Assignment 3

Due Date: 14th November 2012

Please be precise, concise, and legible in your answers; a good idea may be to submit a pdf file with your answers. Discussion is encouraged, and so is its acknowledgement. If you are using a reference then cite it in your solution. *Please write your own solutions; copying of solutions will not be accepted.*

1. Let L be a lattice and $\{A, B, C\}$ a partition of L such that

   (a) if $x \in A$ and $y \leq x$ then $y \in A$, and

   (b) if $x \in C$ and $x \leq y$ then $y \in C$.

   Show that
$$1 + \sum_{x \in A} \sum_{y \in C} \mu(x, y) = \sum_{x,y \in B} \mu(x, y).$$

   Hints: Show that $\sum_{x,y \in L} \mu(x, y) = 1$.

2. Consider two partitions $\mathbf{A} := \{A_1, \ldots, A_k\}$ and $\mathbf{B} := \{B_1, \ldots, B_\ell\}$ of the set $[n]$. We say that
$$\mathbf{A} \leq \mathbf{B}$$
   if each A_i is contained in some B_j.

   (a) Show that the set of all partitions of $[n]$ forms a lattice under the above ordering.

   (b) What is $\mathbf{0}$ and $\mathbf{1}$ in this lattice?

   (c) Show that $\mu(\mathbf{0}, \mathbf{1}) = (-1)^{n-1}(n-1)!$.

   Hints: For third part, first show that if $\mathbf{B} := \{\{1\}, \{2, \ldots, n\}\}$ then $\sum_{\mathbf{C} \wedge \mathbf{B} = \mathbf{0}} \mu(\mathbf{C}, \mathbf{1}) = \mathbf{0}$. How do the partitions $\mathbf{C}$, such that $\mathbf{C} \wedge \mathbf{B} = \mathbf{0}$, look like? Use induction.

3. Let P be a poset and for $x, y \in P$, let n_k be the number of chains $a_0 = x < a_1 < \cdots < a_k = y$. Derive a formula for $\mu(x, y)$ in terms of n_ks.

4. Let $v_1, \ldots, v_n$ be n vertices. Consider a tree T on these vertices, and let $d_i(T)$ denote the degree of v_i in T. Show that the following polynomial
$$p(x_1, \ldots, x_n) := \sum_{\text{all trees } T \text{ on } v_1, \ldots, v_n} x_1^{d_1(T)-1} \ldots x_n^{d_n(T)-1}$$
   is equal to $(x_1 + \cdots + x_n)^{n-2}$. Derive Cayley's formula from this observation.

5. Let T_n be the number of trees on $v_1, \ldots, v_n$.

   (a) Derive a recurrence for T_n.

   (b) Use the recurrence to derive Cayley's formula.

6. A **permutation matrix** is an $n \times n$ matrix where there is exactly one 1 in each row and column. Let M be an $n \times n$ matrix with the following properties:

   (a) all entries are non-negative real numbers, and

(b) the entries in any row and column add up to 1.

Using Hall's theorem, show that

$$M = \sum_{i=1}^{k} \alpha_i P_i,$$

where $\alpha_i \geq 0$, $\sum_i \alpha_i = 1$, and P_is are permutation matrices.

Hint: Construct a bipartite graph (A, B), where A is the row set and B the column set of M. When is there an edge between A and B? What does a matching represent on this graph?

7. Let G be a 3-connected graph.
   (a) Show that every degree three vertex in a 3-connected graph has a contractible edge.
   (b) Let T be a spanning tree for G with no contractible edge. Give arguments for the following:
      i. For every edge uv of T there is a vertex w such that $\{u, v, w\}$ forms a vertex cut in G. Call such a vertex cut as T-separating triple and the resulting components as T-split components.
      ii. A **minimal** T-split component is a set that does not contain another T-split component in it. Show that a minimal T-split component is a single vertex v with degree 3, and is incident to exactly one edge in T.
      iii. Show that v has a neighbor v' that is also a minimal T-split component.
   (c) Does a DFS-tree for G contain a contractible edge?

8. Given a graph $G = (V, E)$, let $v, w \in V$ be two non-adjacent vertices in G, i.e., $vw \notin E$. Let $\mathcal{L}$ be the set of all sets S of minimum cardinality k that separate v and w. Given $S \subseteq V \setminus \{v\}$, let S_v be the set of vertices $z \in S$ such that v and z are connected in the set $G - (S \setminus \{z\})$, i.e., there is a path between v and z that does not go through S. For $S, T \in \mathcal{L}$, define the following operations on $\mathcal{L}$:

$$S \wedge T := (S \cup T)_v \text{ and } S \vee T := (S \cup T)_w.$$

   (a) Are $S \wedge T$ and $S \vee T$ in $\mathcal{L}$?
   (b) Show that $\mathcal{L}$ is a lattice.
   (c) How should we pick an extremal graph G' that has the same $v - w$ vertex connectivity as G?
   (d) Let z be a vertex on a $v - w$ path in G'. Let $x_1, \ldots, x_m$ be neighbors of z on some $v - z$ path. Let K_i be a vertex cut separating v and w, if we remove the edge $x_i z$. Define $S_i := K_i \cup \{x_i\}$ and $T_i := K_i \cup \{z\}$. Show the following.
      i. What is the size of K_is?
      ii. Is $S_i \cup T_i$ in $\mathcal{L}$? If yes, then how are they related in $\mathcal{L}$?
      iii. Define $S := S_1 \vee S_2 \vee \cdots \vee S_m$ and $T := T_1 \vee T_2 \vee \cdots \vee T_m$. Show that $|T - S| = 1$.
      iv. What does $|T - S| = 1$ imply about the degree of z?
      v. From above infer that there exists k independent paths from v to w.

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