Assignment 4

Due Date: 7th December 2012

Please be precise, concise, and legible in your answers; a good idea may be to submit a pdf file containing your answers. Discussion is encouraged, and so is its acknowledgement. If you are using a reference then cite it in your solution. *Please write your own solutions; copying of solutions from any source will not be accepted.*

1. Let $p = 4k + 1$ be a prime. Use pigeonhole principle to show that p is a sum of two squares, i.e., $p = x^2 + y^2$. Hint: Use the fact that there is an $s \in \{1, \dots, p\}$ such that $s^2 + 1 \equiv 0 \mod p$. Consider pairs $(a, b) \in \left\{0, \dots, \lfloor\sqrt{p}\rfloor\right\}^2$.

2. Give a probabilistic method proof of $R(\ell, \ell) > 2^{(\ell-2)/2}$.

3. Either give a direct proof of $\Pr(\chi(G) \le k) < 1/2$ using Markov's inequality, similar to what we did for the girth, or show that this approach does not work. In the latter case, can we make it work for some values of k?

4. Given a graph G with m edges and n vertices, let $B = (b_{ij})_{n\times m}$ be the matrix
$$b_{ij} := \begin{cases} 1 & \text{if } v_i \in e_j \\ 0 & \text{otherwise.} \end{cases}$$
Show the following:
   (a) The kernel of B is the cycle space of G.
   (b) The image of B^t is the cut space of G.
   (c) Let A be the adjacency matrix and D the diagonal matrix with the degree sequence of the vertices on the diagonal. Then $BB^t = A + D$.

5. Let $G = (V, E)$ be an undirected graph and suppose with each vertex v we associate a set $S(v)$ of $8r$ colors. Further suppose that any color in $S(v)$ is shared with at most r neighbors of v. Show that there is a proper coloring of G that colors each vertex v from within the set $S(v)$.

6. A formula for the latin rectangle $L(3, n)$ in terms of the ménage numbers. Let $M_n(x)$ be the ménage polynomial for the $n \times n$ ménage restriction set (crosses on the diagonals and the coordinates $(i, i+1)$ mod n). Define the hit polynomial $E_n(x) := M_n((x-1)\Delta^{-1}(n))$, where $\Delta^{-j}(n) := (n-j)!$. The nth menage number $e_n := E_n(0)$. Show the following.
   (a) Let π be a permutation that consists of a single cycle of length n (e.g., $(1 \to 4 \to 3 \to 2 \to 1)$, or succinctly (1432)). Let B be the $n \times n$ board with the restriction set (i, i) and $(i, \pi(i))$. Show that the rook polynomial corresponding to the board B is $M_n(x)$.
   (b) Now consider a permutation σ that has k_2 cycles of length 2, k_3 cycles of length 3, and so on k_n cycles of length n (therefore, $\sum_{i=2}^n ik_i = n$), i.e., has cycle structure $(k_2, \dots, k_n)$. Consider the $n \times n$ board with the restriction set (i, i) and $(i, \sigma(i))$. What is the rook polynomial for this board in terms of the ménage polynomials? What is the hit polynomial for this board?
   (c) Let $C(\mathbf{k})$ be the number of permutations with cycle structure $\mathbf{k} := (k_2, \dots, k_n)$. Show that $L(3, n) = \sum_{\mathbf{k}} C(\mathbf{k}) e_2^{k_2} \dots e_n^{k_n}$.