Assignment 2

Due Date: 16 October 2023

Try to solve all the problems. Be precise, concise, and legible in your answers; a good idea may be to submit a pdf file with your answers. Discussion is encouraged, and so is its acknowledgement. Please cite your sources, if you are referring to one. In any case, *write the solutions in your own words to express your understanding; copying of solutions will not be accepted.*

1. Let $\widetilde{D}_n$ denote the number of permutations of $[n]$ that have cycles of length three or more. Derive the egf for $\widetilde{D}_n$ and use it to get an asymptotic estimate.
2. An involution is a permutation π that is its own inverse, i.e., π^2 is identity. Let P_n be the number of involutions of $[n]$. Derive the egf for P_n and an asymptotic estimate using the saddle point method. Please give the details of the three steps (Tails pruning, central approximation and tails completion).
3. Use the saddle point method to derive an asymptotic estimate for $\binom{2n}{n}$. Please give the details of the three steps.
4. When the determinant of a symmetric $n \times n$ matrix is evaluated, let $M(n)$ denote the number of distinct monomials which appear. For instance, $M(2) = 2$, $M(3) = 5$.
   (a) What is the egf for $M(n)$?
   (b) Given an asymptotic estimate for $M(n)$?
5. Use generating functions to prove the following identities.
   (a) $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.
   (b)
   $$n\binom{n+m-1}{m-1} = m\binom{n+m-1}{m}.$$
   (c)
   $$\sum_{k=0}^{n}\binom{m+k}{k} = \binom{m+n+1}{n}.$$
   (d)
   $$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n.$$
6. Let $\left[{n \atop k}\right]$ be the number of permutations of $[n]$ with *exactly* k cycles. Derive the ogf and egf for the sequence $\left[{n \atop k}\right]$. Using them, derive asymptotic estimates for $\left[{n \atop k}\right]$.
7. Show that if the roots of the polynomial $f(x) = \sum_i a_i x^i$, where $a_i \in \mathbb{R}$, $a_n > 0$ and $a_0 \neq 0$, have their arguments in the range $[2\pi/3, 4\pi/3]$ then the coefficient sequence $a_0, \ldots, a_n$ is log-concave.