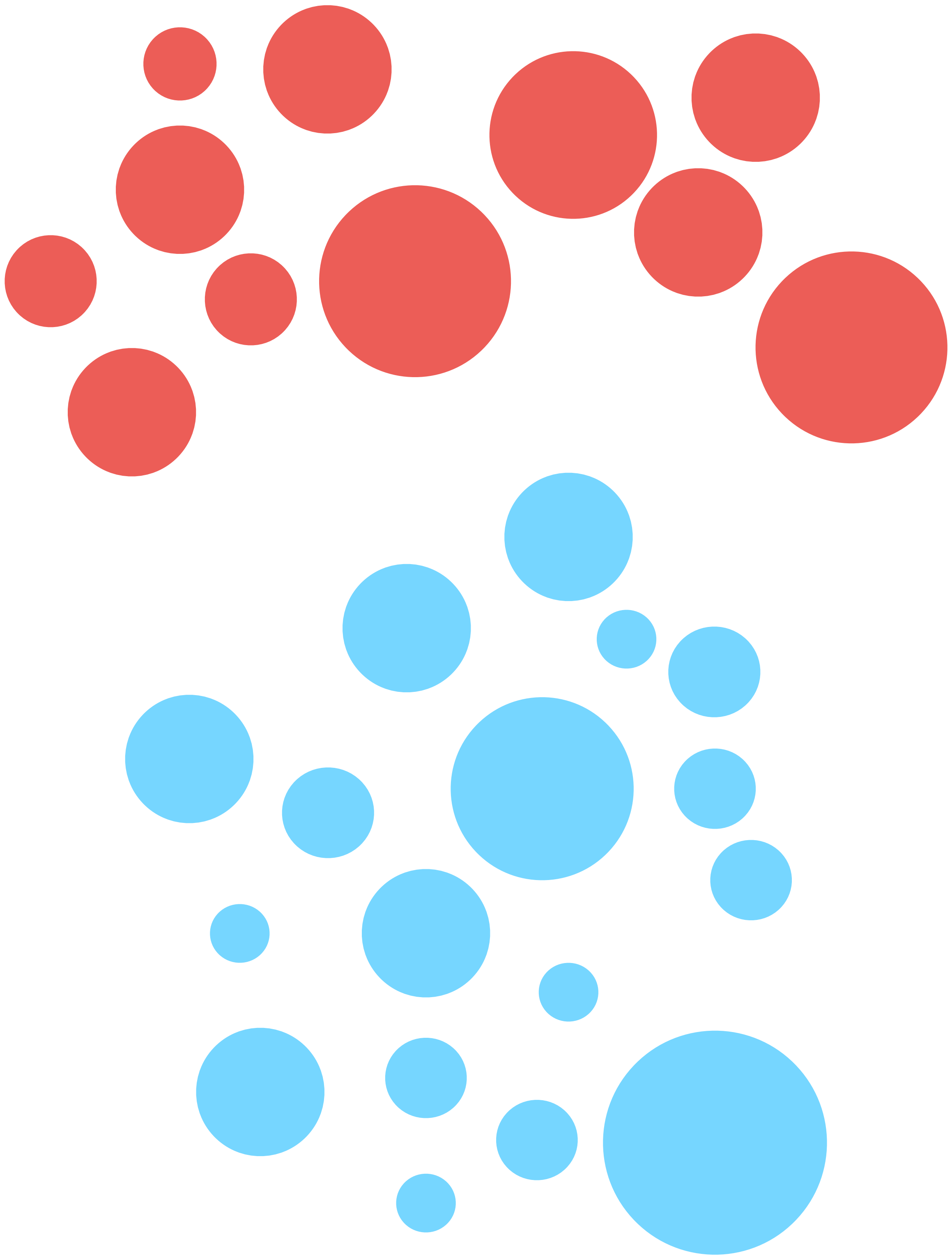

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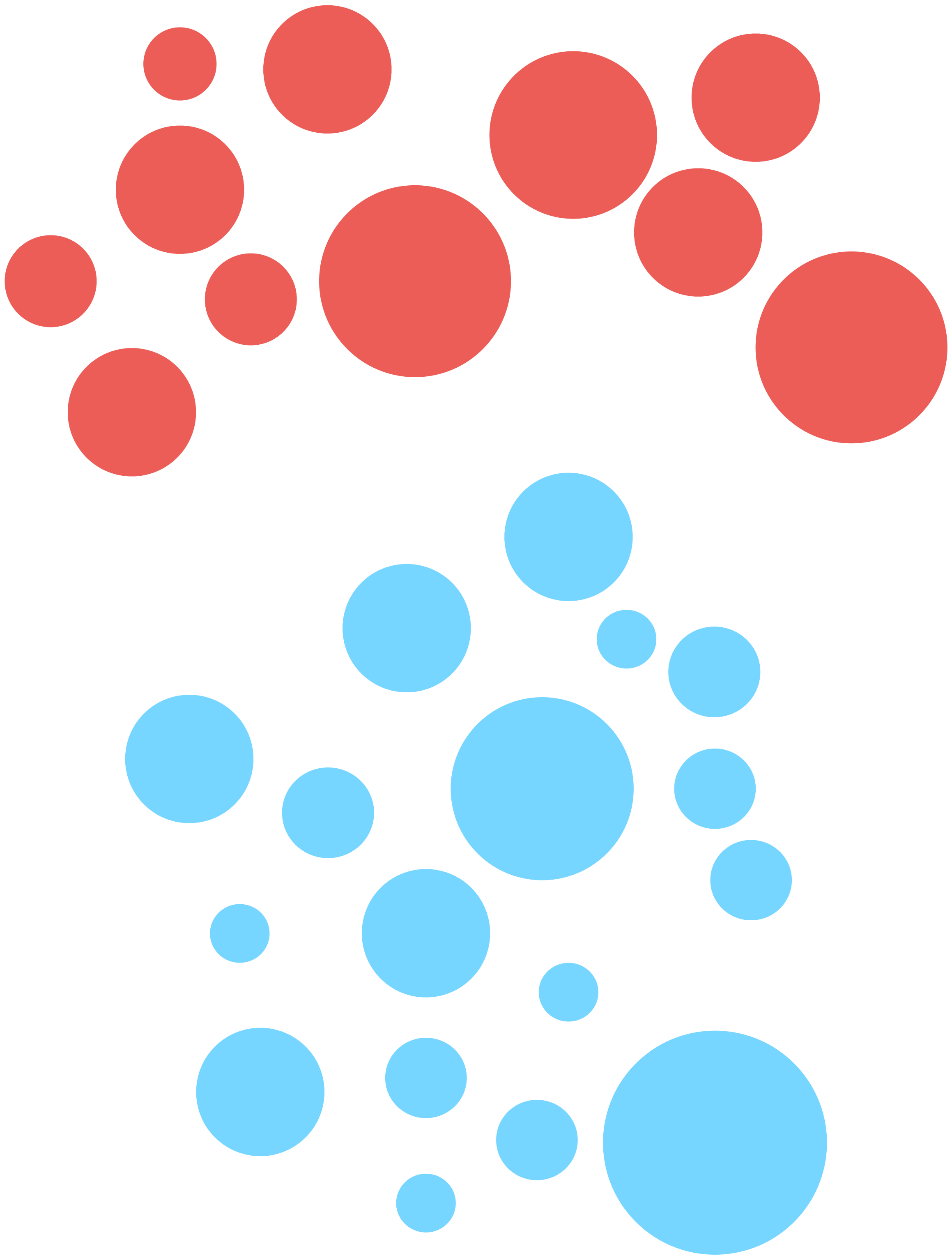

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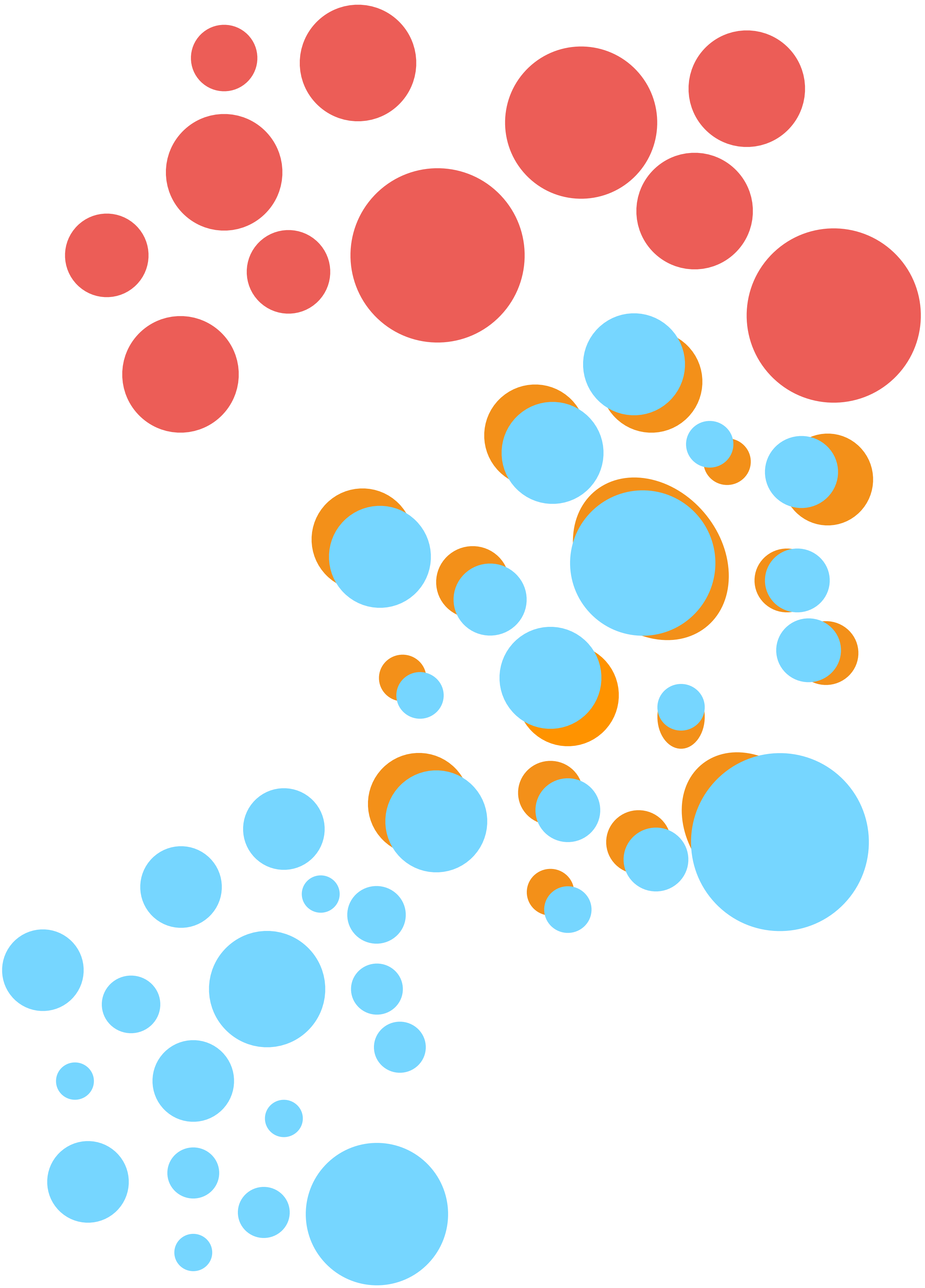

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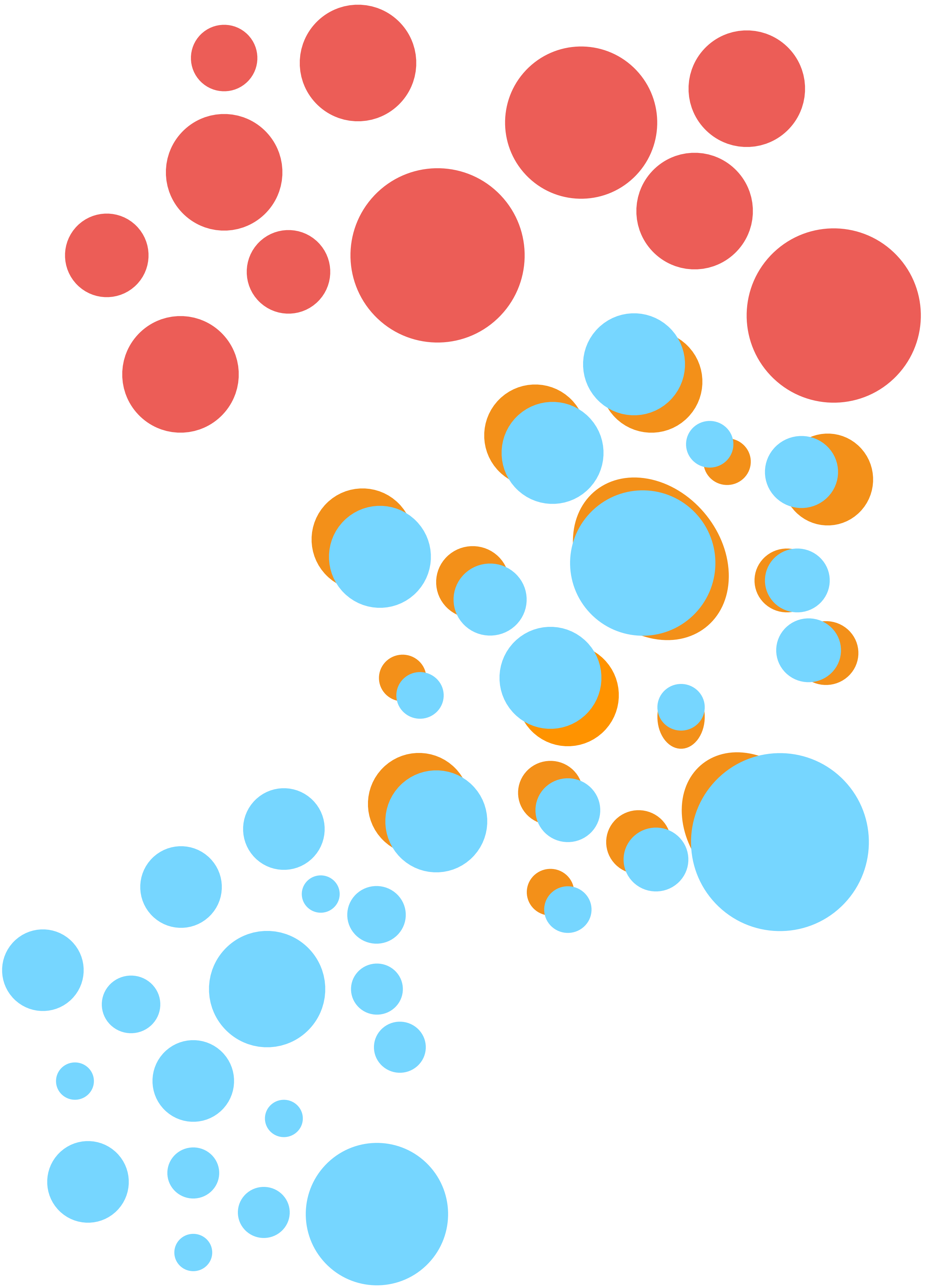

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Now: the graph modification problem has boiled down to a membership testing problem, which can be done in cubic time, given the finite forbidden set.

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Can the node deletion question for hereditary properties always be answered with the "graph minors hammer"?

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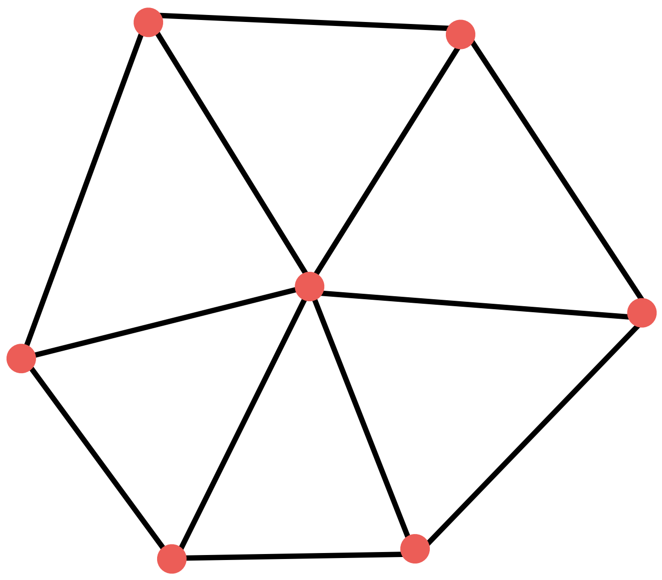
Consider the property π of being wheel-free, that is, not having any wheel as an induced subgraph.



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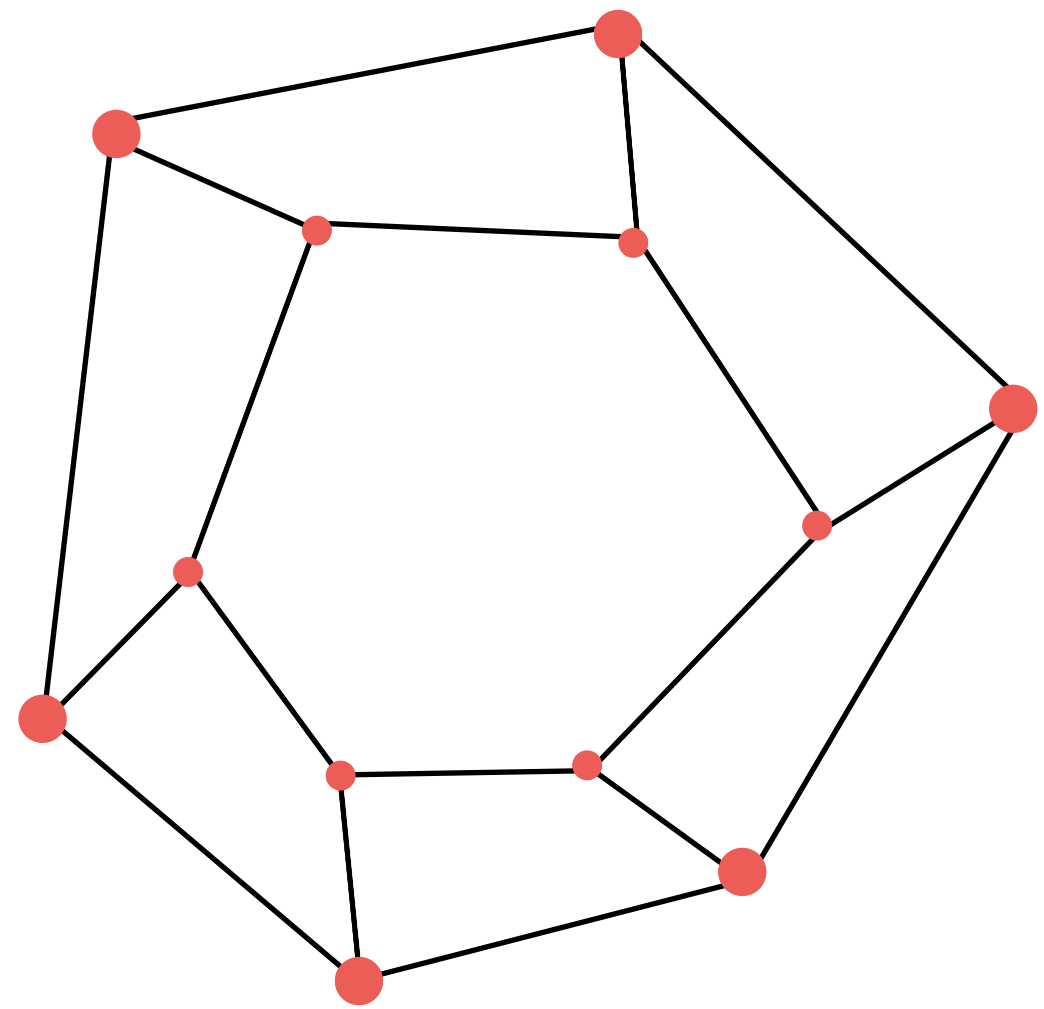

The property is hereditary, but **not** minor-closed.

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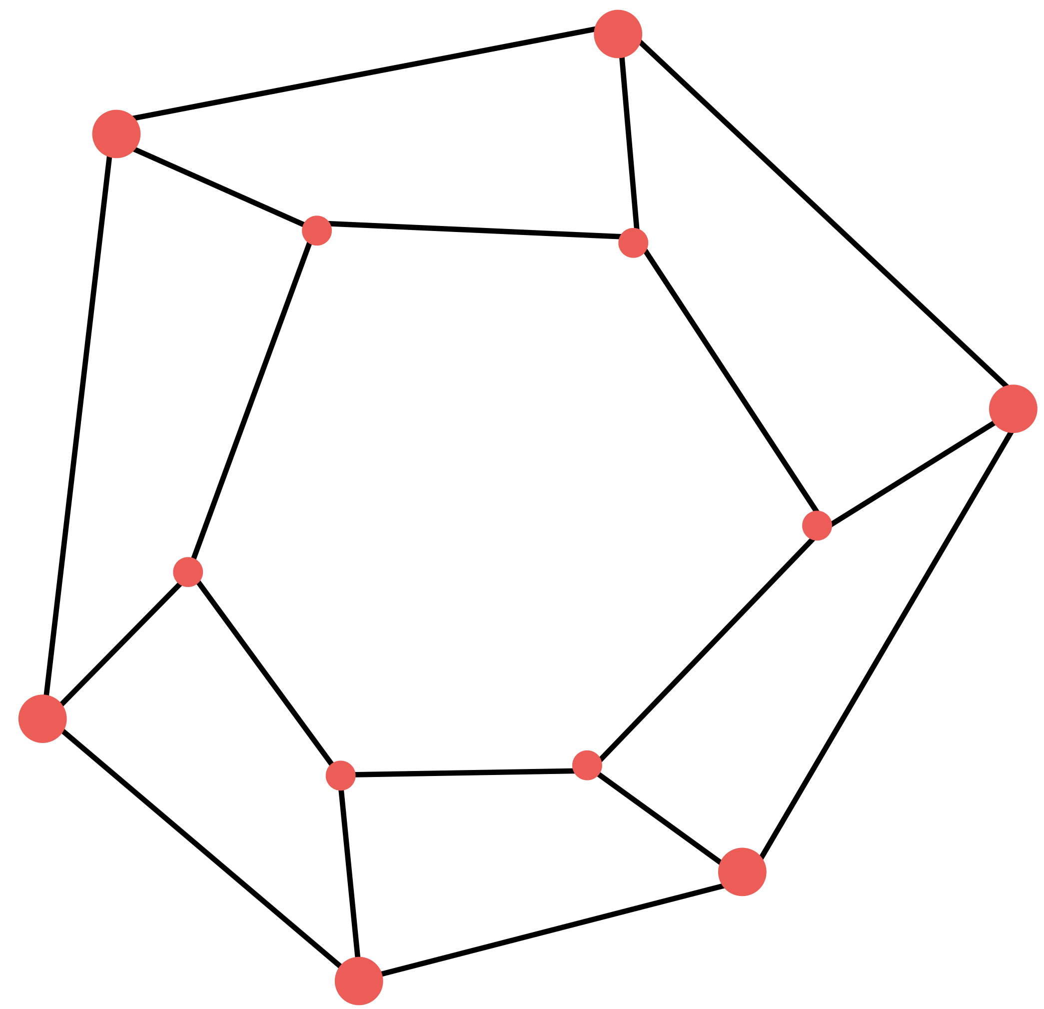

Lokshtanov (2008)

Wheel-Free Deletion and Wheel-Free Vertex Deletion are **W[2]-hard.**

Finding large induced subgraphs with property π.

Khot and Raman, (2002)

Finding large induced subgraphs with property **π**.

Khot and Raman, (2002)

If **π** is a hereditary property that contains all independent sets and all cliques,
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Ramsey Numbers!

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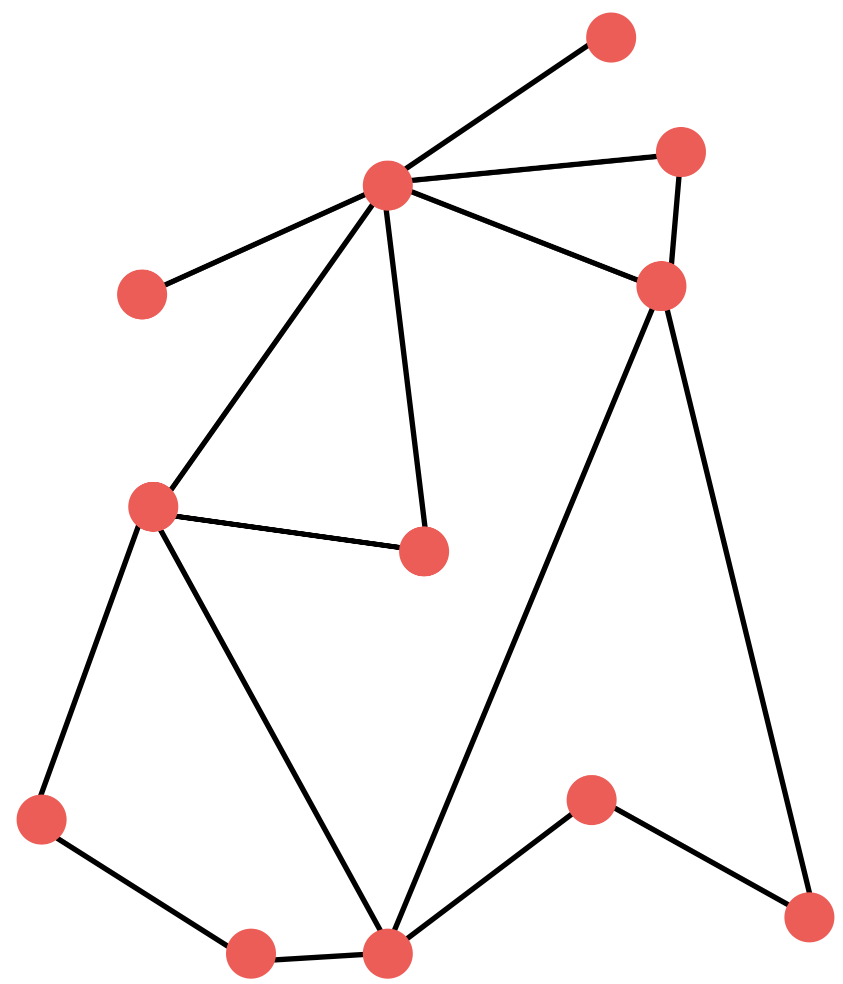
Vertex Cover

Can G be made edgeless by the removal of at most k vertices?



A simple randomized algorithm with an error probability of 2^{-k}.

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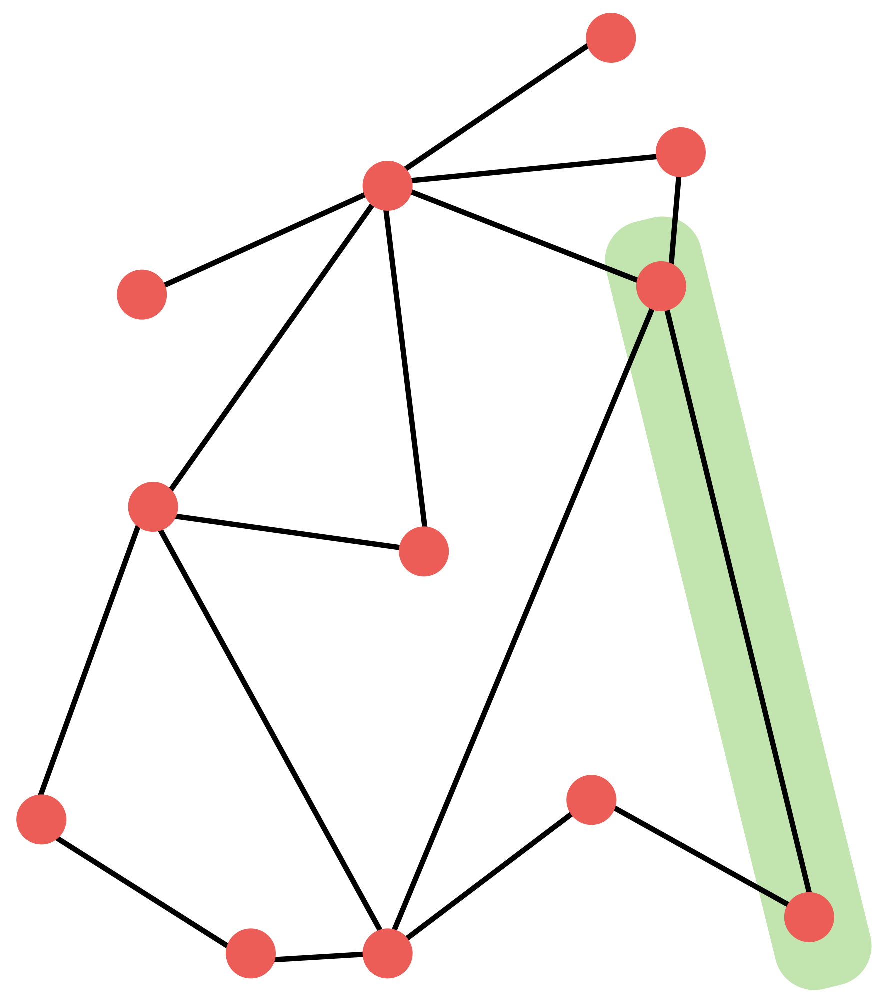

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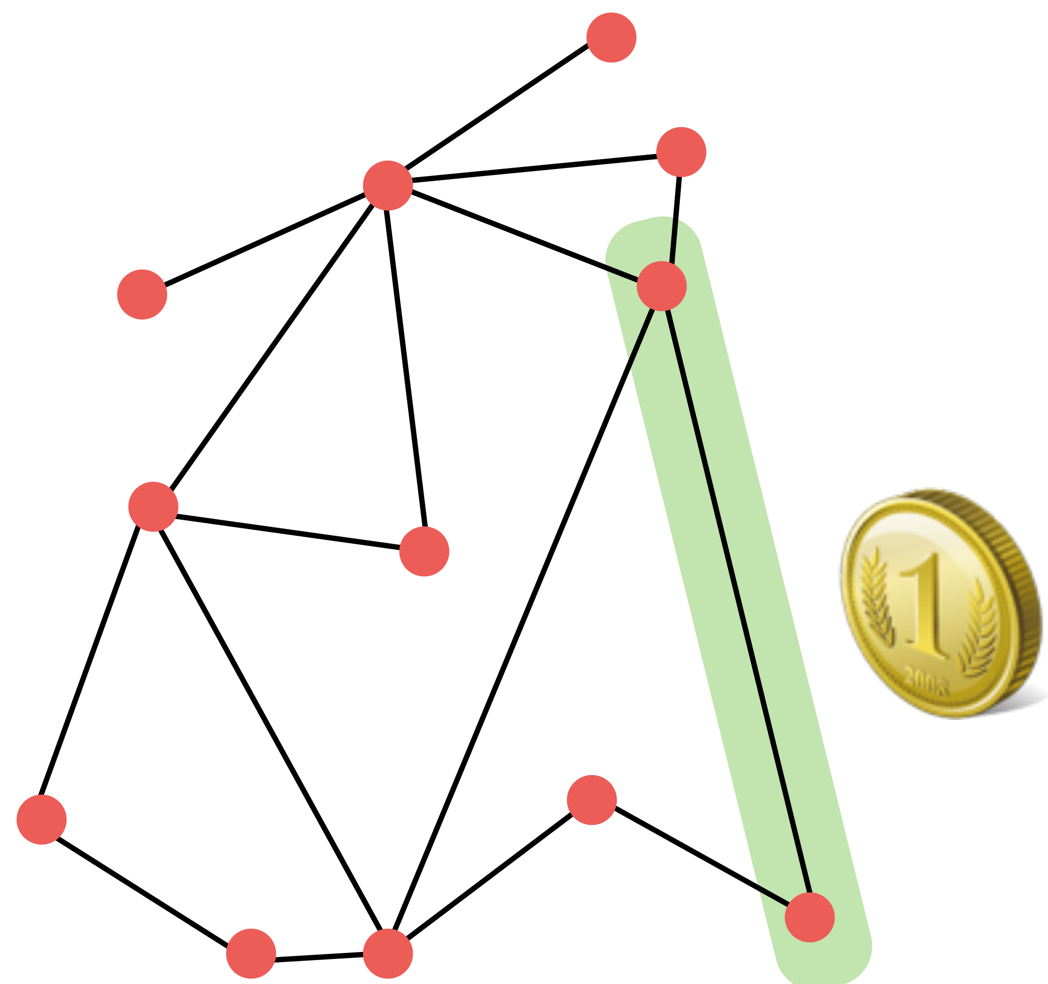

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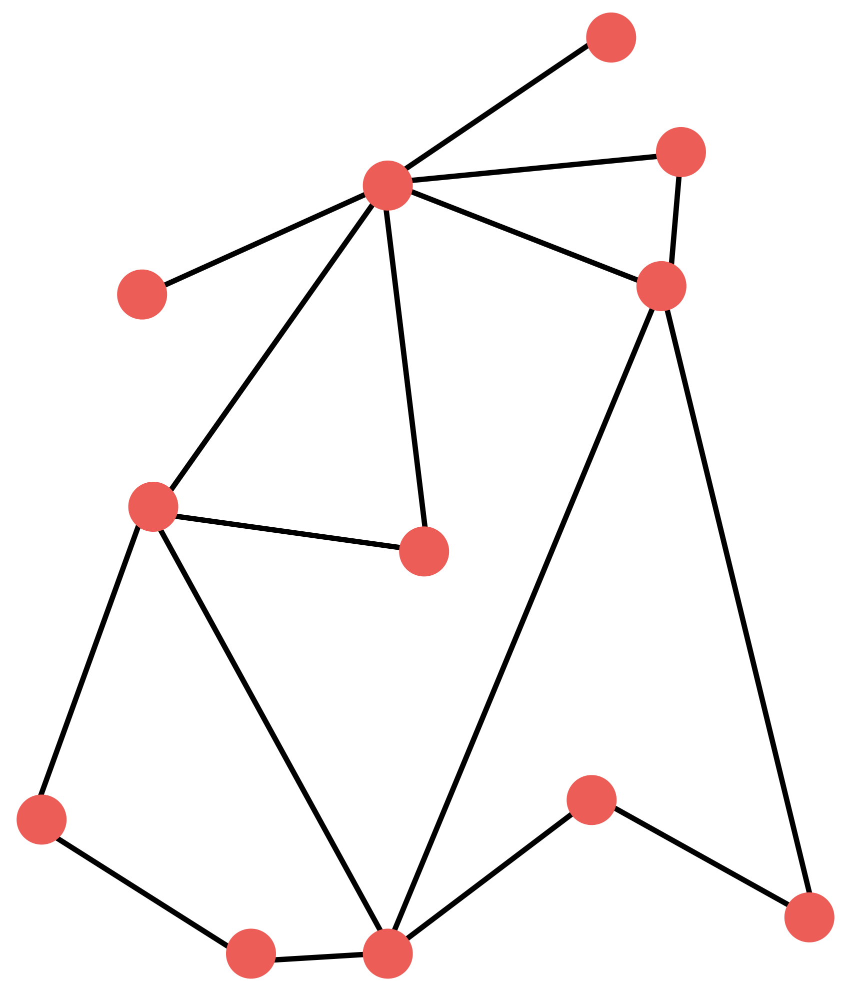

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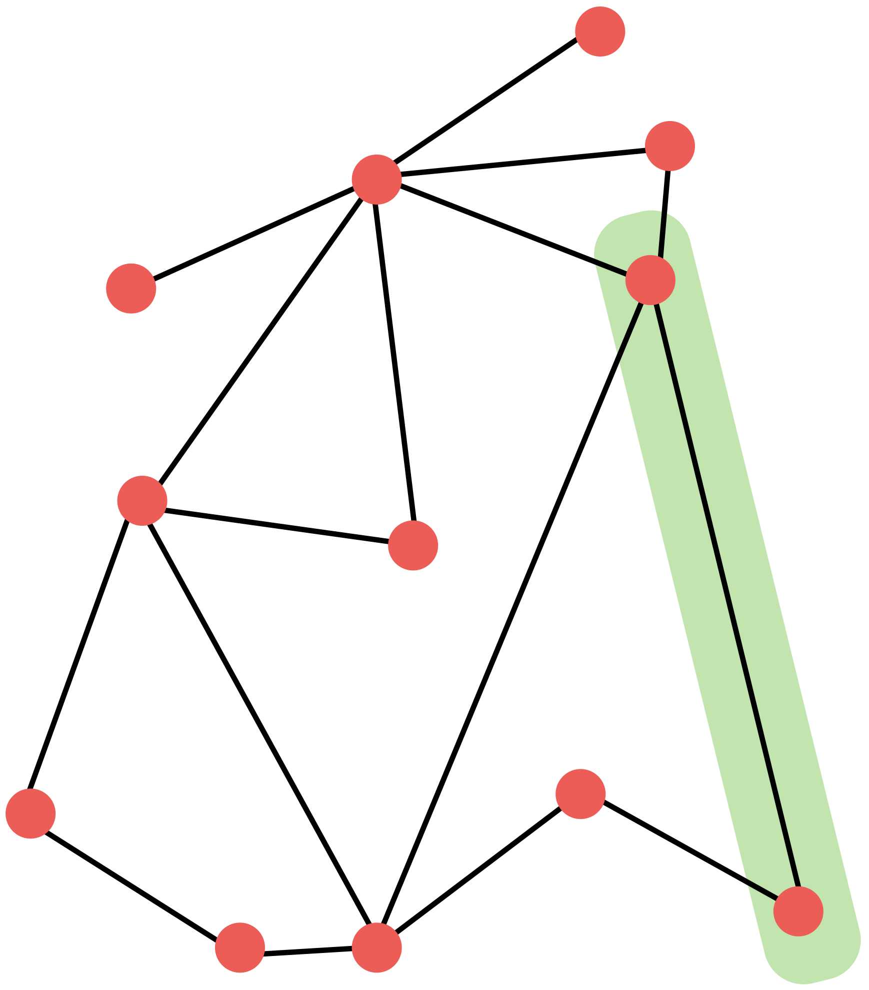

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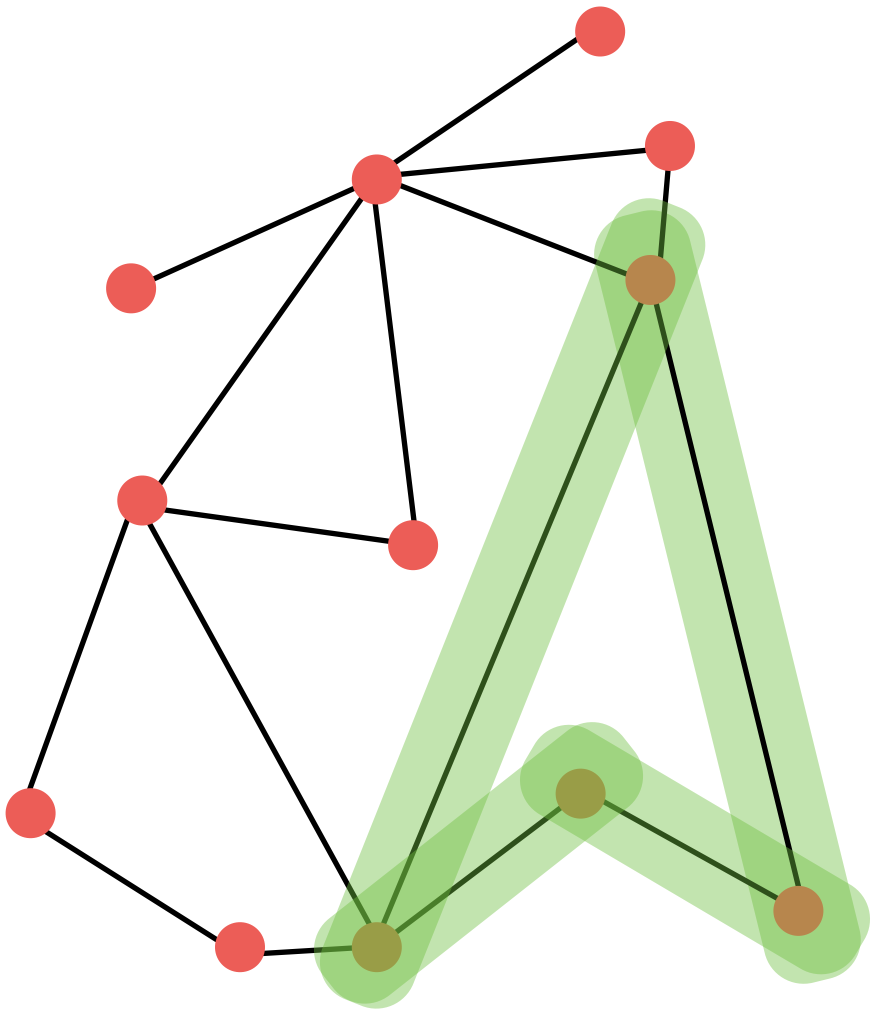

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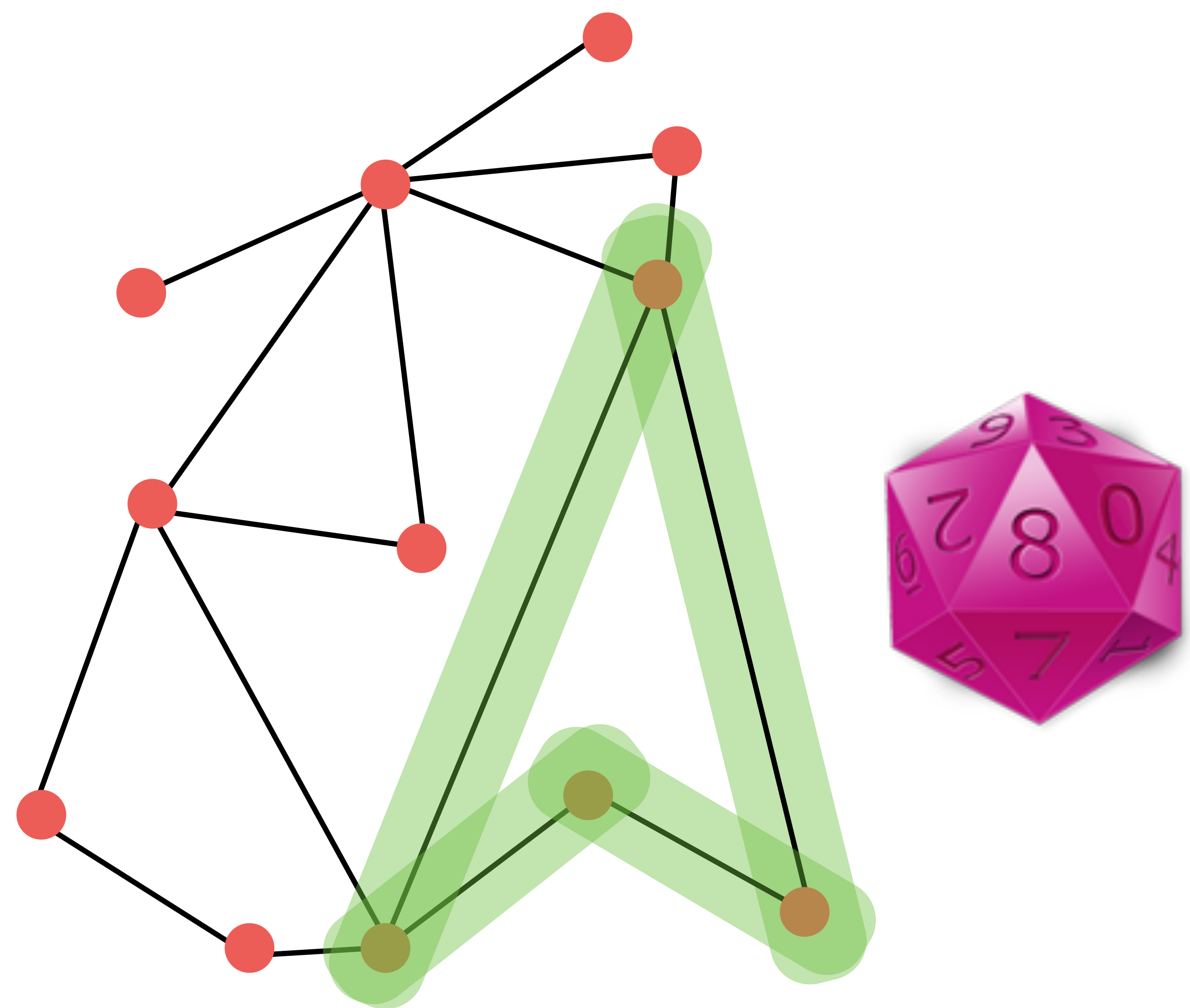

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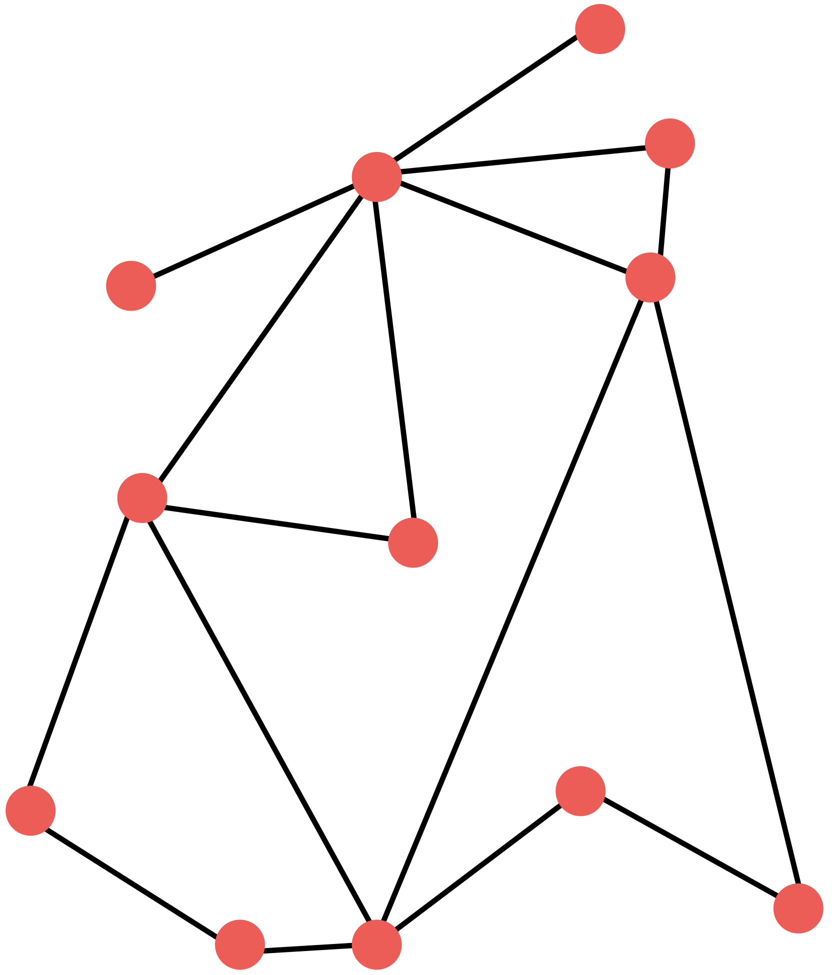

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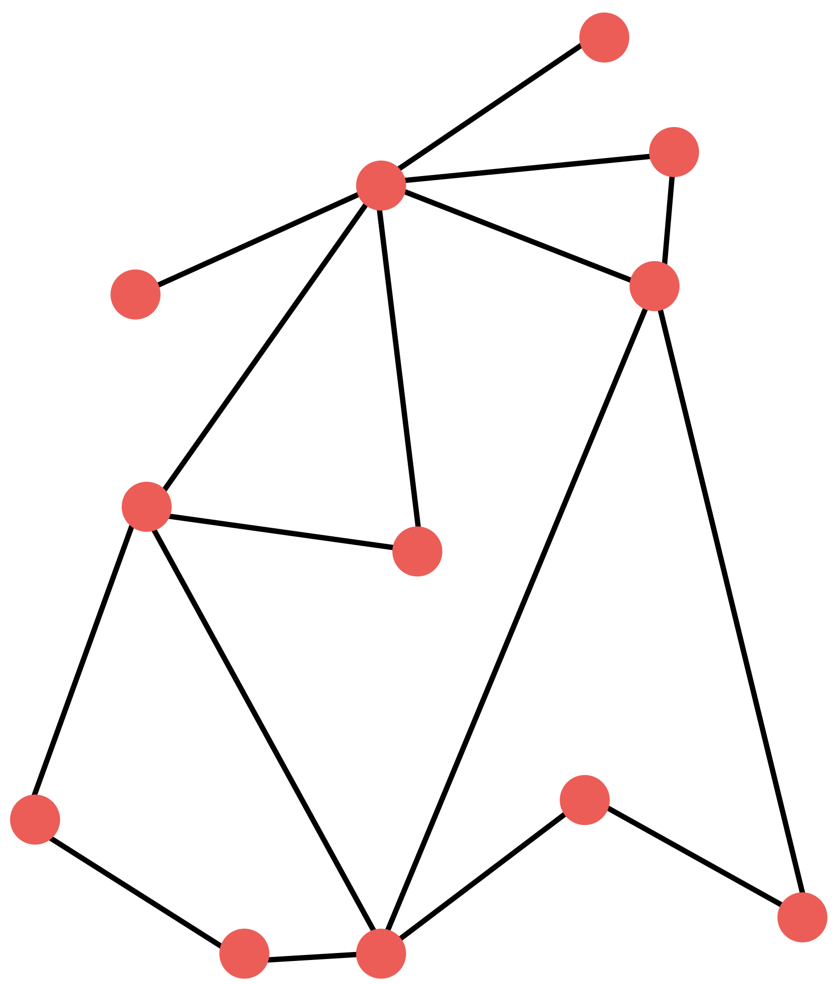
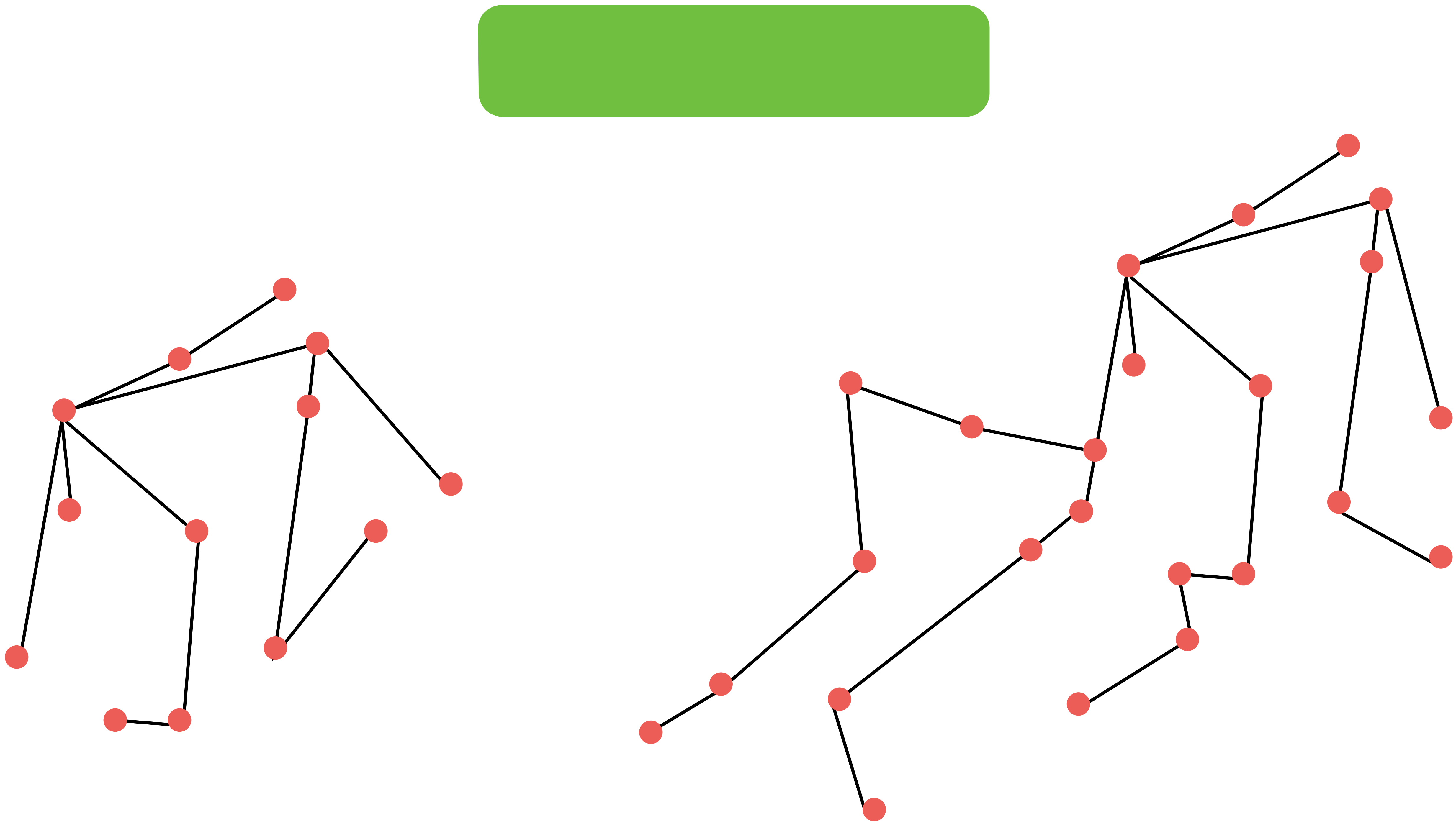


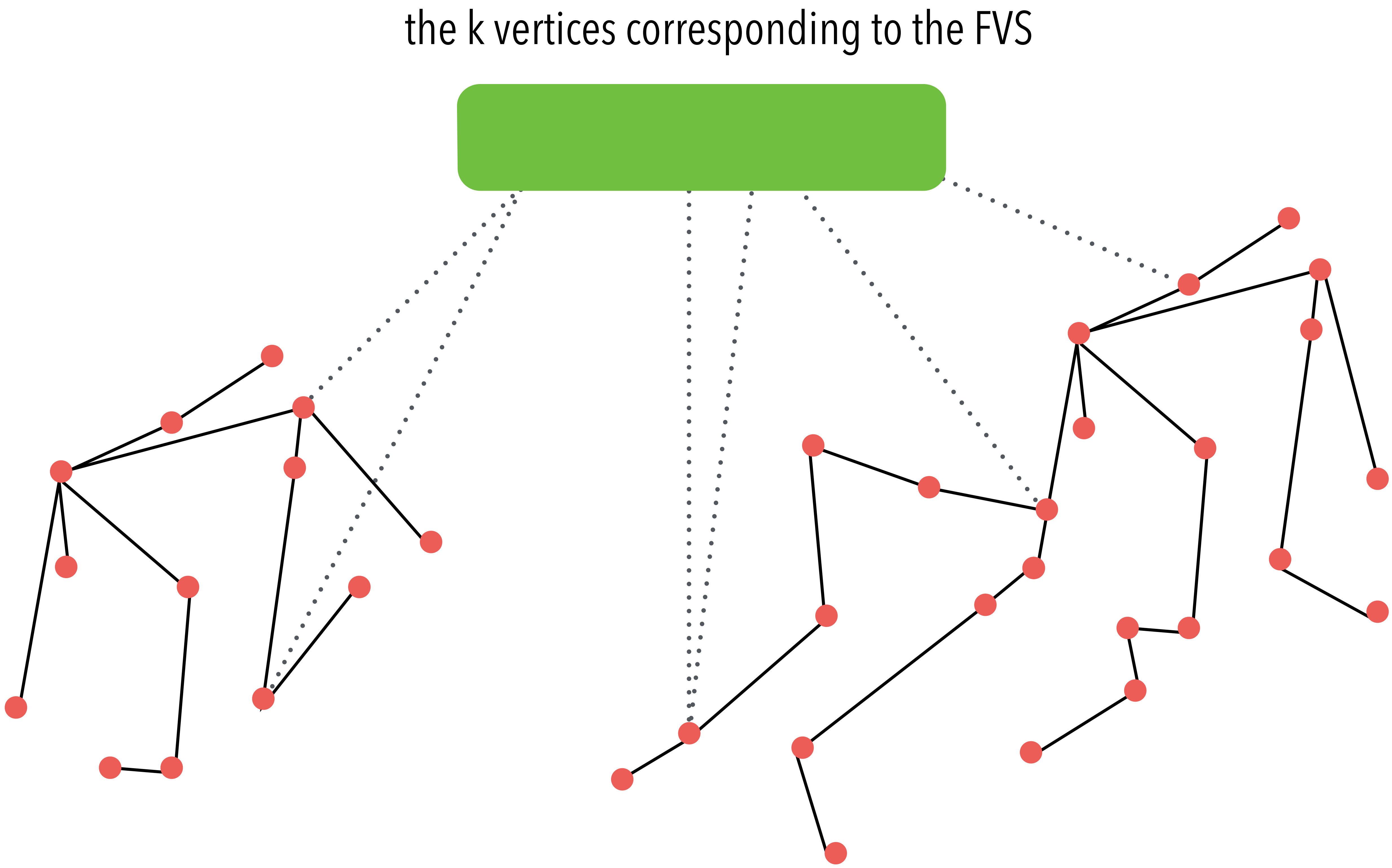
Let's stare at the *structure* of a YES-instance.

#whisper: Let us also restrict ourselves to graphs of constant maximum degree, say five.

the k vertices corresponding to the FVS



(n-k) vertices and at most (n-k-1) edges "outside".



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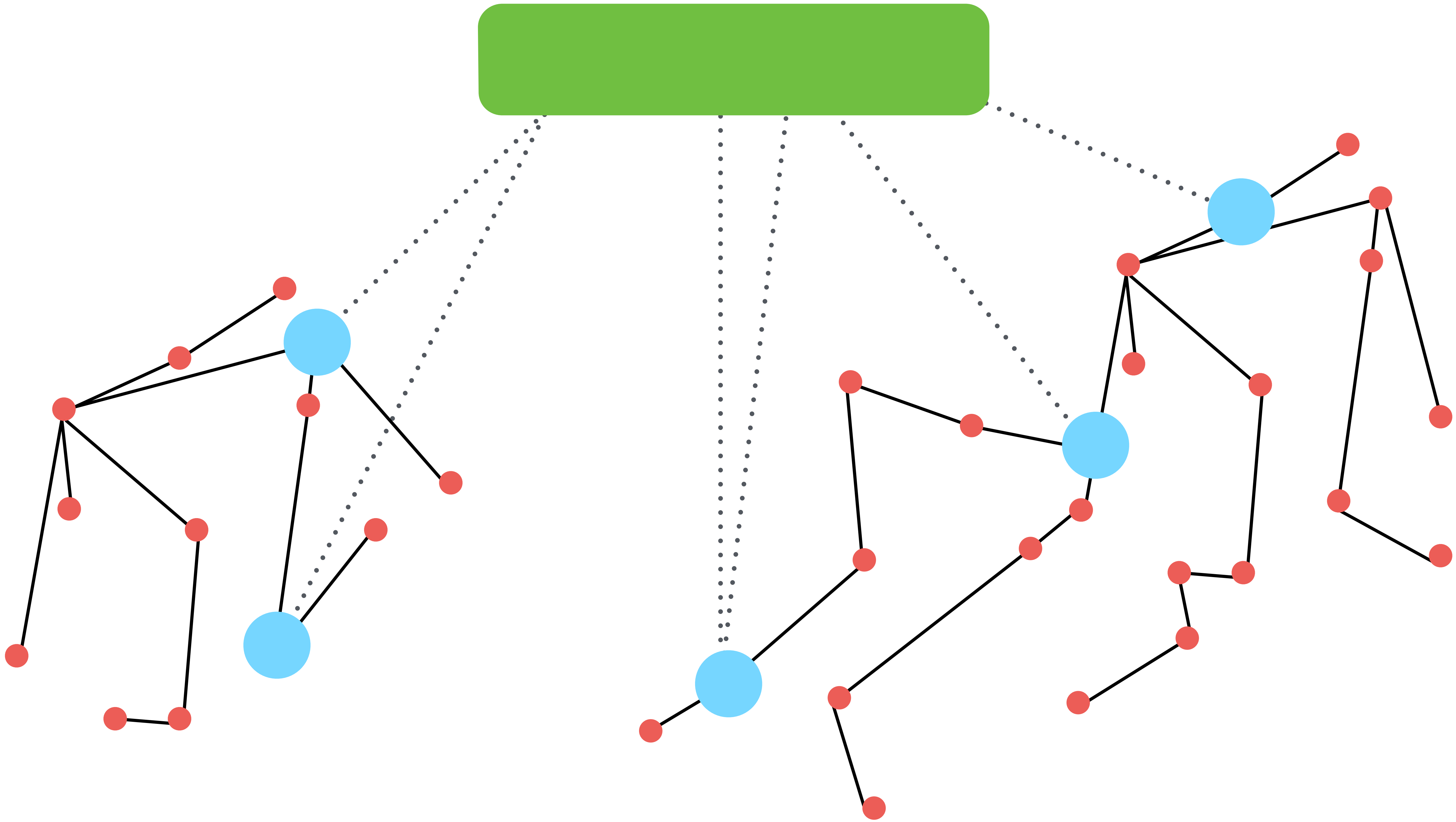
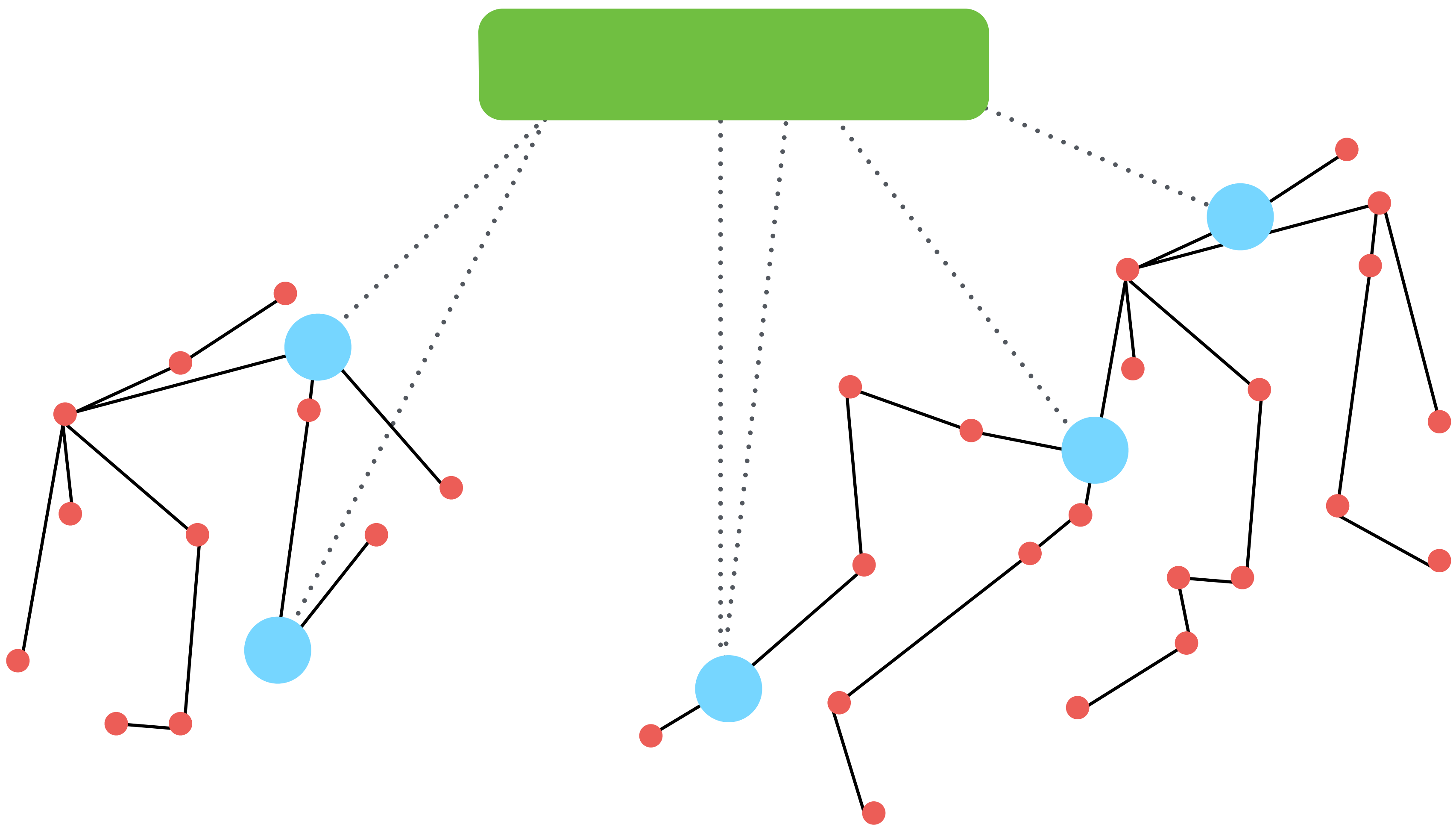
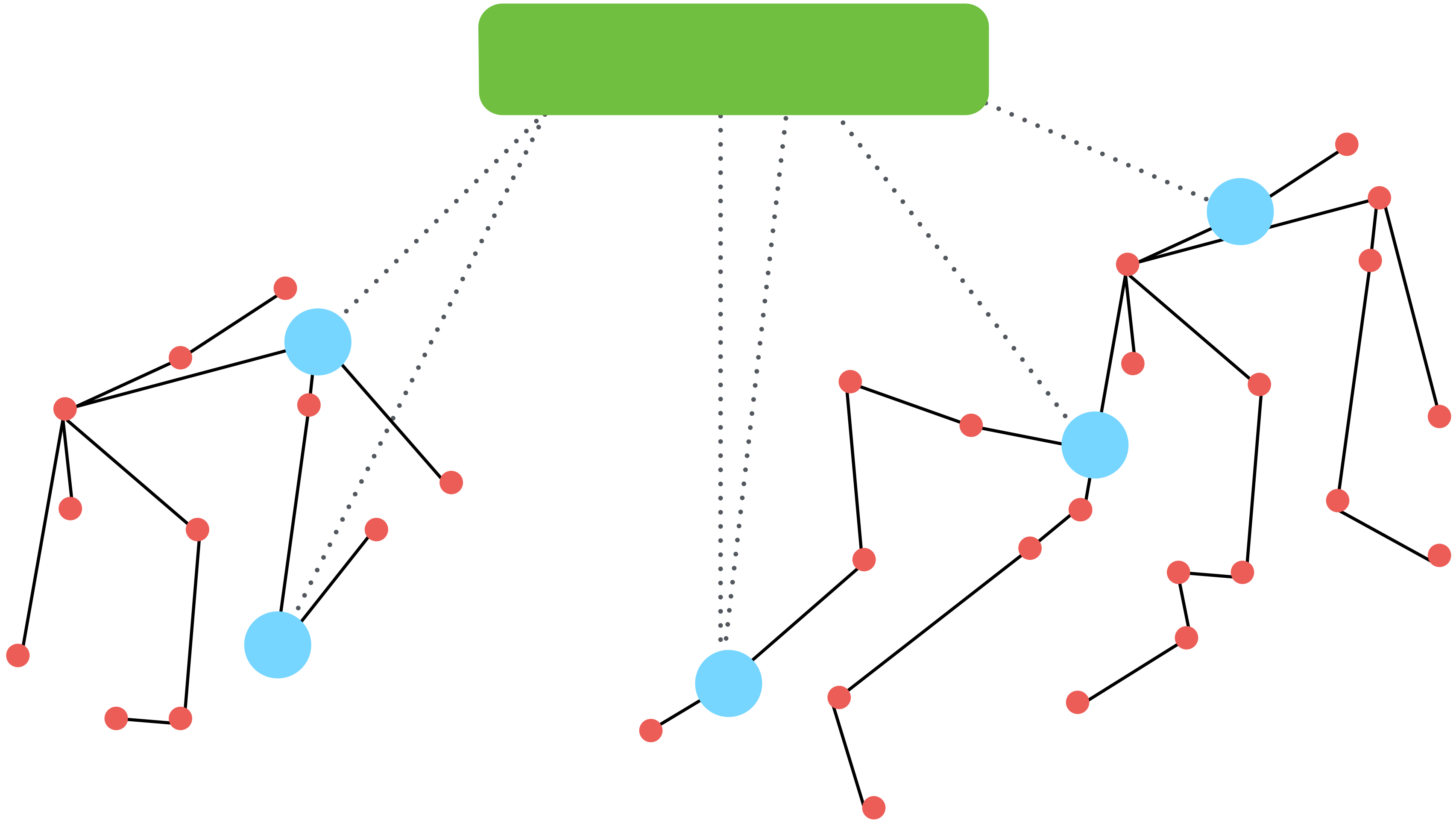
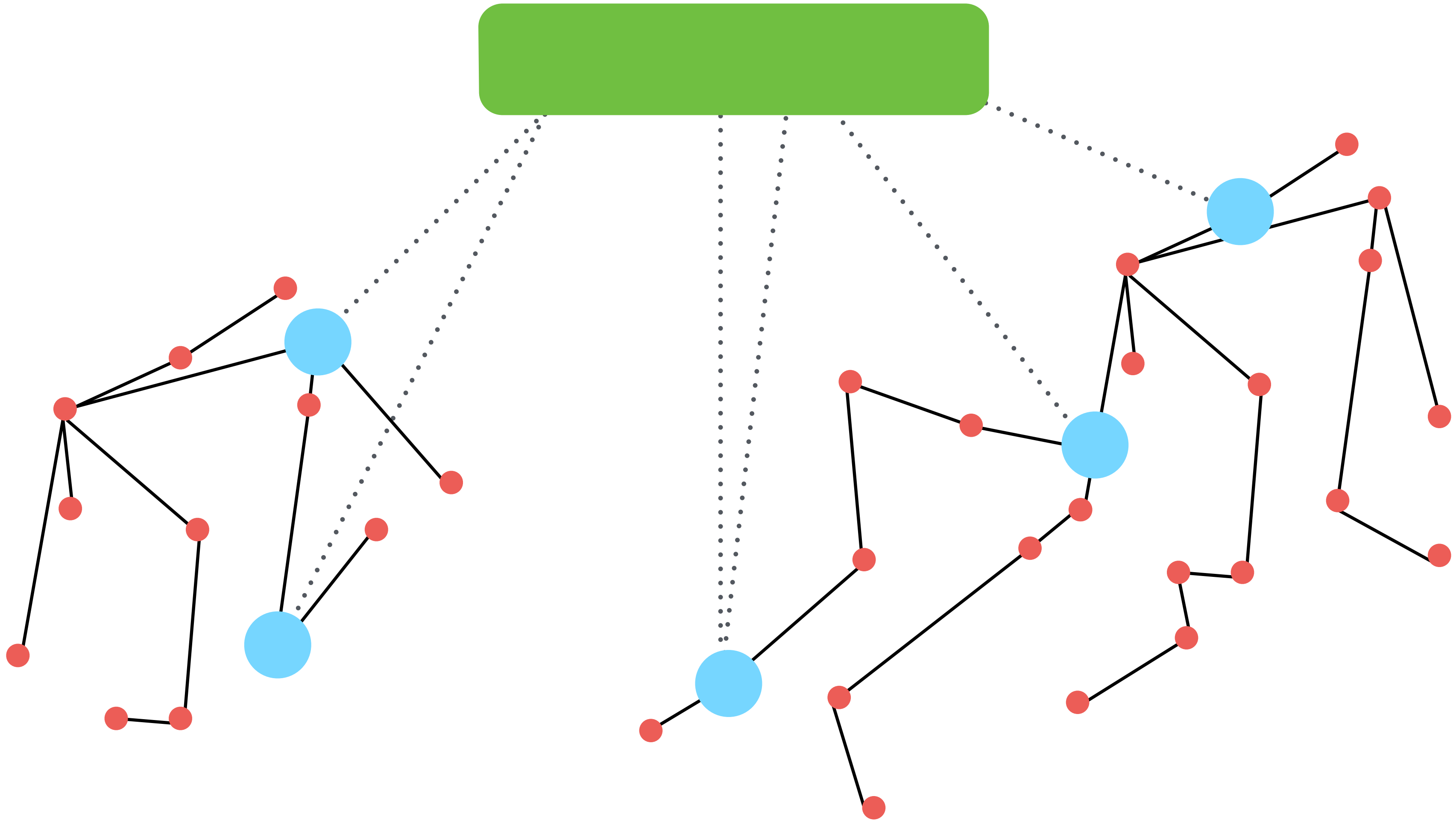


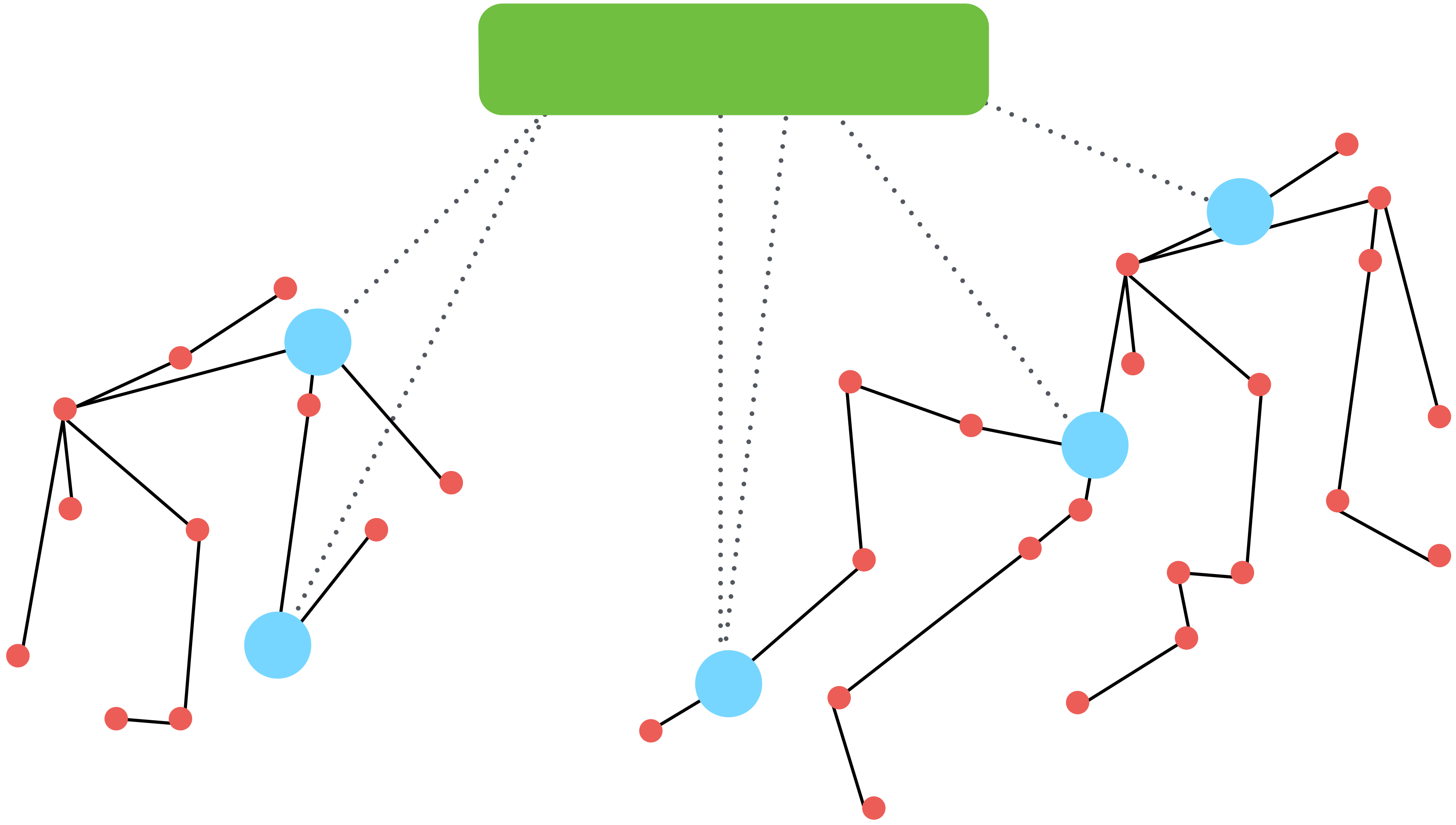




t “points of contact” between S and G\S…



t “points of contact” between S and G\S…   at least t edges going across the two sets.



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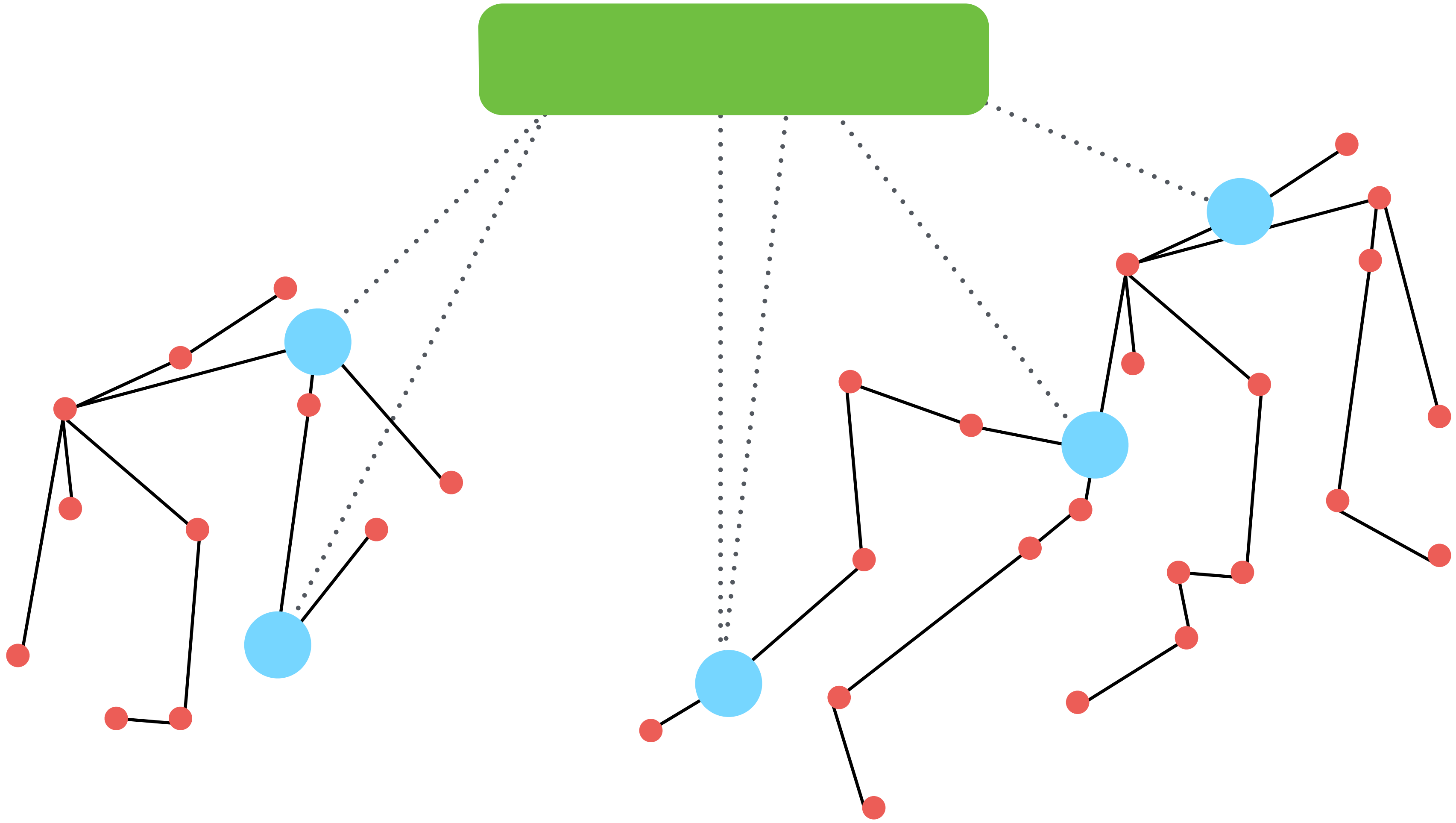

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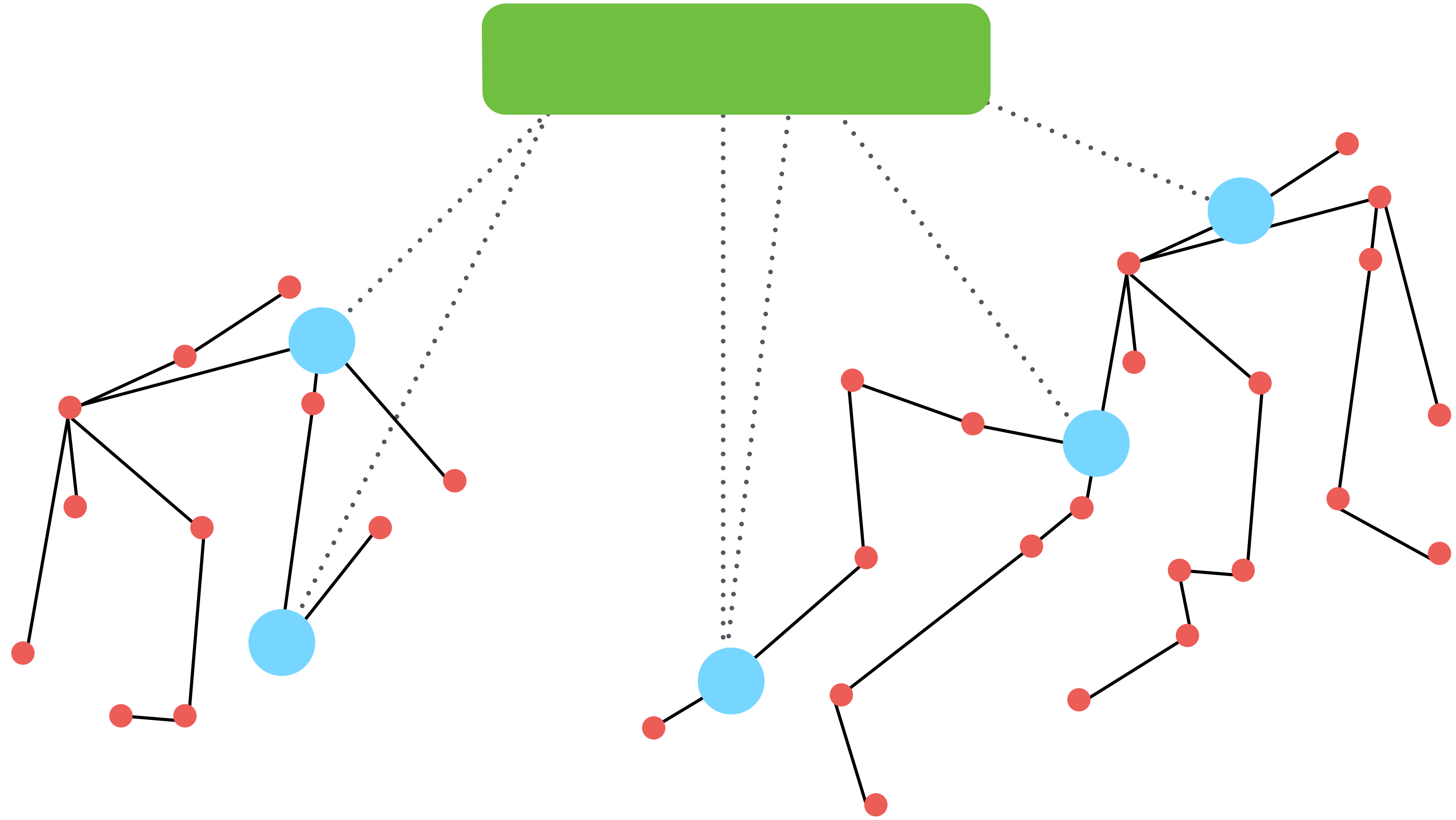

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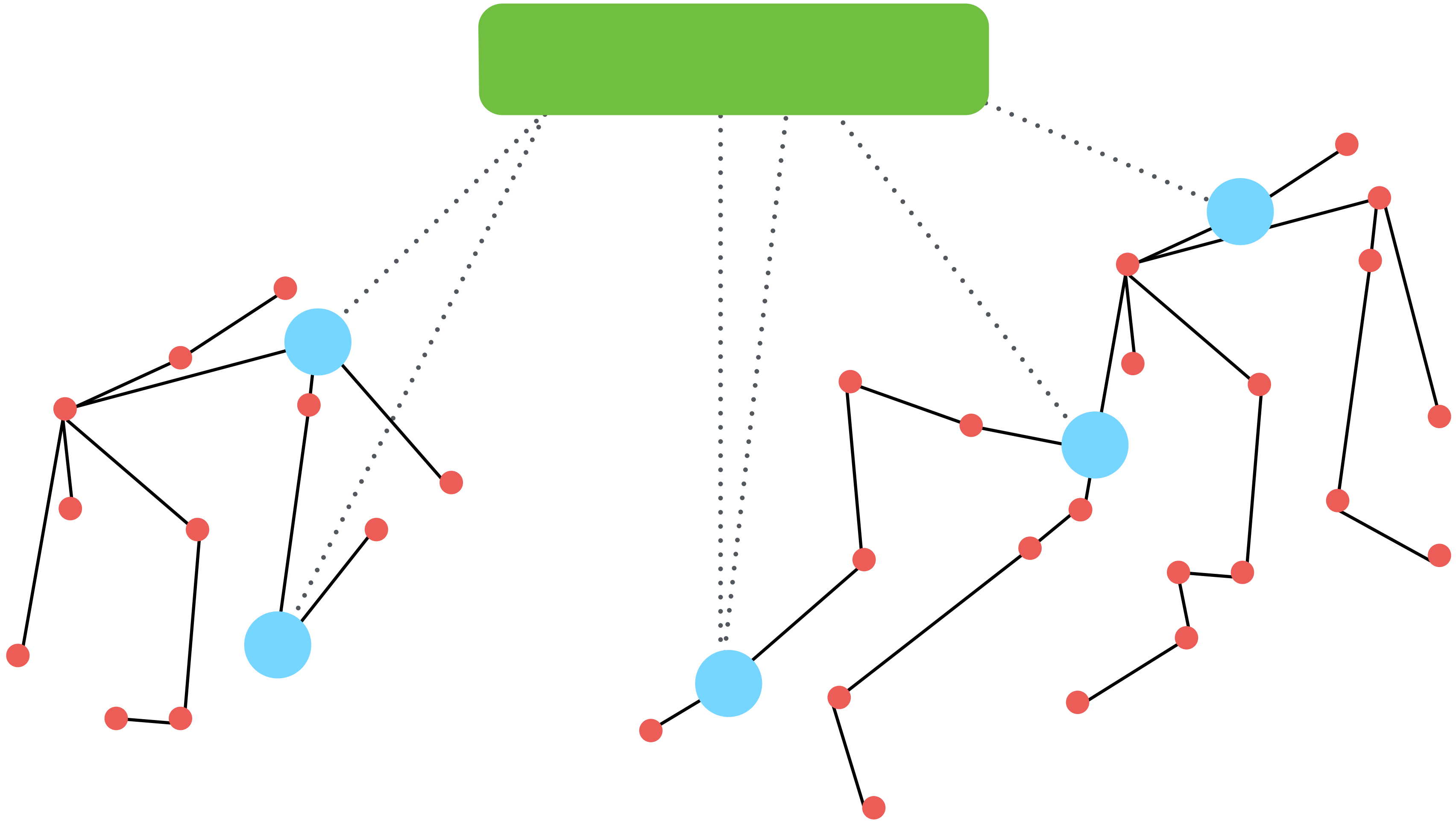

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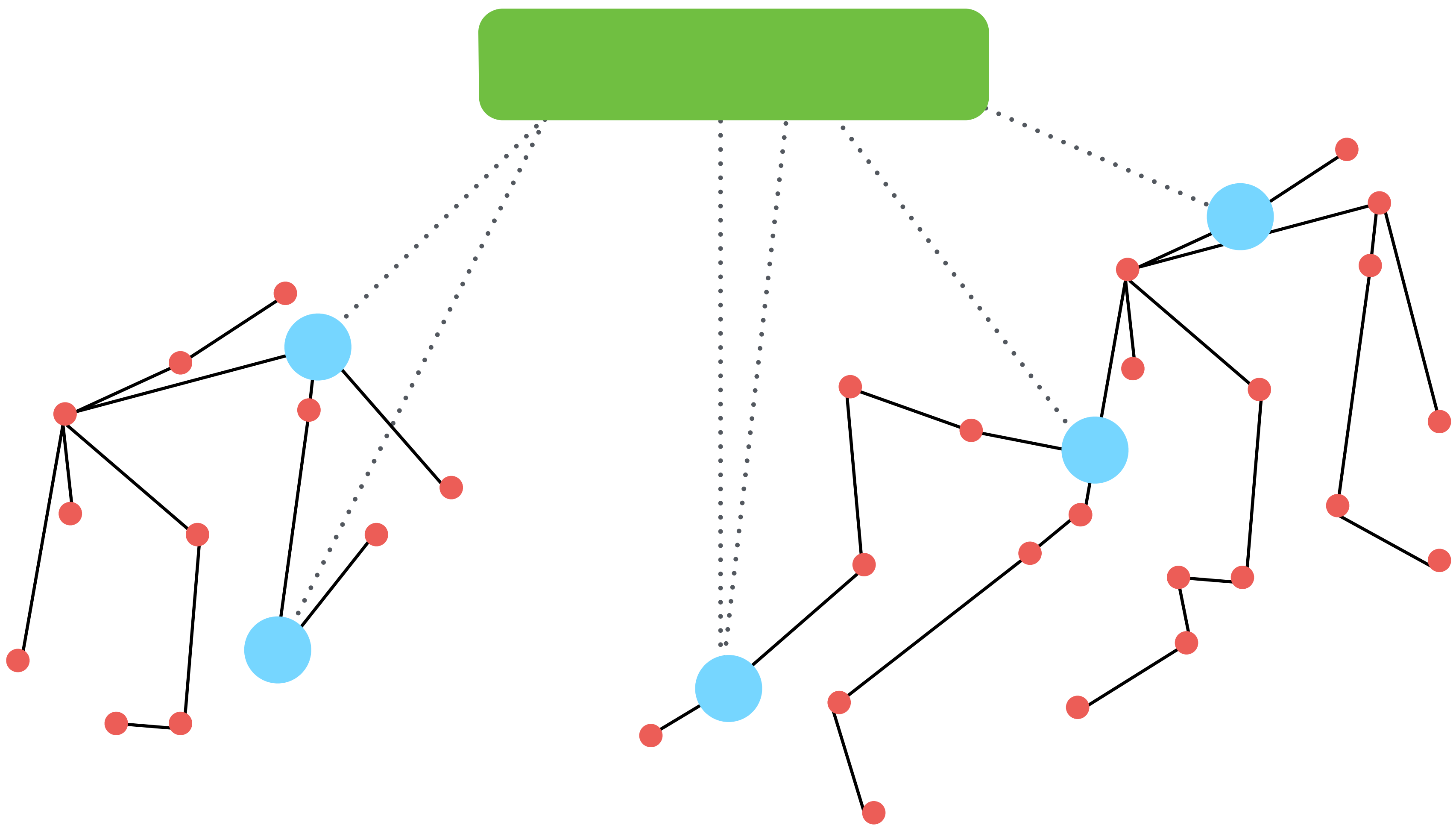

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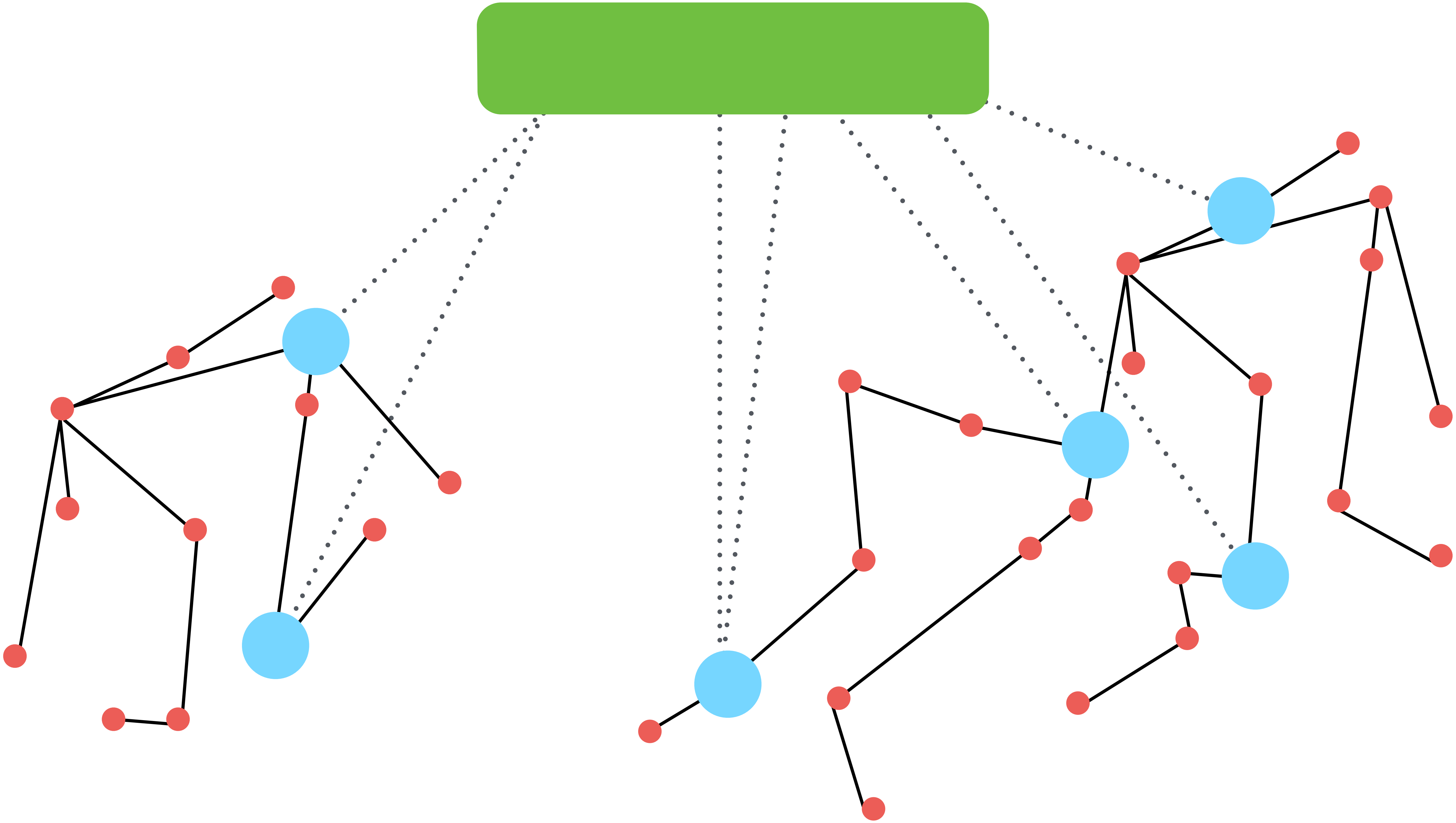
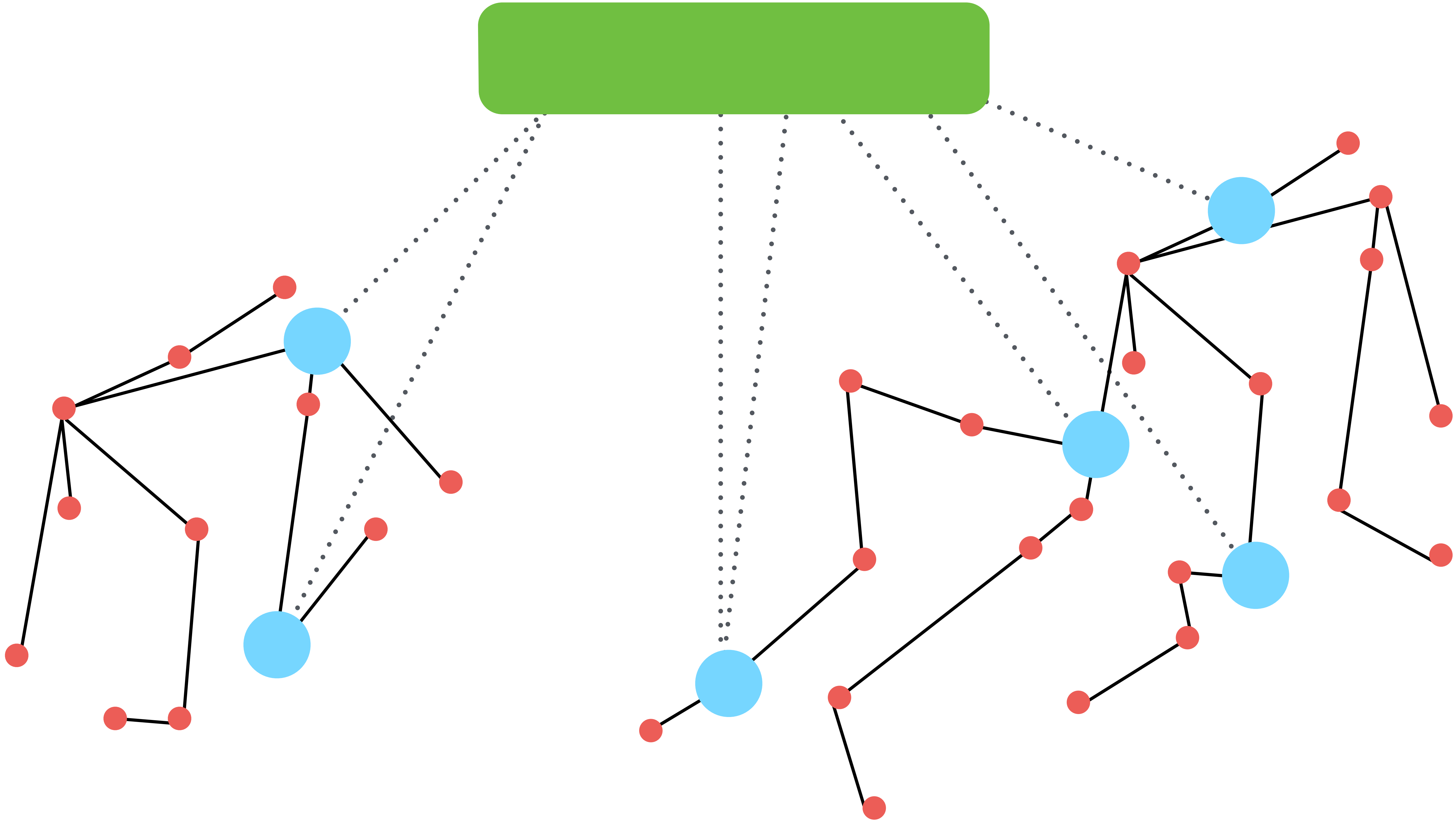
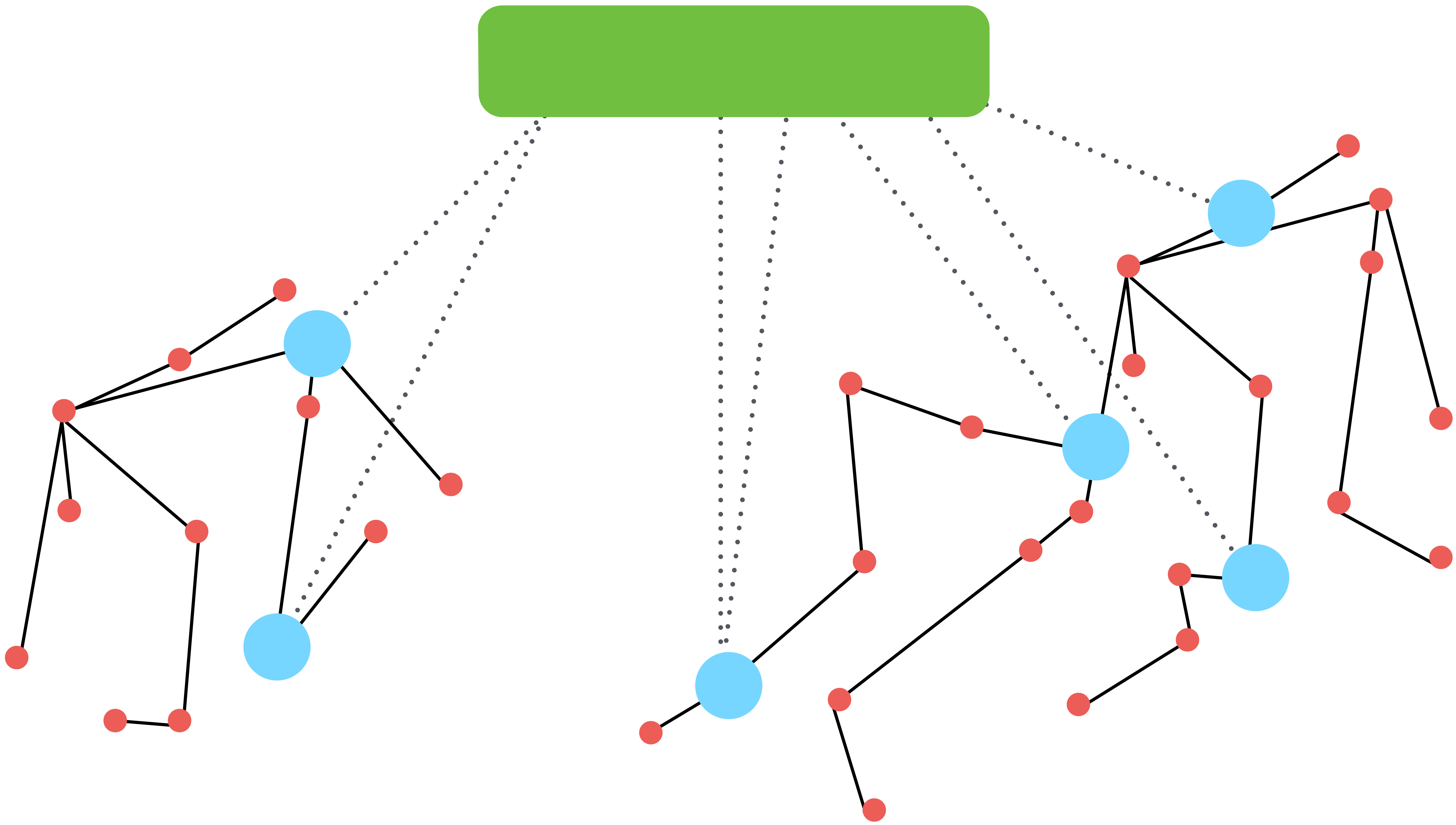
In this scenario, a randomly chosen edge will be **good with constant probability!**



Bad scenario: **one of the pieces is large.**



These pieces have **simple structure** and **bounded interaction** with the outer world.



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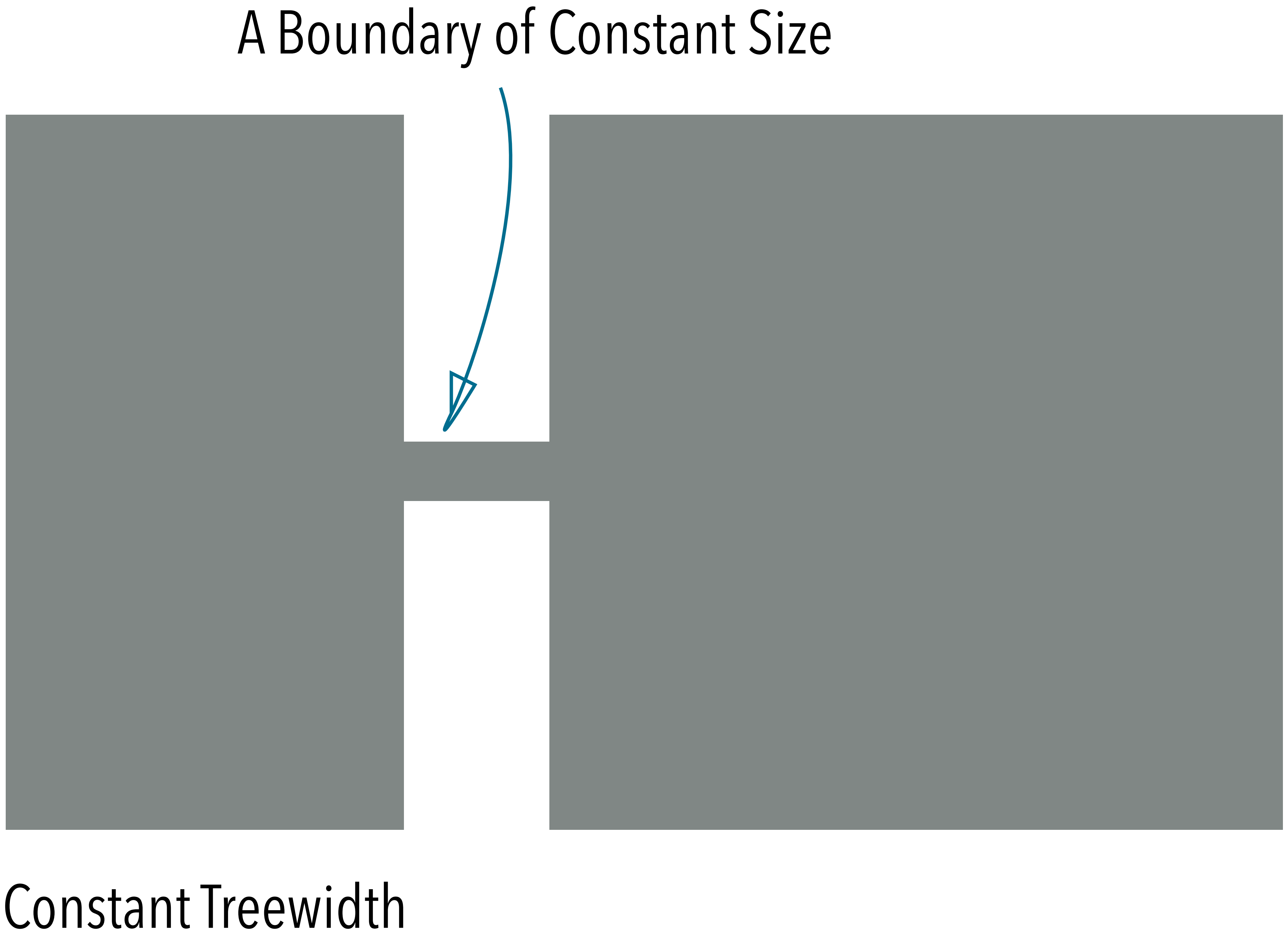
Such pieces are called **“protrusions”.**


A Boundary of Constant Size
Constant Treewidth

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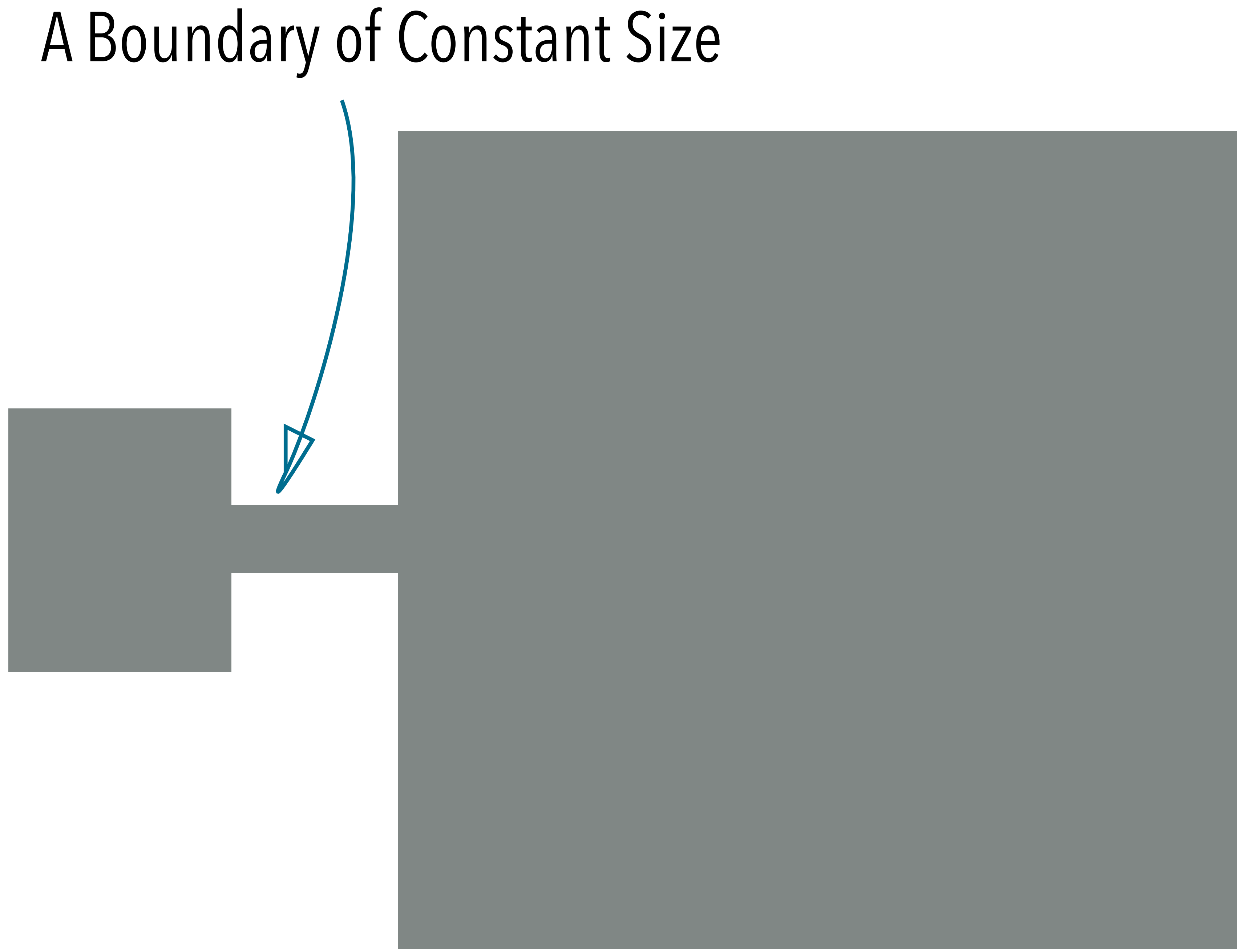

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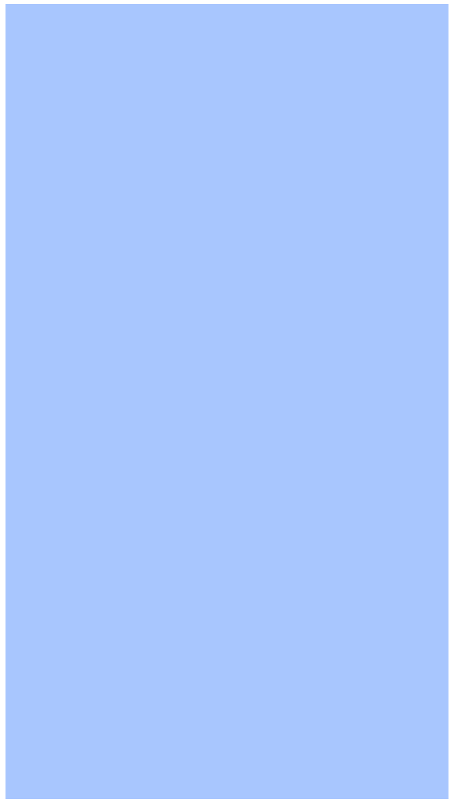

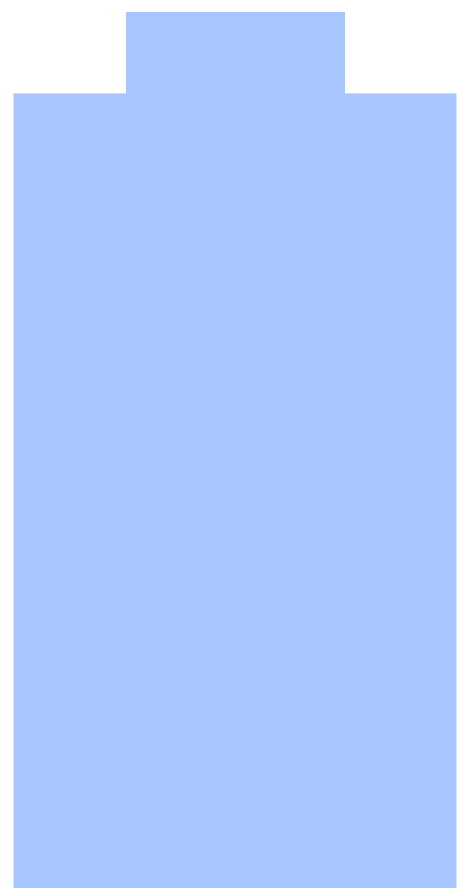

The space of t-boundaried graphs
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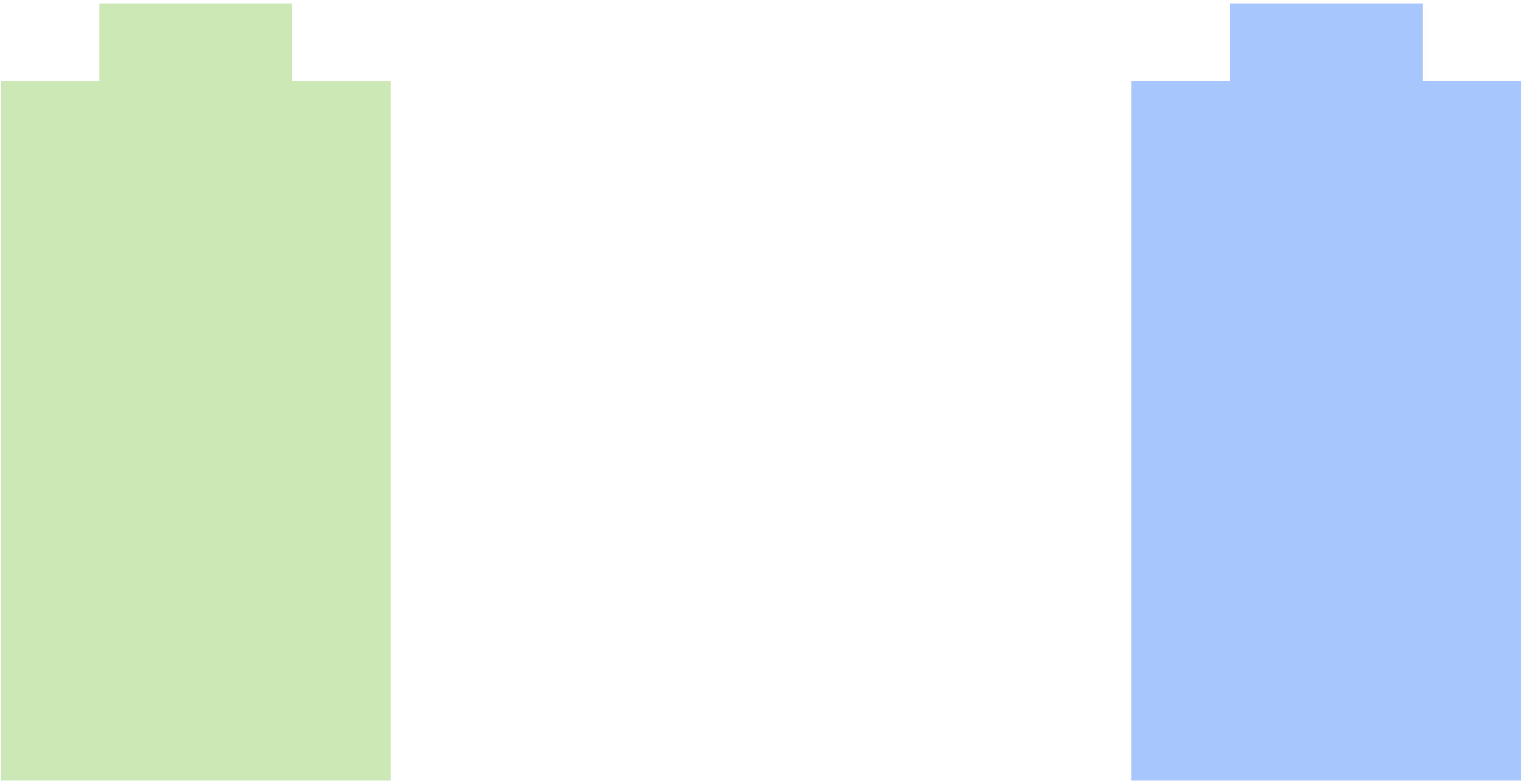
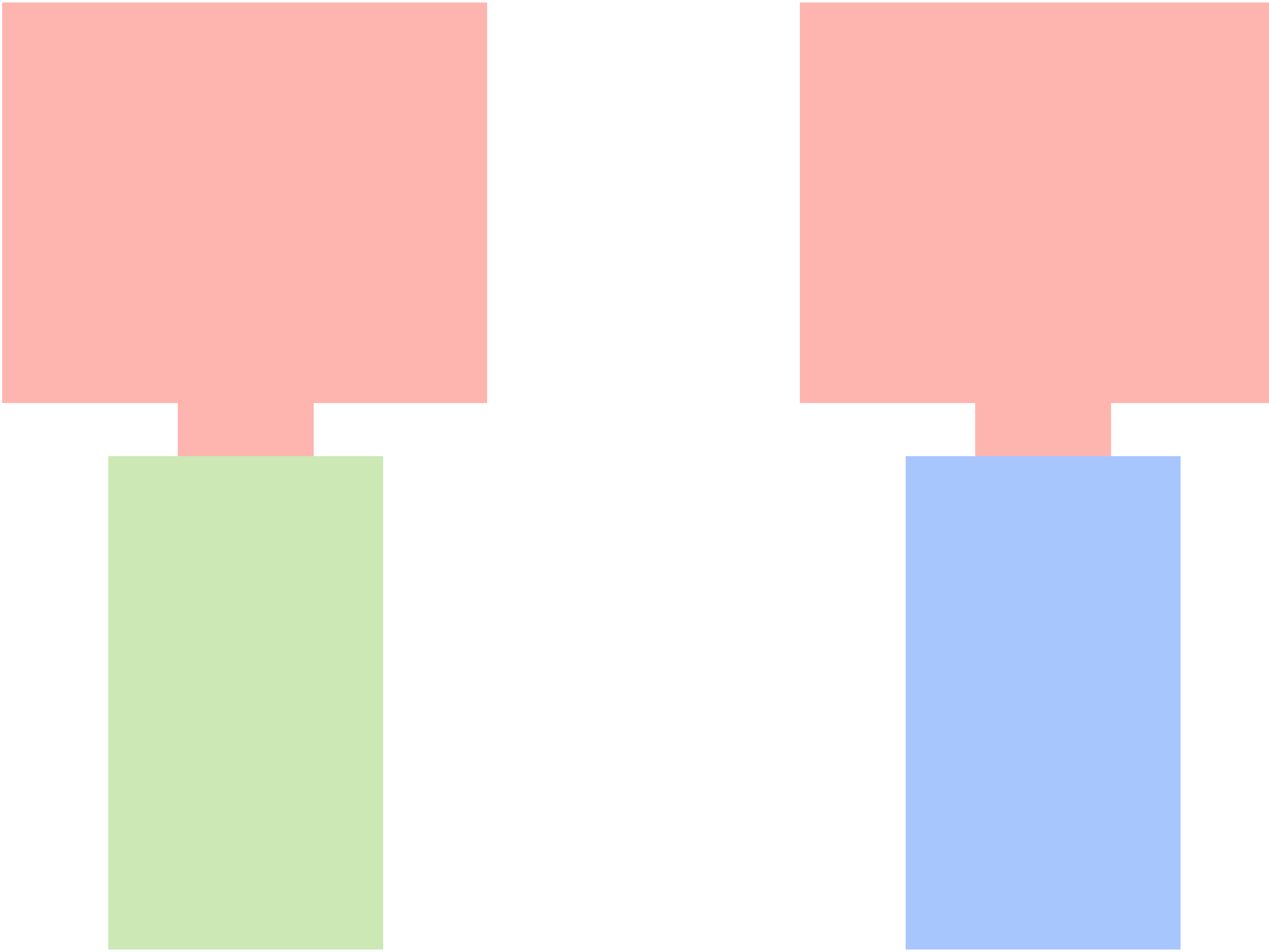
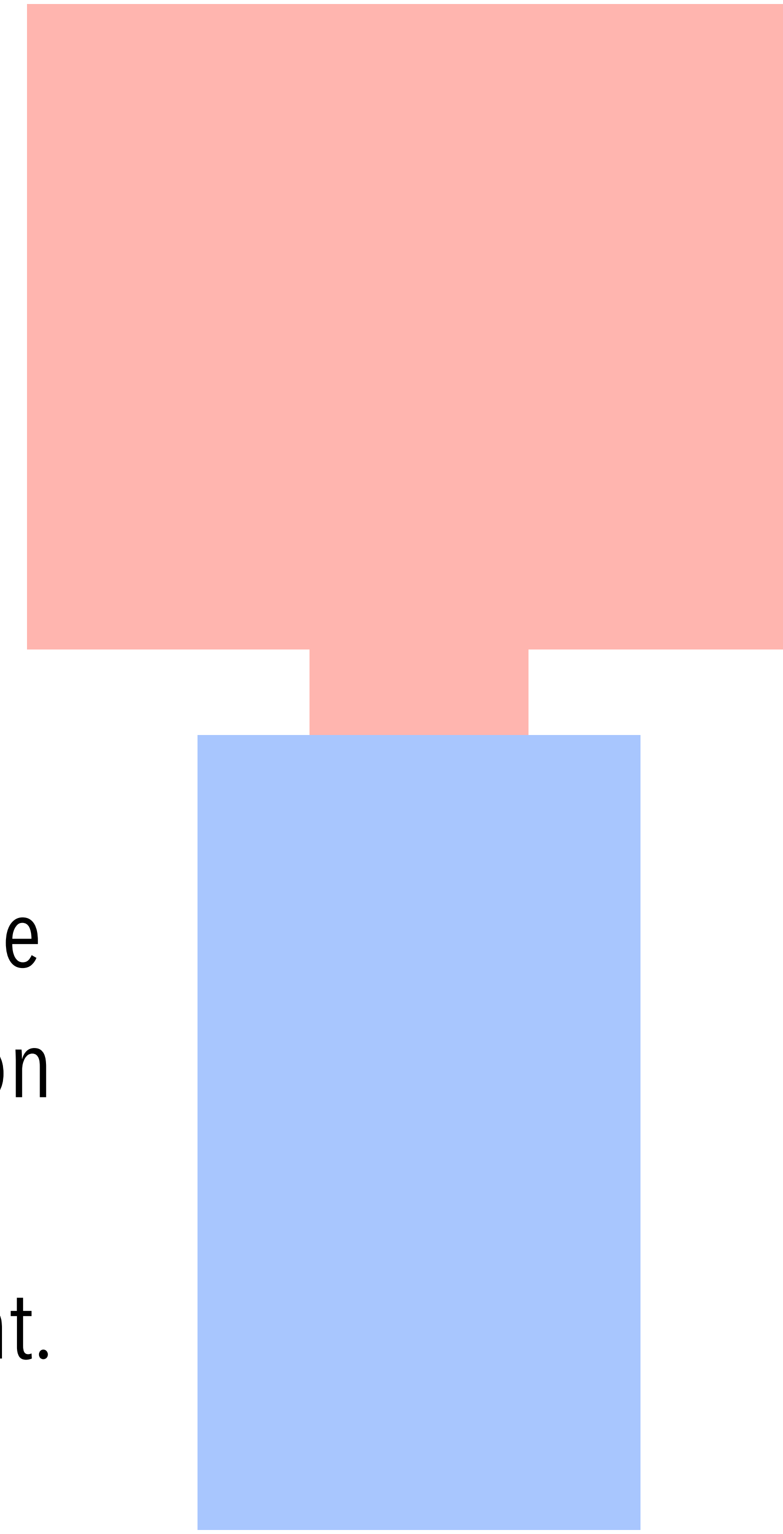
based on how they "behave" with
the "other side" of the boundary.





The value of the
optimal solution
is the same
up to a constant.



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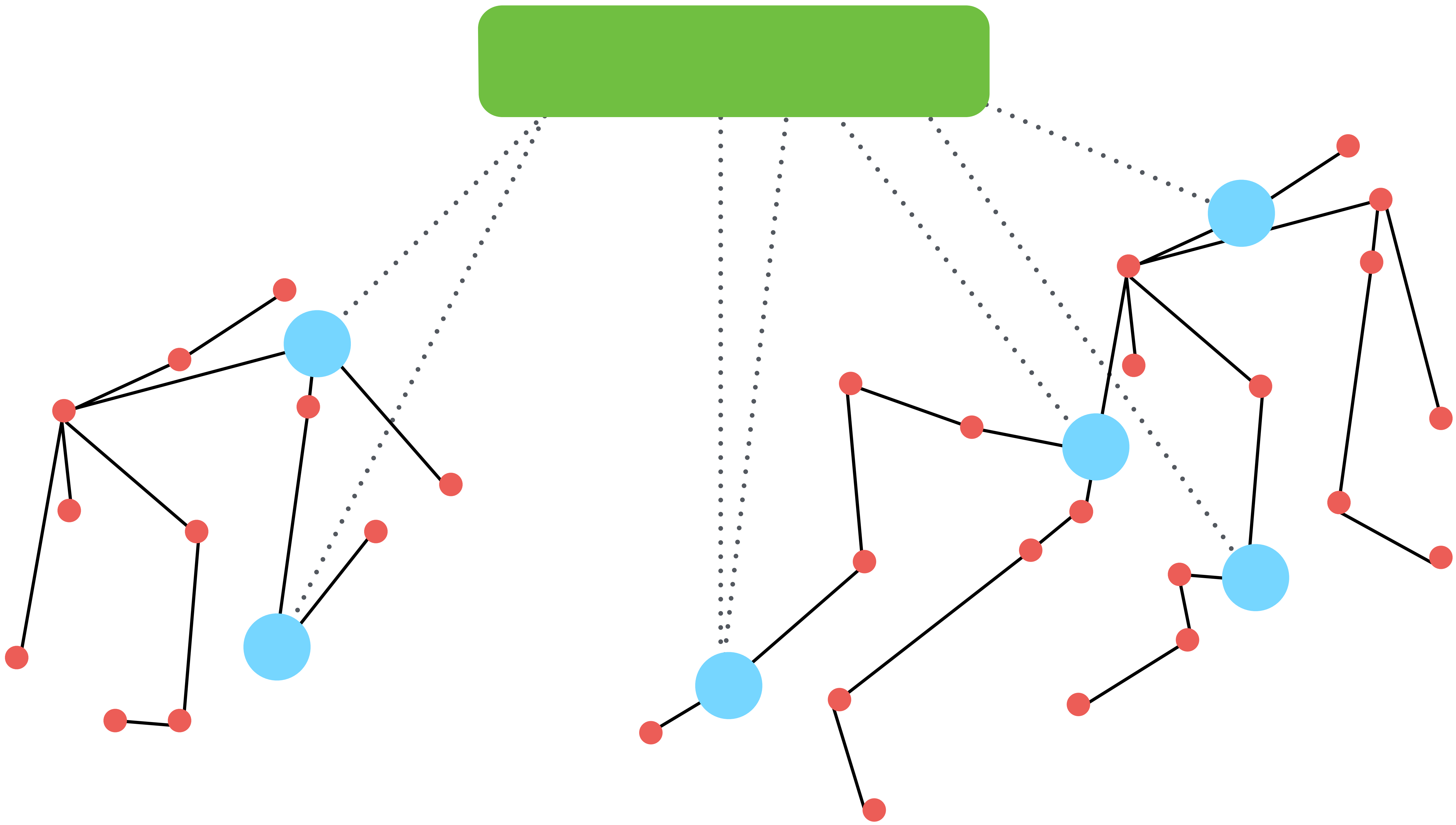
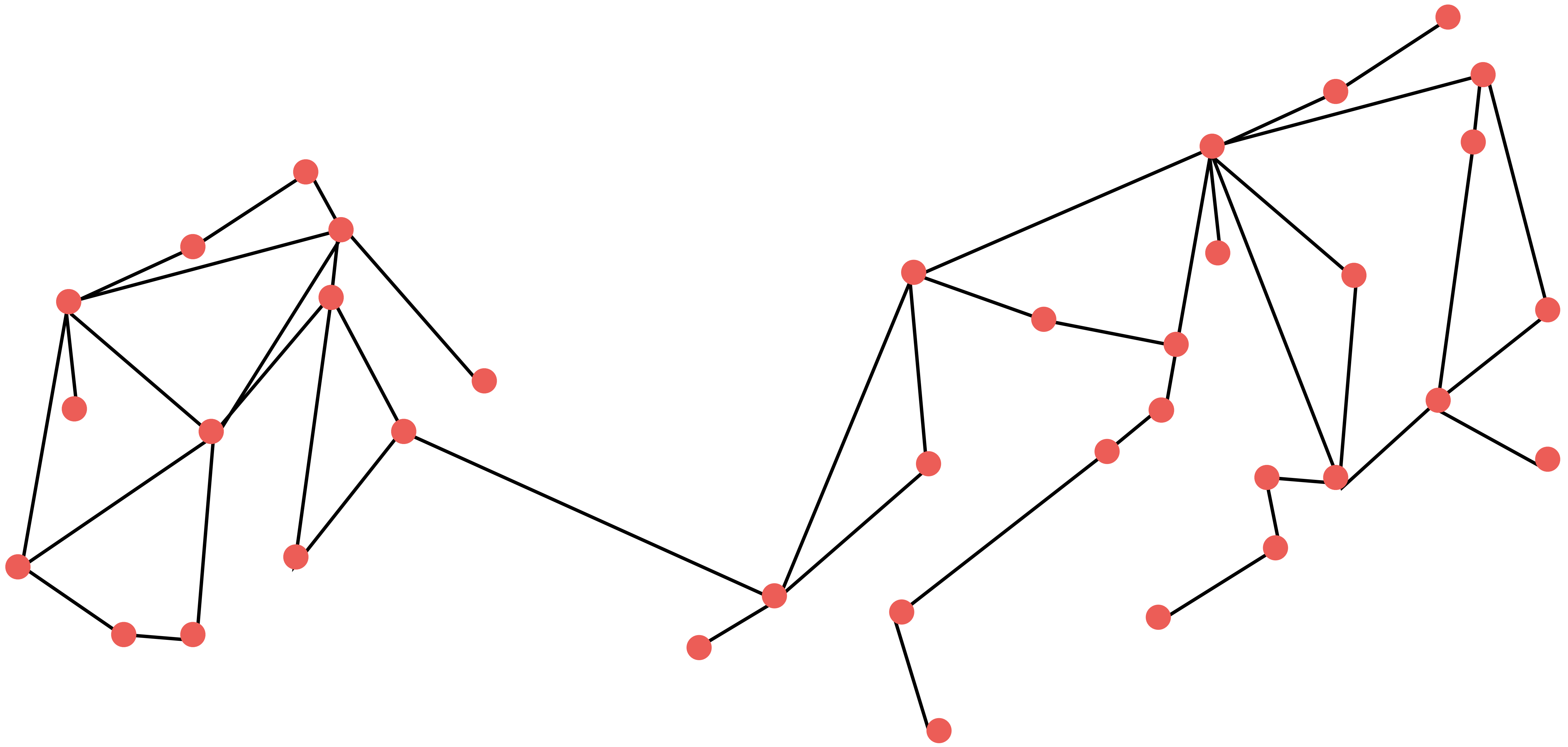
For some problems,
the number of equivalence classes is **finite**,
allowing us to replace protrusions in graphs.





Finding protrusions? What do we replace them with?

The $\mathcal{F}$-Deletion Problem

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Can G be made H-minor free, for all H in $\mathcal{F}$,
by the removal of at most k vertices?

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The Planar $\mathcal{F}$-Deletion Problem

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Pick an edge uniformly at random.

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Repeat till G is $\mathscr{F}$-free.

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Include both endpoints.

(Approximation algorithm.)

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(Approximation algorithm.)

Pick an endpoint u.a.r.

(Randomized algorithm.)

Repeat till G is $\mathcal{F}$-free.

The Planar $\mathscr{F}$-Deletion Problem accounts for several specific problems…

The Planar $\mathcal{F}$-Deletion Problem accounts for several specific problems…

$\mathcal{F}$	
K_2	Vertex Cover
K_3	Feedback Vertex Set
((k+1)-by-(k+1)) Grids	Treewidth k
$K_{2,3}$, K_4	Outerplanar
K_4	Series-Parallel
K_3,T_2	Pathwidth-One

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The question of kernels is **completely open.**

Graph Modification Problems

When the operation induces hardness

Contraction Problems

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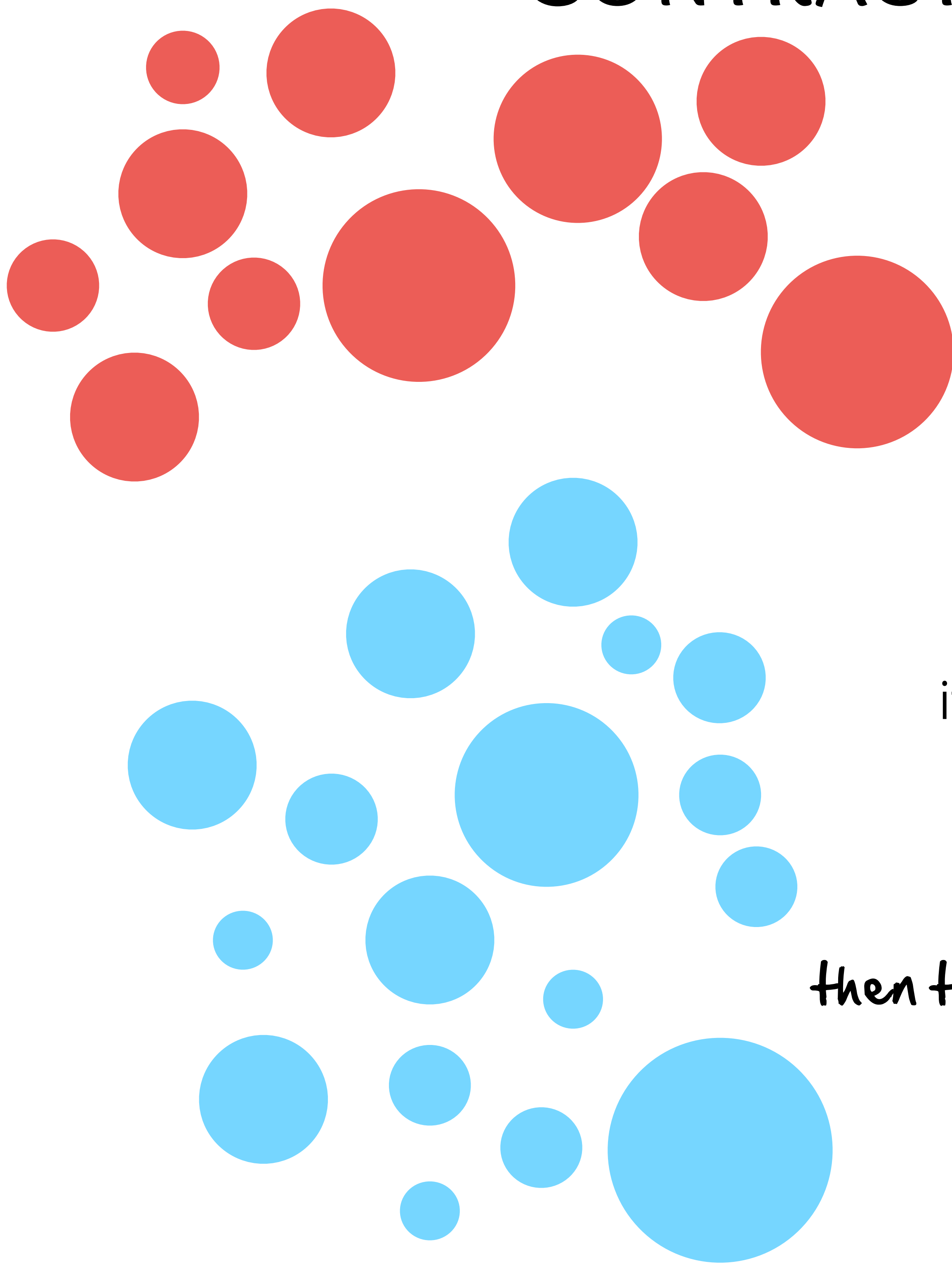

Graph Class or "property", π

Forbidden Subgraph Characterization

A graph admits a **forbidden subgraph characterization**
if, and only if,
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that is, the property is preserved under vertex deletions.

If there are finitely many forbidden graphs,
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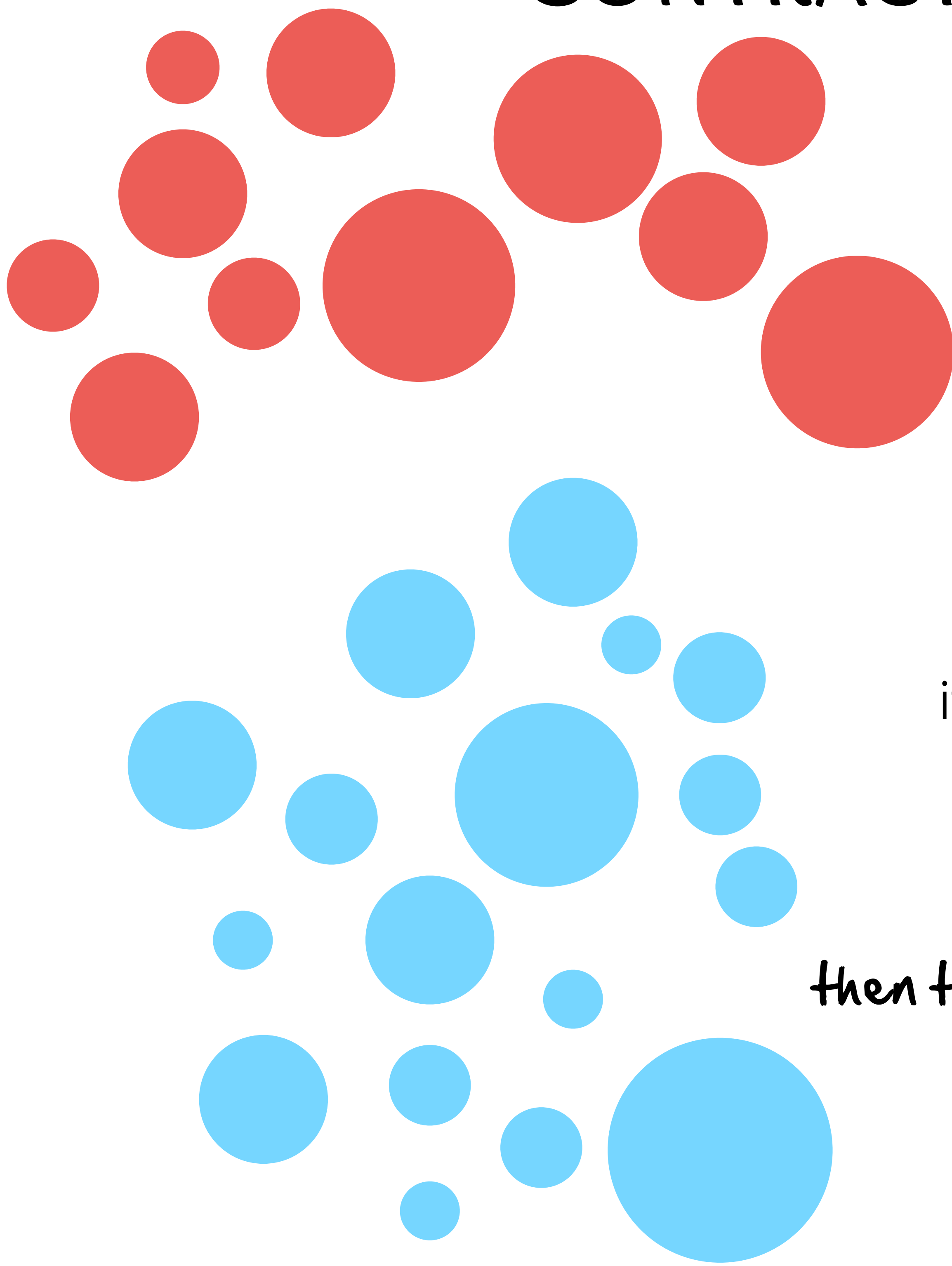

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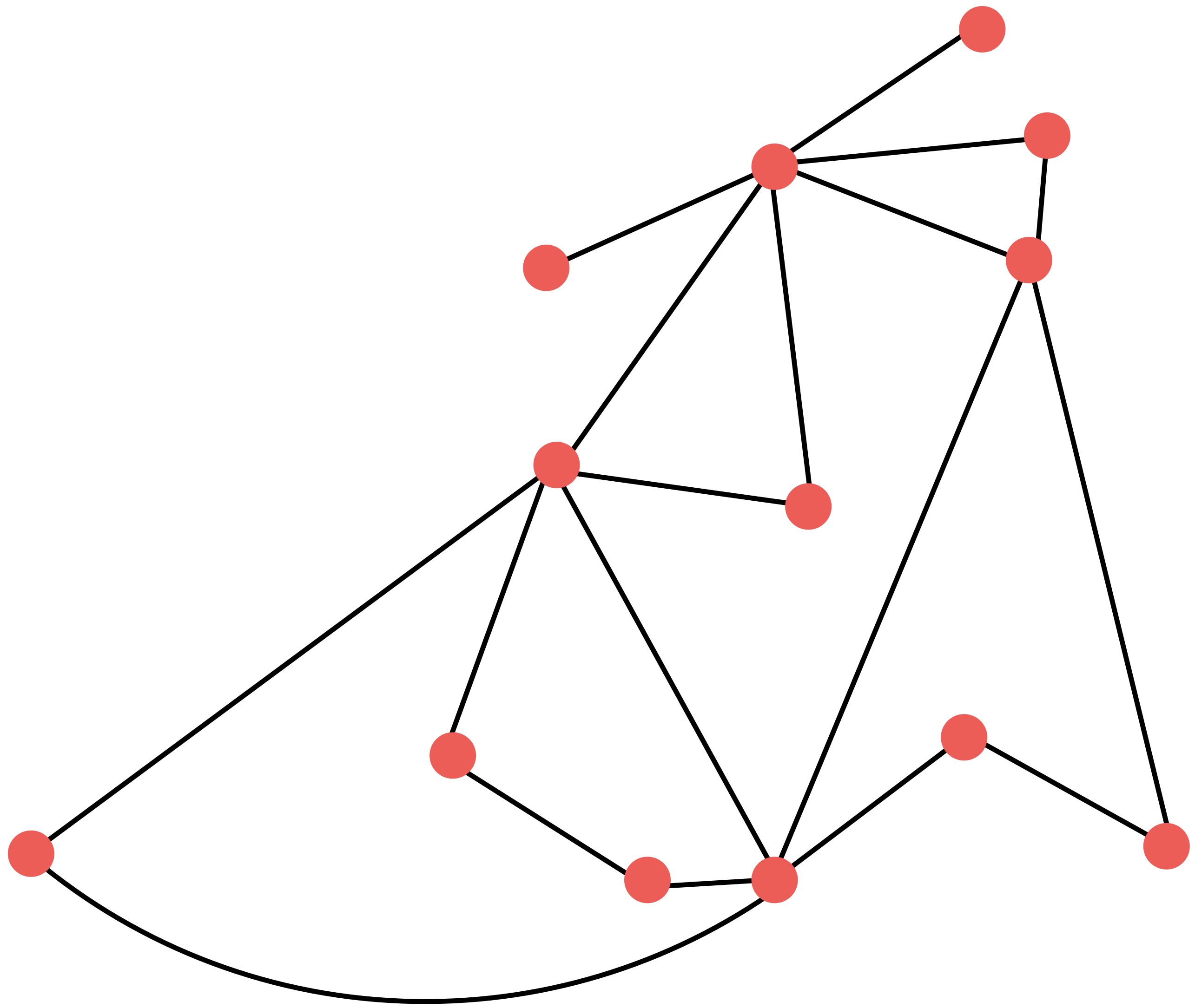
Not true any more!

Contraction To C_4-free graphs

Can G be made C_4-free by the contraction of at most k edges?



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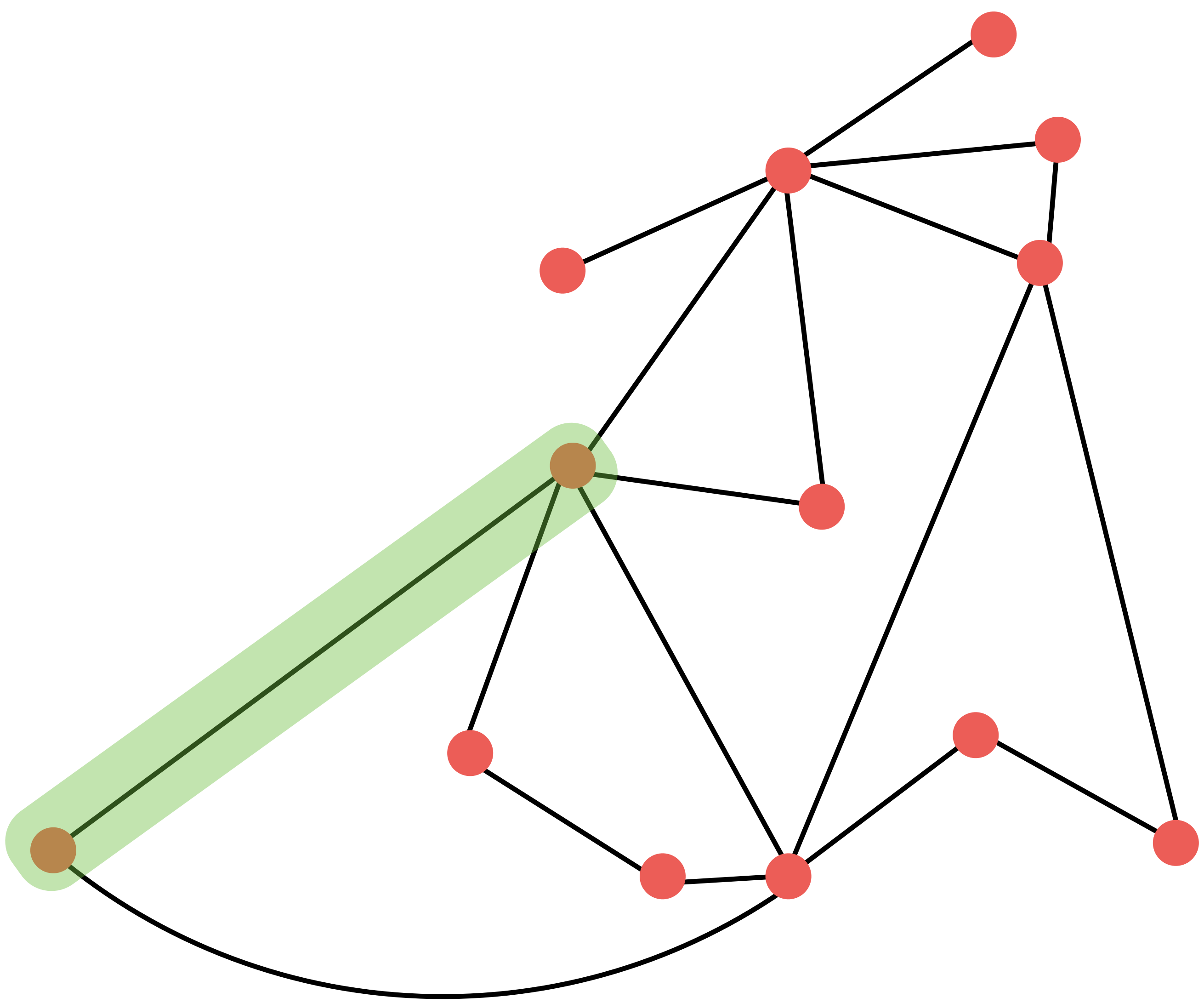

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Contracting to paths is **FPT** and has a **linear-vertex kernel**; while contracting to trees is **FPT** but is **unlikely to admit a polynomial kernel**.

Graph Modification Problems

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Graph Modification Problems

A Summary

Edge Deletions

Kernels on Planar Graphs & the Meta-Kernel project

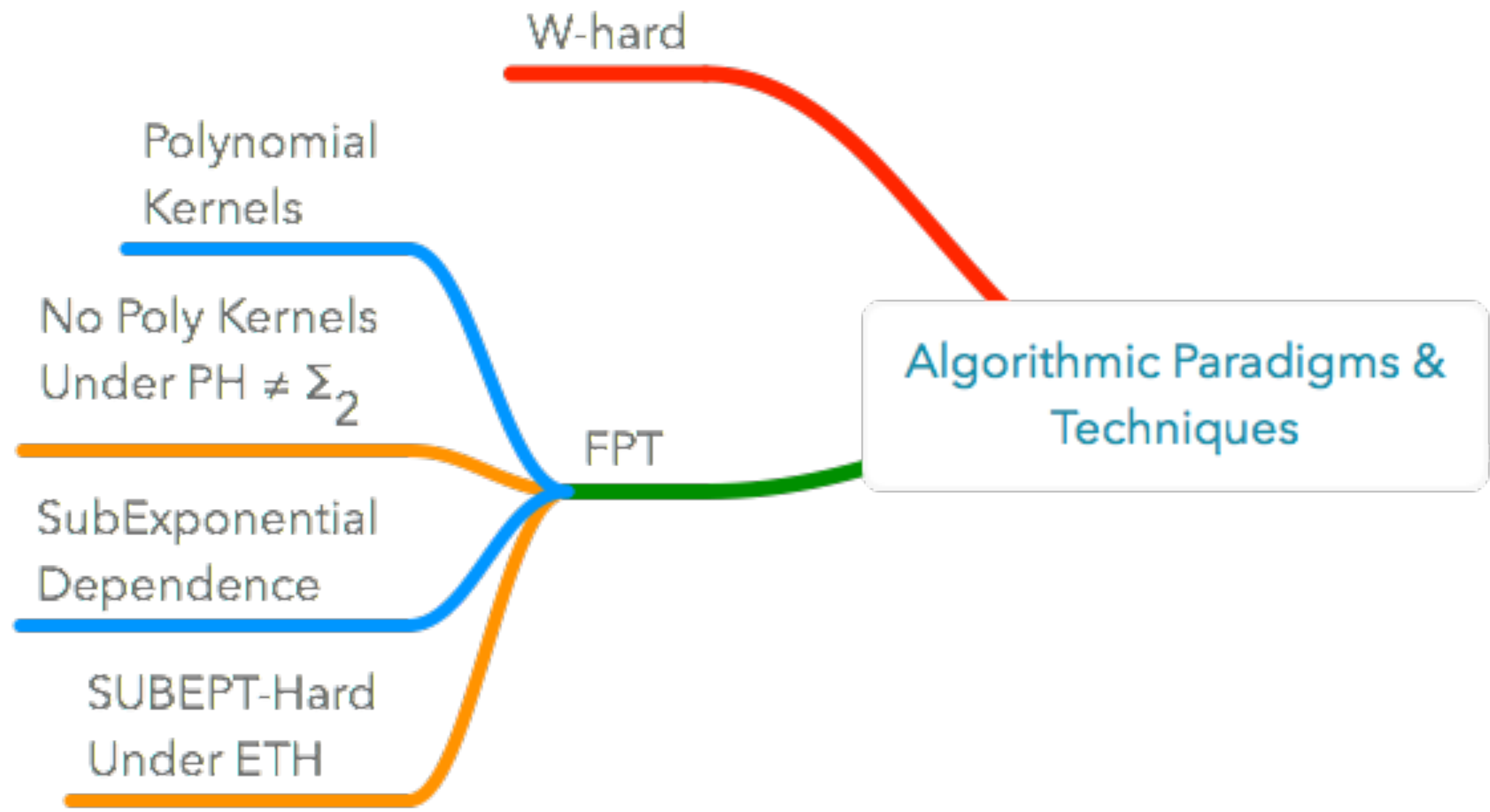
Special Graph Classes, eg, Feedback Arc Set on Tournaments

Connectivity Augmentation

The descriptive complexity of graph modification

Structural parameters and other objective functions

Backdoor Sets!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FPT
Polynomial Kernels
No Poly Kernels Under PH ≠ Σ2
SubExponential Dependence
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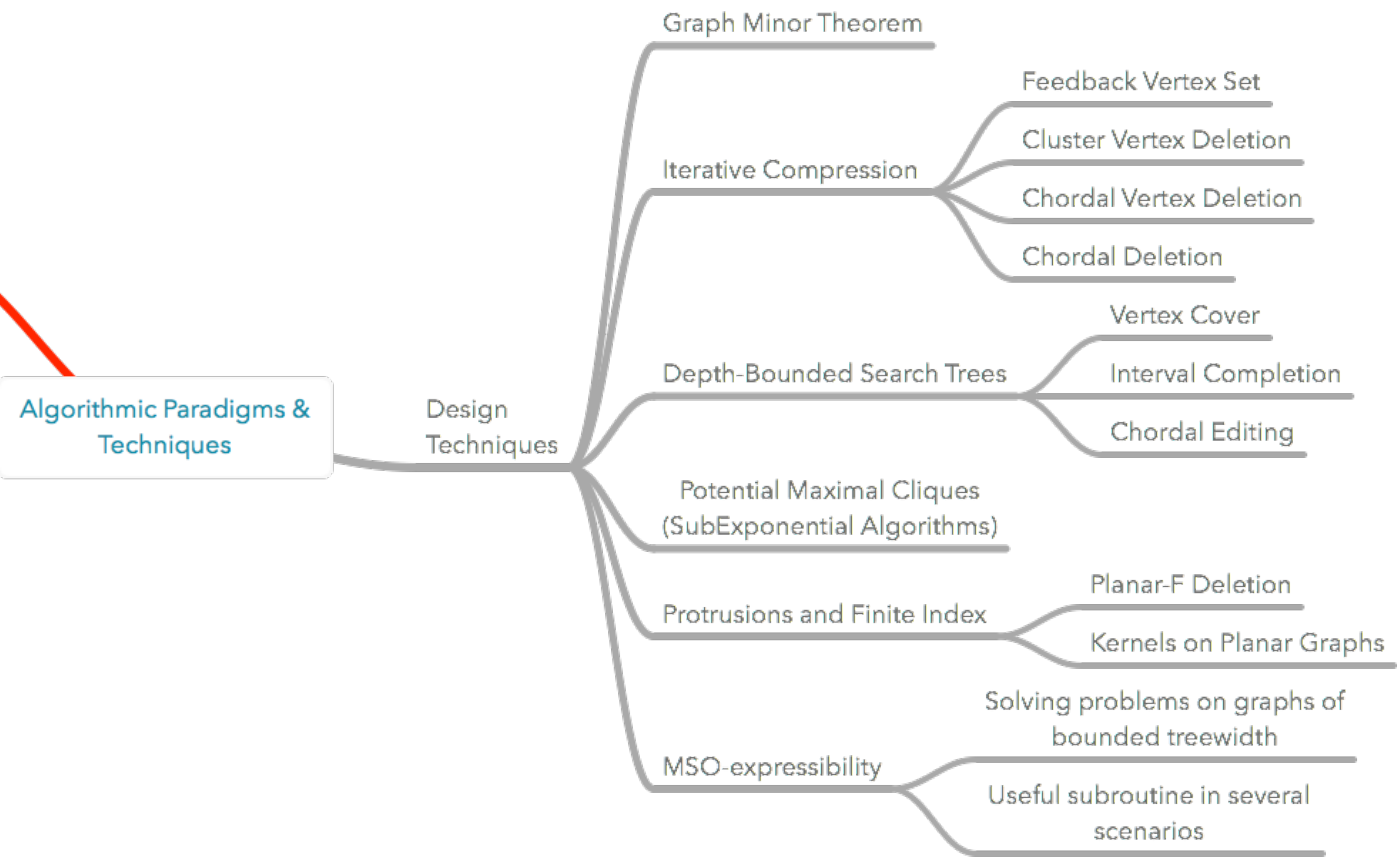
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Cluster Vertex Deletion
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Vertex Cover
Interval Completion
Chordal Editing
Potential Maximal Cliques (SubExponential Algorithms)
Protrusions and Finite Index
Planar-F Deletion
Kernels on Planar Graphs
MSO-expressibility
Solving problems on graphs of bounded treewidth
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