The Twelvefold Way

The Twelvefold Way[1] is a uniform approach to classify many standard combinatorial problems, such as counting permutations, equivalence classes, surjective functions, partitions, compositions etc. Throughout this write up we assume $f : X \to Y$, and $|X| = m$ and $|Y| = n$; x_i's will denote elements of X and y_i's elements of Y.

The first combinatorial question we can ask is, How many functions are there from X to Y? The number is n^m, since for each $x \in X$ there are m possiblities for to map it in Y. An application of this result is to count the number of subsets of X: each subset S of X cooresponds to a functions $\chi_S : X \to \{0, 1\}$, where χ_S is the characteristic function corresponding to S; the result then says that there are 2^m such functions, and hence subsets.

How many *injective or one-to-one* functions are there from X to Y? The element x_1 has n choices; for each such choice of x_1, the element x_2 has $n-1$; for each such choices of x_1 and x_2 the element x_3 has $n-2$ choices. Continuing in this manner we see that the number of injective functions from X to Y are $n(n-1)(n-2)\dots(n-m+1)$. If $|X| = |Y|$ then this gives us the number of permutations of the set X.

How many *surjective or onto* functions are there from X to Y? The argument is a bit involved. A surjective function f should have the property that the n inverse sets $f^{-1}(y_1), f^{-1}(y_2), \dots, f^{-1}(y_n)$ are all non-empty, or in other words, they partition X. In how many ways can we partition X? We know it is $\left\{ {m \atop n} \right\}$. Now the function f acts as an injective map from a partition of X to Y; thus from the second result above, each partition can be mapped in $n!$ ways to Y. So the number of injective maps from X to Y is $\left\{ {m \atop n} \right\} n!$.

An implicit assumption in our definition of the sets X and Y is that each element of X is different. In a "bins-and-balls" interpretation of f (where $f(x_i) = y_j$ means that the ball x_i is placed in the bin y_j) it means that our balls are all of different colours/labels and so are the bins. If, however, our balls were of the same colour/label then our answers above do not apply since permutations within balls in the same bin does not matter. Give instances where f is the same when say X is indistinguishable, Y is indistinguishable, and both X and Y are indistinguishable. We will enumerate the total number of functions, injective functions and surjetive functions for these three cases. These nine enumeration problems, along with the three that we have already mentioned form the classification called the Twelvefold Way. Before we proceed with the enumeration, we will formalize the notion of indistinguishability, and and what it means to count functions over indistinguishable sets. Figure 1 illustrates functions that are different when either X or Y or both are indistinguishable.

Given an indistinguishable set X, we say that two functions f, g from X to Y are equivalent if there exists a permutation π of X such that for all $x \in X$, $f(\pi(x)) = g(x)$. We claim that this is an equivalence relation (reflexive is the identity permutation, symmetric is the inverse permutation, and transitivity is by composition). Thus the set of all functions from X to Y is partitioned into equivalence classes. Similarly, we can define equivalence between functions f, g when only Y is indist. as there exists a permutation σ such that for all $x \in X$, $\sigma(f(x)) = g(x)$; when both X and Y are indist. as there exists permutations $\pi : X \to X$, and $\sigma : Y \to Y$ such that for all $x \in X$ $\sigma(f(\pi(x)) = g(x)$. It can be verified that these relations are indeed equivalence relations.

To count functions that are injective or surjective when say X is indistinguishable, it would suffice to count the number of equivalence classes if we can show that the notions of injectiveness and surjectiveness remain invariant over an equivalence class. More precisely, we have to show that if f and g are equivalent then f is injective iff g is injective; similarly for surjectiveness.

¶1. X Indistinguishable Total: Number of non-negative solutions to $i_1 + i_2 + \cdots + i_n = m$, which is the coefficient of m in $(1-x)^{-n}$, i.e., $\binom{-n}{m}$.

Injective: $\binom{n}{m}$.

Surjective: Number of positive solutions to $i_1 + i_2 + \cdots + i_n = m$, or equivalently non-negative solutions to $i_1 + i_2 + \cdots + i_n = m - n$, which is $\binom{-n}{m-n}$.

[1]The classification was introduced by Rota; the terminology by Joel Spencer as a reference to the EIghtfold Way of Buddhism'

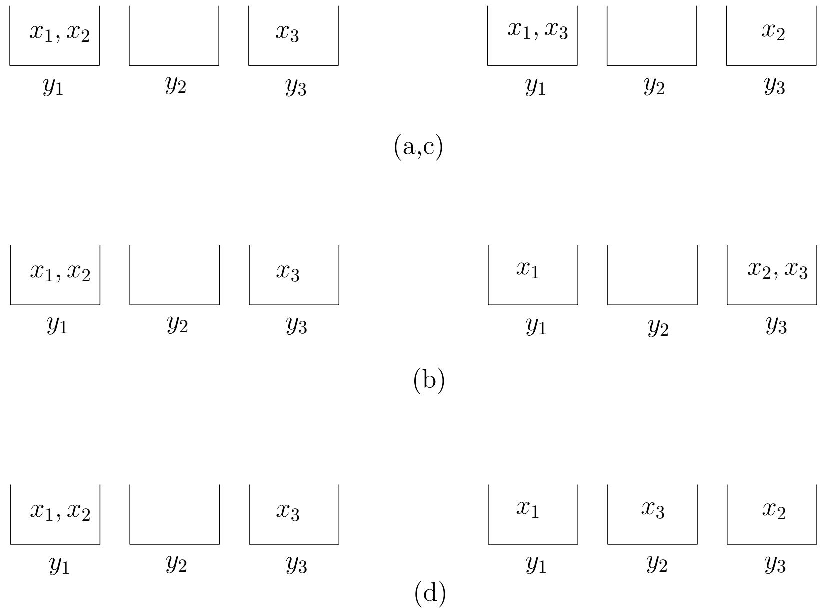

Figure 1: (a) Two functions different when X and Y are distinguishable, (b) Two functions different when X is indistinguishable, (c) Two functions different when Y is indistinguishable, and (d) Two functions different when both X and Y are indistinguishable.

¶2. Y **Indistinguishable** Total: Every function $f : X \to Y$ corresponds to a partition of X. The number of functions that partition X into k boxes is $\left\{ {m \atop k} \right\}$. Thus total functions $B_m = \sum_{k=1}^{m} \left\{ {m \atop k} \right\}$.

Injective: 1.

Surjective: $\left\{ {m \atop n} \right\}$.

¶3. X **and** Y **Indistinguishable** Since neither the permutations of the balls, neither of the boxes matters, what matters is the number of ways to write m as sum of n, or the number of non-negative solutions to $i_1 + i_2 + \cdots + i_n = m$, where the ordering of the solutions does not matter. We cheat here and introduce the solution as a definition, $p(m, k)$, as the number of partitions of m into *exactly* k non-zero parts. Then the answer to our question is $\sum_{k=1}^{n} p(m, k)$.

Injective: 1.

Surjective: $p(m, n)$.

	X and Y distinguishable	X indistinguishable	Y indistinguishable	X and Y indist.
$f : X \to Y$	n^m	$\binom{-n}{m-n}$	B_n	$\sum_{k=1}^{n} p(m, k)$
Injective	$(n)_m = n(n-1)\ldots(n-m+1)$	$\binom{n}{m}$	1/0	1/0
Surjective	$\left\{ {m \atop n} \right\} n!$	$\binom{-n}{m-n}$	$\left\{ {m \atop n} \right\}$	$p(m, n)$