Hierarchy of pushdown graphs

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The hierarchy of pushdown graphs

recursive transition graphs

Corresponding hierarchies of

languages, terms, ordinals, infinite words

higher order recursive schemes

Graphs

alphabet L of edge labels

alphabet C of vertex labels: colours

Graph :

$$G \subseteq V \times L \times V \cup C \times V$$

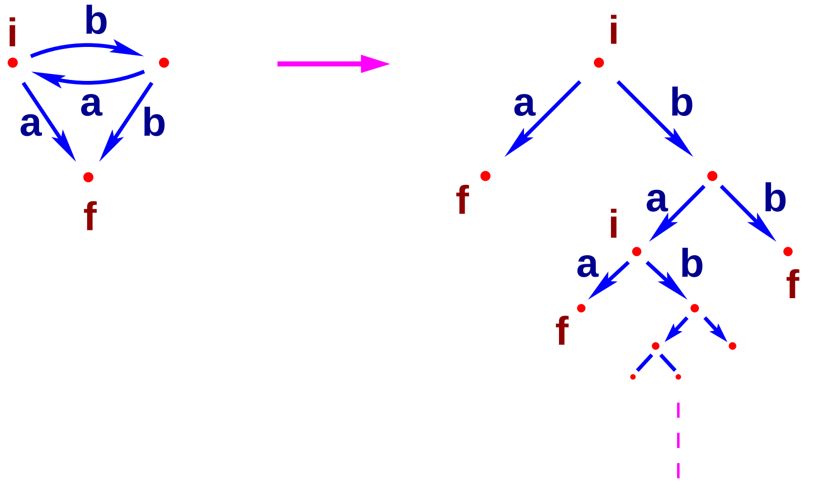
on an arbitrary countable set V of vertices

A hierarchy of graph families 2002

Two basic graph operations :

unfolding and path functions

Unfolding



Path functions

set Exp of path expressions

$$\mathsf{C} \cup \mathsf{L} \cup \{\varepsilon\} \subseteq \mathsf{Exp}$$

for any u,v $\in$ Exp

$$\mathsf{u}^{-1}, \mathsf{u} \cdot \mathsf{v}, \mathsf{u}^{+}, \neg \mathsf{u}, \mathsf{u} \vee \mathsf{v}, \mathsf{u} \wedge \mathsf{v} \in \mathsf{Exp}$$

path $\mathsf{s} \xrightarrow[G]{u} \mathsf{t}$ for u $\in$ Exp

$$s \xrightarrow{a} t \quad \text{for} \quad (s,a,t) \in G$$

$$s \xrightarrow{c} t \quad \text{for} \quad s = t \ \wedge \ (c,s) \in G$$

$$s \xrightarrow{\varepsilon} t \quad \text{for} \quad s = t$$

$$s \xrightarrow{u^{-1}} t \quad \text{for} \quad t \xrightarrow{u} s$$

$$s \xrightarrow{u \cdot v} t \quad \text{for} \quad \exists\, r\ (s \xrightarrow{u} r \ \wedge \ r \xrightarrow{v} t)$$

$$s \xrightarrow{u^{+}} t \quad \text{for} \quad s\ (\xrightarrow{u})^{+}\ t$$

$$s \xrightarrow{\neg\, u} t \quad \text{for} \quad \neg\ (s \xrightarrow{u} t)$$

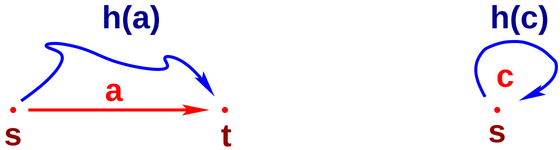
$$s \xrightarrow{u \vee v} t \quad \text{for} \quad s \xrightarrow{u} t \ \vee \ s \xrightarrow{v} t$$

For instance $s \xrightarrow{\varepsilon \ \wedge \ a \cdot a^{-1}} t$ means that $s = t \wedge s \xrightarrow{a}$

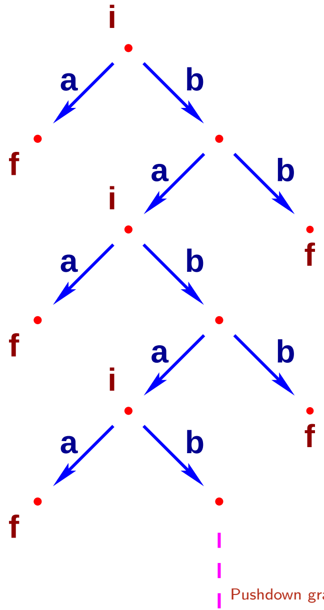
Path function $h : L \cup C \longrightarrow Exp$

applied by inverse on a graph G

$$h^{-1}(G) = \{ (s,a,t) \mid s \xrightarrow[G]{h(a)} t \} \cup \{ (c,s) \mid s \xrightarrow[G]{h(c)} s \}$$

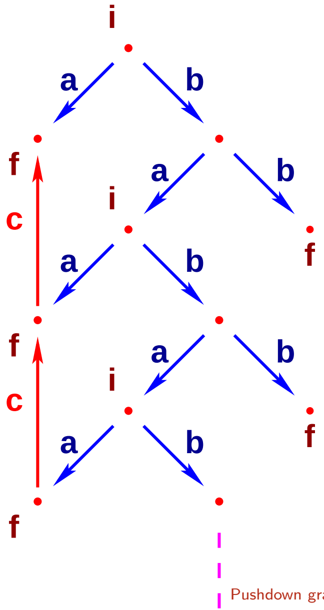


Inverse path function



Inverse path function

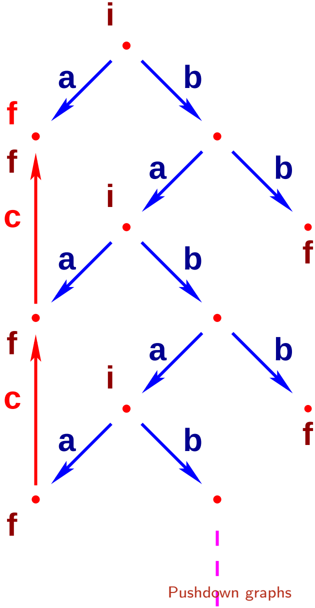
$c \longrightarrow a^{-1}a^{-1}b^{-1}a$



Inverse path function

$c \longrightarrow a^{-1}a^{-1}b^{-1}a$

$f \longrightarrow a^{-1}(i + \text{not}(a^{-1}a))a$



Inverse path function

c ⟶ $a^{-1}a^{-1}b^{-1}a$

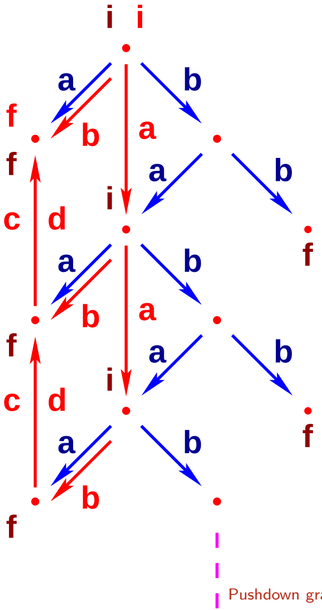
f ⟶ $a^{-1}(i + not(a^{-1}a))a$

a ⟶ $b\ a$

b ⟶ $a\ f$

d ⟶ $a^{-1}a^{-1}b^{-1}a$

i ⟶ $i + not(a^{-1}a)$



Inverse path function

$c \longrightarrow a^{-1}a^{-1}b^{-1}a$

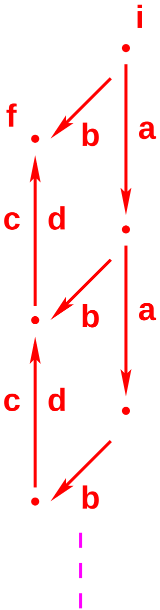
$f \longrightarrow a^{-1}(i + \mathrm{not}(a^{-1}a))a$

$a \longrightarrow b\ a$

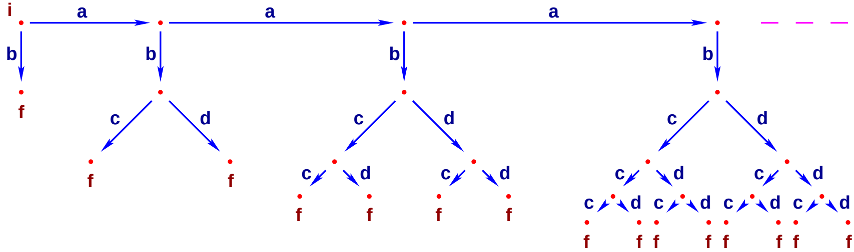
$b \longrightarrow a\ f$

$d \longrightarrow a^{-1}a^{-1}b^{-1}a$

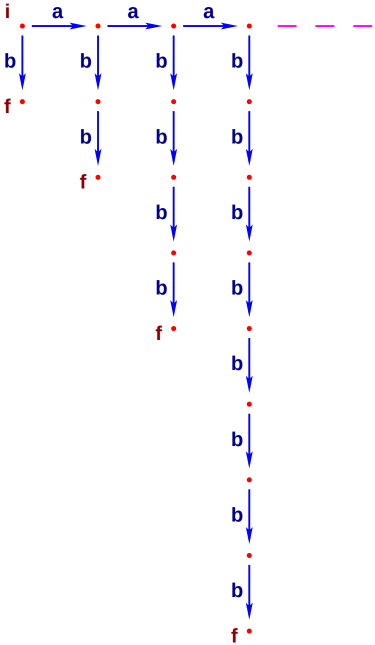
$i \longrightarrow i + \mathrm{not}(a^{-1}a)$



Unfolding



Inverse path function



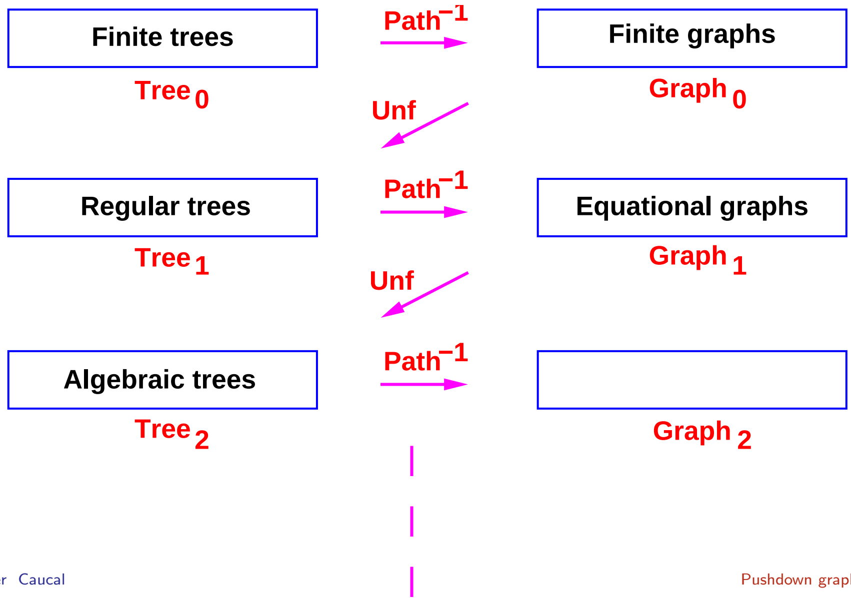
A hierarchy of graph families

$$\mathsf{Tree}_0 \quad = \quad \text{family of finite trees}$$

$$\mathsf{Graph}_n \quad = \quad \mathsf{Path}^{-1}(\mathsf{Tree}_n)$$

$$\mathsf{Tree}_{n+1} \quad = \quad \mathsf{Unf}(\mathsf{Graph}_n)$$

A hierarchy of graph families



VR-equational graphs Courcelle 1989

Same hierarchy

Path functions = regular substitutions

Exp = set of regular expressions

a^{-1} for $a \in L$; $u \cdot v$, u^+ , $u \vee v$ for $u,v \in Exp$

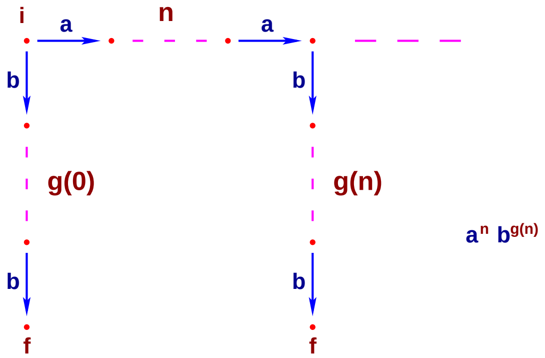
$Path^{-1}$ = monadic interpretations

Sub-hierarchy of finite degree graphs

Path functions = finite substitutions

$graph_1$ = HR-equational graphs Courcelle 89

Tree(g) for some integer mapping g



$$\text{Tree}(2^n) \in \text{Graph}_2$$

$$\text{Tree}(n!) \;,\; \text{Tree}(2^{2^n}) \in \text{Graph}_3$$

$$\text{Tree}(2\uparrow n) \notin \text{hierarchy}$$

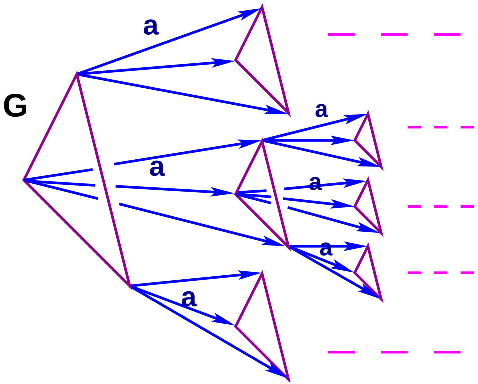
Generators

For each level n

find $gen_n \in Graph_n$ such that

$$Graph_n = Path^{-1}(gen_n)$$

Graph iteration Shelah, Stupp 1975

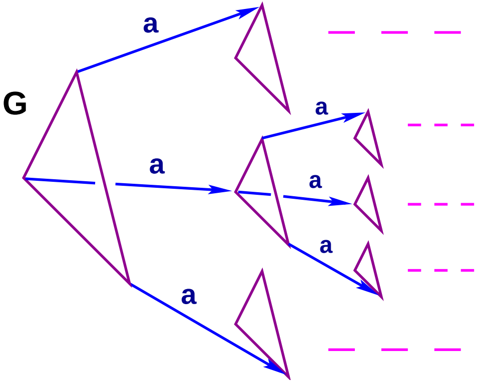


Proposition with Knapik 2011

The iteration operation preserves Graph_n

for each $n > 0$

Tree-graph



Proposition   Colcombet 04, Carayol, Wöhrle 03

$$\mathrm{TreeGraph}(\mathrm{Graph}_n) \subset \mathrm{Graph}_{n+1}$$

Theorem Muchnik 1984, Walukiewicz 1996

The tree-graph operation preserves the decidability of the monadic theory

Corollary Courcelle Walukiewicz 1998

The unfolding operation preserves the decidability of the monadic theory

Corollary

Any graph of the hierarchy has a decidable monadic theory

Generators

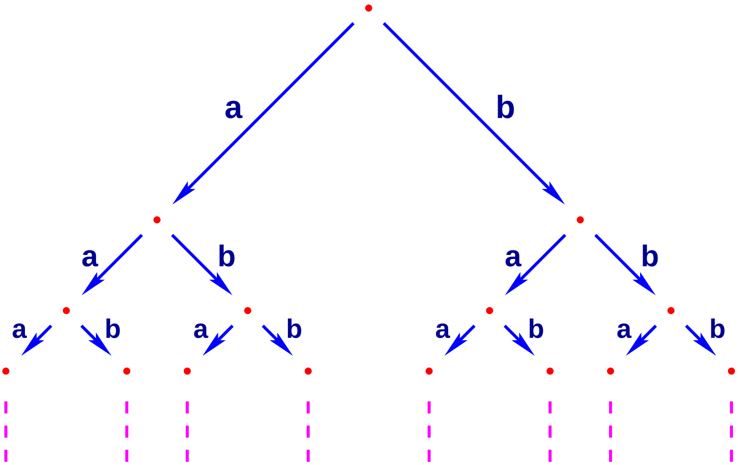
$$\mathrm{Graph}_{n+1}$$

$$= \quad \mathrm{Path}^{-1}(\mathrm{Unf}(\mathrm{Graph}_n \cap \mathrm{Det} \cap \mathrm{CoDet}))$$

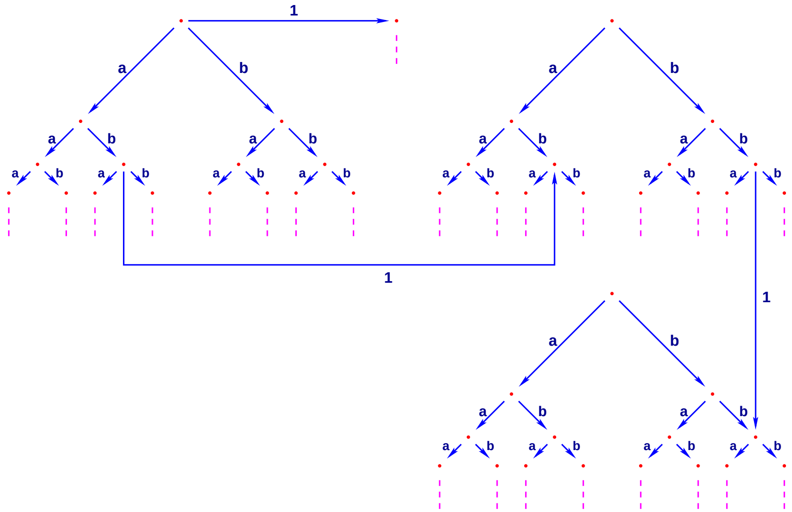
$$= \quad \mathrm{Path}^{-1}(\mathrm{gen}_{n+1})$$

$$\mathrm{gen}_{n+1} \in \mathrm{Tree}_{n+1} \cap \mathrm{Det}$$

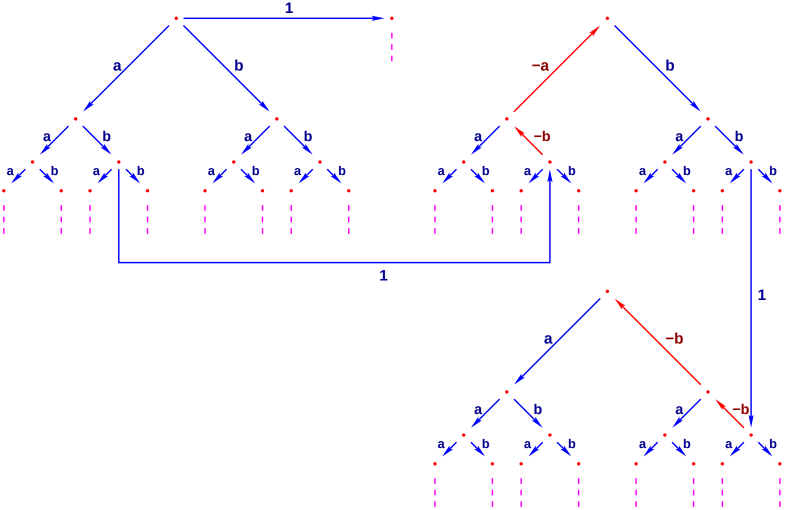
Tree generator at level 1



Graph generator at level 2



Tree generator at level 2



Higher order pushdown automata

Theorem Carayol 2006

$$\mathsf{Graph}_n \quad = \quad \varepsilon\text{-closure}((\mathsf{Pda}_n)_{|\mathsf{Reg}})$$

$$= \quad \bigcup_i W_i\,(U_i \xrightarrow{a_i} V_i)$$

pusdown hierarchy

Hierarchies of

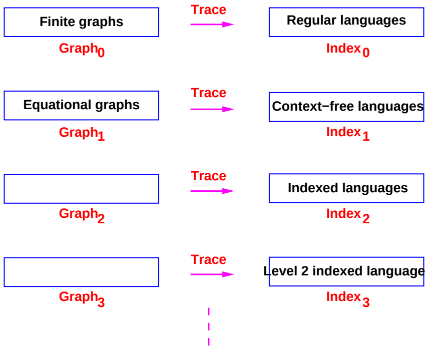
- languages

- ordinals

- infinite words

- terms

A hierarchy of languages



Indexed languages Aho 1968

The hierarchy of indexed languages Maslov 1974

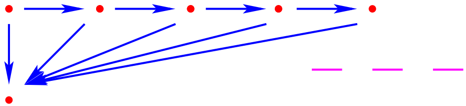
Proposition Maslov 1976

For each $n \geq 0$, *the language family* Index_n *is closed under*

- *intersection by any regular language*
- *inverse regular substitution*
- Index_n *substitution*

The hierarchy on ordinals

Ordinal $\omega + 1$: transitive closure of

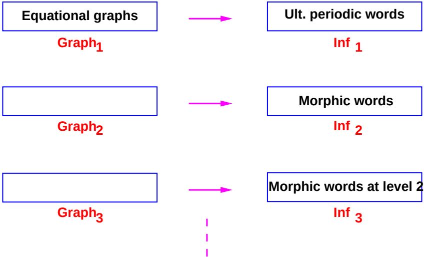


Theorem Braud 2009

For each n , *the ordinals in* $\mathrm{Graph}_{\mathsf{n}}$ *are the ordinals* $< \omega \uparrow (\mathsf{n}+1)$

Graph_ω ? ϵ_0 ? MSO ?

The hierarchy on infinite words



Level 2 morphic words: Champernowne number

$$0\ 1\ 10\ 11\ 100\ 101\ 110\ 111\ \ldots$$

The Liouville number:

$$\sum_{i \geq 1} 10^{-i!} = ,110001000000000000000001\ldots$$

Questions

- characterize morphic words at level 2
- for $\alpha = 0{,}u$ with $u \in \mathrm{Inf}_n$ and $n > 0$

  is α rational or transcendental number ?

  is $\sqrt{2}$ in the hierarchy ?

The hierarchy on terms

By first order substitutions

Theorem 2002

$$\mathsf{Tree}_{n+1} \cap \mathsf{Term} \;=\; \mathsf{Subst}_n$$

Courcelle, Knapik 2002

$$\mathsf{Subst}_0 \;=\; \text{regular terms}$$

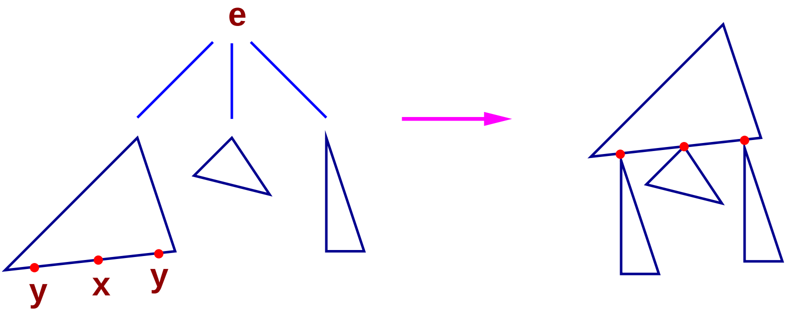
$$\mathsf{Subst}_{n+1} \;=\; \bigcup S_{e,u}\,(\mathsf{Subst}_n)$$

u word of distinct constants

e symbol of arity $|u|+1$

First order substitution (evaluation)

function e constant word x y



Finite terms

$$S_{e,u}(e \;\; t_0 \; t_1 \ldots \;\; t_n)$$

$$= \; S_{e,u}(t_0) \; [S_{e,u}(t_1 \;)/u(1),\ldots,S_{e,u}(t_n)/u(n)]$$

$$S_{e,u}(f \; t_1 \ldots \; t_m) \; = \; f \; S_{e,u}(t_1) \ldots S_{e,u}(t_m)$$

Infinite terms

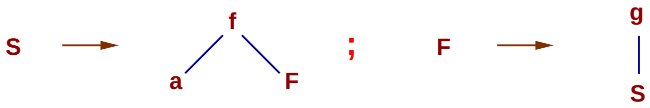
$$S_{e,u}(t) \; = \; \sup_n S_{e,u}({t_n}^{\Omega}) \quad \text{without} \;\; \Omega$$

The hierarchy on terms

By higher order schemes Damm 1982

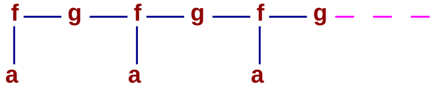
Scheme at level 0

R



Solution

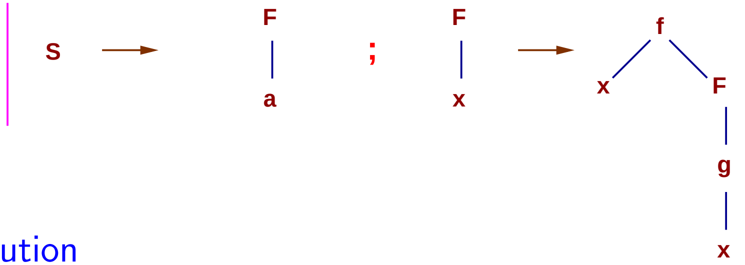
R^{ω}



Scheme at level 1

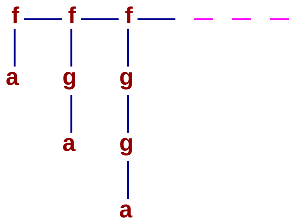
Nivat 1975, Guessarian 1987, Courcelle 1990

R



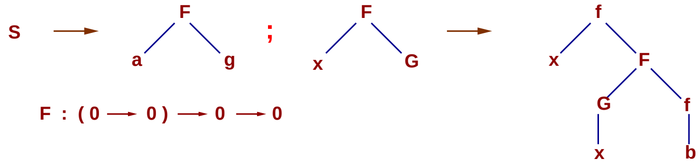
Solution

R^ω

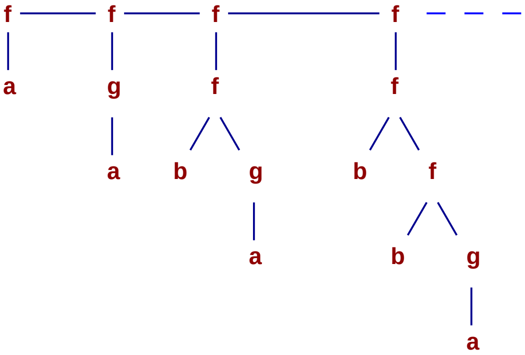


Scheme at level 2

R

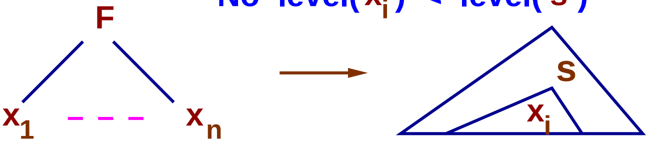


Solution R^ω

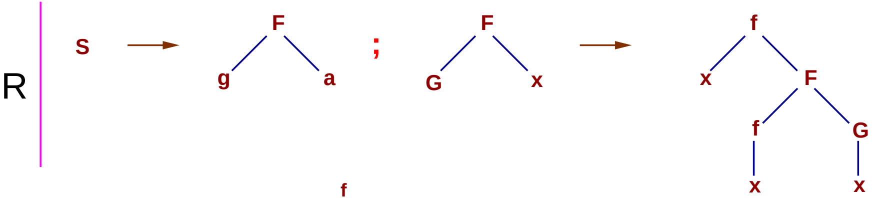


Safety condition: scope of variables Damm 1982

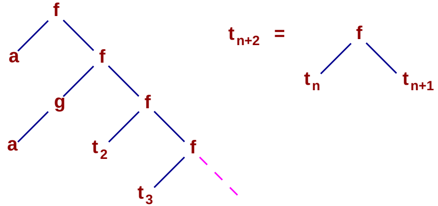
No level(x_i) < level(s)



Unsafe scheme



Solution R^{ω}



Theorem 2002

$$\mathrm{Tree}_{n+1} \cap \mathrm{Term} = \mathrm{SafeScheme}_n$$

Knapik, Niwiński, Urzyczyn 2002

Any safe scheme R at level n+1 is transformed

into a safe scheme S at level n such that

$R^\omega = \mathrm{Unf}(h^{-1}(S^\omega),r)$ for some path function h

Conjecture

The non rational algebraic numbers

are not in the pushdown hierarchy