Homework 1

Due Date: 18 September 2023

Try to solve all the problems. Be precise, concise, and legible in your answers; a good idea may be to submit a pdf file with your answers. Discussion is encouraged, and so is its acknowledgement. Please cite your sources, if you are referring to one. In any case, *write the solutions in your own words to express your understanding; copying of solutions will not be accepted.*

1. For $n > 0$, show that the set $\{1, \dots, 2n\}$ can always be partitioned into n pairs (a_i, b_i) such that $a_i + b_i$ is a prime.

2. Show that all natural numbers greater than six can be expressed as sum of *distinct* primes.

3. Show that for every integer $m \geq 3$, there exists infinitely many primes $p \not\equiv 1 \mod m$.

4. Let $\pi(n)$ be the number of primes not exceeding n. Use ideas from Erdös's proof to show that $\pi(n) = \Theta(n/\log n)$.

5. Let R_n be the smallest number such that for all $x \geq R_n$, $\pi(x) - \pi(x/2) \geq n$. You may use the asymptotic property of $\pi(n)$ mentioned above in the following.
   (a) What is R_1?
   (b) Show that R_n exists. Is it a prime? Justify your arguments.
   (c) Let p_n be the nth prime. Show that $p_{2n} < R_n < p_{4n}$.

6. Show that if a simple graph G on n vertices has all vertices of degree at least $n/2$ then it contains a Hamiltonian cycle.

7. Recall that $\chi(G)$ is the chromatic number of G. Prove that G must have at least $\binom{\chi(G)}{2}$ edges.

8. Prove or disprove: For any two graphs G_1, G_2, $\chi(G_1 \cup G_2) \leq \chi(G_1) \cdot \chi(G_2)$.

9. Recall that $R(n_1, n_2, \dots, n_k)$ is the least number N such that on any edge coloring of the complete graph K_N using k colors there is a monochromatic K_{n_i}, for some $i \in \{1, \dots, k\}$. Prove the generalized Ramsey theorem: for all $n_1, n_2, \dots, n_k \in \mathbb{N}$, $R(n_1, n_2, \dots, n_k) < \infty$.

10. Prove that $R(3, 4) \leq 9$.

11. Given m real numbers $\alpha_1, \dots, \alpha_m$ and $N \in \mathbb{N}$ show that there exists an integer $1 \leq q \leq N^m$ and integers $p_1, \dots, p_m$ such that $\left|\alpha_i - \frac{p_i}{q}\right| < 1/(qN)$.

12. Use Louville's Theorem to prove that $\alpha = \sum_{k=1}^{\infty} 10^{-k!}$ is transcendental.

13. Show that in any poset P of at least $(sr + 1)$ elements there exists a *chain* of length $(s + 1)$ or an *anti-chain* of length $(r + 1)$.