Introduction to Automatic Numbers

Automatic Presentations of Graphs and Numbers

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IMSc, Chennai, October, 2016

Numbers

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Representations

Number	Decimal	Binary	Continued Fractions
3	3	11	3
$\frac{1}{3}$	.3333...	.0101...	$0+\frac{1}{3}$
$\sqrt{2}$	1.4142...	1.011...	$1+\frac{1}{2+\frac{1}{2+\cdots}}$
e	2.7182...	10.101..	$2+\frac{1}{1+\frac{1}{2+\frac{1}{1+\cdots}}}$

Representation of Real Numbers

- Injective map from $\mathbb{R}$ to a set of infinite strings on an alphabet.
- Alphabet $\Sigma_2 := \{0, 1\}$.
- $\Sigma_2^* := \{\text{finite strings on } \Sigma_2\} = \{\varepsilon, 0, 1, 00, 01, 10, 11, ...\}$.
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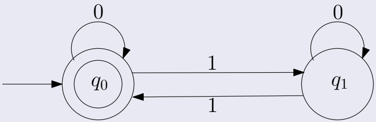
- Σ finite input alphabet (e.g., Σ_2).
- $\delta : Q \times \Sigma \to Q$ transition function.
- q_0 initial state.
- $F \subseteq Q$ set of accepting states.

Example



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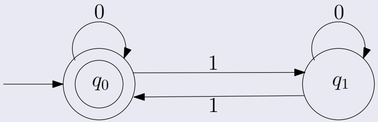

$L(M) = \{w \in \Sigma^* | \delta(q_0, w) \in F\}$.
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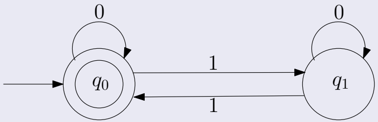

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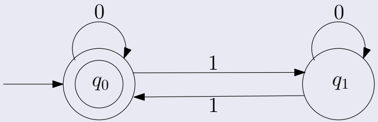

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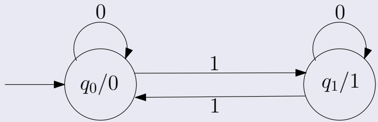

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Thue-Morse sequence:

$n = \ 1 \ \ 2 \ \ 3 \ \ 4 \ \ 5 \ \ 6 \ \ 7 \ \ \ldots$

$t_n = 1 \ \ 1 \ \ 0 \ \ 1 \ \ 0 \ \ 0 \ \ 1 \ \ \ldots$

$T = \frac{1}{2} + \frac{1}{4} + 0 + \frac{1}{2^3} + 0 + 0 + \frac{1}{2^7} + \ldots$

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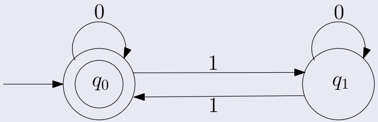

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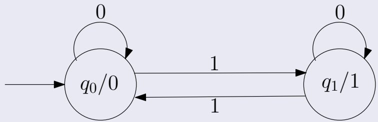
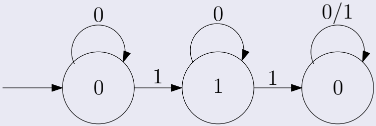


Automatic Numbers – More examples

Fredholm number $= \sum_{n\geq 1} 2^{-2^n}$



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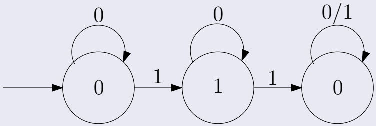
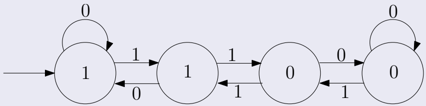


Rudin-Shapiro

nth bit is '1' iff the number of (overlapping) occurrences of "11" in $[n]_2$ is even.



Non-Automatic Numbers

- $L \subseteq \Sigma_2^*$ a non-regular language. Then nth bit is one iff $[n]_2 \in L$.
- E.g., the nth bit is one iff $[n]_2$ is of the form $0^k 1^k$, for some $k \geq 0$.

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Characteristic sequence of Squares

- nth bit is one iff n is a square: $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \dots \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & \dots \end{bmatrix}$.
- Suppose Squares is accepted by a DFA (i.e., it is regular).

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- Claim: n is a square iff $k = \ell$.
- Thus Squares $\cap (11)^*(00)^*01 = \left\{1^{2k}0^{2k+1}1\right\}$, which is not regular.

Properties of Automatic Numbers

Let $(a_n)_{n\geq 1}$ be a sequence of bits, and $F_i := \{[n]_2 | a_n = i\}$, $i \in \{0, 1\}$.

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Rationals are k-automatic

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- Inverse: $\mathbf{x}^2 = \mathbf{x} + \frac{1}{\frac{1}{\mathbf{x}-1} - \frac{1}{\mathbf{x}}}$.

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- Multiplication can increase the subword complexity to $|\Sigma|^n$.

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