Partition of Numbers

1 Introduction

Given a number n, let $p(n,k)$ be the number of ways of expressing n as $\lambda_1 + \lambda_2 + \cdots + \lambda_k$, where λ_i's are positive numbers, and the order of the summands does not matter. The nth partition number $p(n)$ is the total number of partitions of n, i.e., $\sum_{k=1}^{n} p(n,k)$. It appears frequently in number theory, combinatorics, representation theory and other fields, and has been the subject of study by many celebrated mathematicians. Some of Ramanujan's best work is in partition of numbers. For instance, with Hardy he had showed that

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}. \tag{1}$$

And also the various conguence relations, such as $p(5k+4) \equiv 0 \mod 5$, $p(7k+5) \equiv 0 \mod 7$. Despite Rademacher's convergent series expression for $p(n)$, finding a finite formula for $p(n)$ has been one of the challenges in number theory. Recently, Ono and Bruinier have claimed such a formula.

Here we study some interesting elementary properties of $p(n)$, compute bounds on it, and describe algorithms to compute it.

We first get a generating function for $p(n)$. Before we do that, we introduce a representation for a partition of n: $1^{a_1} 2^{a_2} \dots n^{a_n}$ is a partition of n, where i^{a_i} denotes that i appears a_i times in the partition. Thus $\sum_i i a_i = n$. Taking the two terms on both sides of the equation as exponents to x we get

$$x^{a_1} x^{2a_2} x^{3a_3} \dots x^{na_n} = x^n.$$

From this we infer that

$$\sum_{n \geq 0} p(n) = \prod_{i \geq 1} \frac{1}{(1 - x^i)}.$$

The coefficient of x^n are precisely the terms in the LHS of the equation above.

¶1. Reprensetation: Ferrer's or Young's Diagram A more graphical description of the partition $(\lambda_1, \dots, \lambda_k) \vdash n$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ is a matrix representation where the ith row either contain λ_i dots or squares abutting each other; e.g., $(3,3,2,1) \vdash 9$ can be represented as shown in Figure 1.

The **dual**, λ^*, of a partition $\lambda := (\lambda_1, \dots, \lambda_k)$ is obtained by taking the "transpose" of the corresponding Ferrer's diagram. More precisely, the ith entry of λ^* is the number of λ_j's greater than or equal to i. Thus the dual of $(3,3,2,1)$ is $(4,3,2)$ as shown in Figure 1.

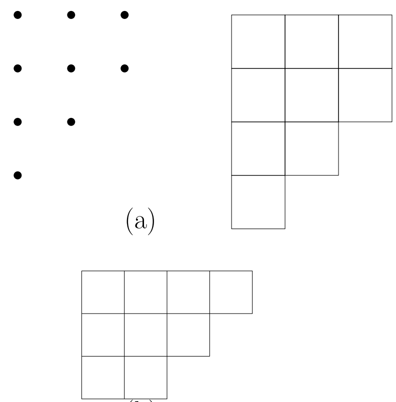
2 Bound on the nth partition number

We can derive an upper bound on $p(n)$ very close to what Hardy-Ramanujan gave in (1) using elementary means. We will show that $p(n) \leq e^{\pi\sqrt{2n/3}}$. From the generating function of $p(n)$, it follows that for all $x \in (0,1)$,

$$p(n) \leq \frac{1}{x^n} \prod_{k \geq 1} \frac{1}{1 - x^k}.$$

Taking logarithm on both sides we obtain

$$\ln p(n) \leq -n \ln x - \sum_{k \geq 1} \ln(1 - x^k).$$



(a) (b)

Figure 1: (a) Ferrer's diagram showing the partition $(3, 3, 2, 1)$ of 9, and (b) its dual

Using the power series

$$-\ln(1-y) = y + \frac{y^2}{2} + \frac{y^3}{3} + \cdots$$

which is well defined when $|y| < 1$, we obtain

$$\ln p(n) \le -n \ln x \sum_{k\ge1}\sum_{j\ge1} \frac{x^j k}{j} = -n\ln x \sum_{j\ge1}\frac{1}{j}\sum_{k\ge1} x^j k = -n \ln x \sum_{j\ge1} \frac{1}{j}\frac{x^j}{1-x^j}.$$

We can show that

$$\sum_{j\ge1}\frac{1}{j}\frac{x^j}{1-x^j} \le \frac{x}{1-x}\sum_{j\ge1} j^{-2} = \frac{x}{1-x}\frac{\pi^2}{6}.$$

Thus

$$\ln p(n) \le -n\ln x + \frac{x}{1-x}\frac{\pi^2}{6}.$$

Find the value of $x \in (0,1)$ that minimizes the LHS.

3 Computing the nth Partition Number

We know that $\sum_{n\ge0} p(n)x^n = \prod_{k\ge1} 1/(1-x^k)$. From this relation there is a straightforward algorithm to compute $p(n)$, namely compute the coefficient of x^n on the RHS. We don't have to consider all the terms in the expansion, since the terms $k > n$ do not contribute to $p(n)$. So we have to find the coefficient of x^n in the product

$$(1 + x + x^2 + \cdots + x^n)(1 + x^2 + x^4 + \cdots + x^{2\lfloor n/2\rfloor})\ldots(1 + x^n).$$

Let f_i be the ith term in this product. To compute the coefficient of x^n we do not want to compute the complete product $f_i f_j$, as terms greater than x^n do not matter. So what we want instead is to compute the product $f_i f_j \mod x^{n+1}$. Let $g_0 := f_1$, $g_1 := g_0 f_2 \mod x^{n+1}$, $g_2 := g_1 f_3 \mod x^{n+1}$ and so on $g_{n-1} := g_{n-2} f_n \mod x^{n+1}$. What is the complexity? Each multiplication involves $O(n^2)$ operations and there are $O(n)$ many such steps. Thus the complexity is $O(n^3)$, or $O(n^2 \log n)$ if we use FFT, or even by a more careful analysis. Can we improve this result? The answer is yes! A natural way to compute $p(n)$ is to obtain a nice recursion. Can we obtain such a nice recursive formula? Yes, and here we see another beautiful application of generating functions. Recall that an inverse of a power series $\{a_n\}$ is a series $\{b_n\}$ such that

$$\sum_{k\ge0} a_k b_{n-k} = 0 \tag{2}$$

and it exists iff $a_0 \neq 0$. The equation above implies that $a_n = \sum_{k\geq 1} a_k b_{n-k}$, which gives a natural recurrence for a_n in terms of a_k, $k = 1, \ldots, n-1$ if we can get hold of the coefficients of the inverse. The inverse of the GF corresponding to $\{p(n)\}$ is well-defined as $p(0) = 1$. Moreover, and herein lies the insight, it is simply

$$\prod_{k\geq 1}(1-x^k). \tag{3}$$

We know that in the $\prod_{k\geq 1}(1+x^k)$ the coefficient of x^n is the number of partitions of n into distinct parts. The product in (3) is different. Let us expand the product a little to see the coefficients appearing in it

$$\prod_{k\geq 1}(1-x^k) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \cdots .$$

There is no apparent pattern in the exponent of x. Nevertheless the signs seem to come in pairs, so let's group the terms according to this pairing

$$\prod_{k\geq 1}(1-x^k) = 1 - (x + x^2) + (x^5 + x^7) - (x^{12} + x^{15}) + (x^{22} + x^{26}) - \cdots$$

and possibly try to get a precise form for the exponents. Let $f(j)$ be the smaller and $s(j)$ the larger exponent in the jth pair, $j \geq 1$; thus $f(1) = 1, s(1) = 2, f(2) = 5, s(2) = 7\ldots$. From the exponents in the expansion above, we can conjecture that $s(j) = f(j) + j$. More precisely, Euler showed that

THEOREM 1 (Euler's Pentagonal Theorem). *Let $f(j) := j(3j-1)/2$, $j \geq 1$. Then*

$$\prod_{k\geq 1}(1-x^k) = \sum_{k\in\mathbb{Z}}(-1)^k x^{f(k)}.$$

Note that $s(j) = f(-j) = j(3j+1)/2 = f(j) + j$, as conjectured. The numbers $f(j)$ are called **pentagonal numbers**, since they are obtained as the number of points in a sequence of subsuming pentagoins as shown in Figure 2; the jth pentagon is obtained from the $(j-1)$th pentagon by "incrementing" each side by one, where the points are equispaced along each edge.

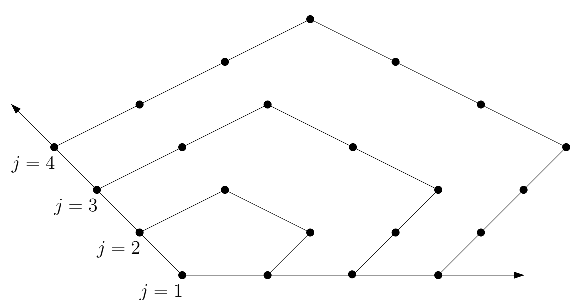


Figure 2: Pentagonal Numbers: $f(j) = j(3j-1)/2$ counts the number of points in the interior and on the boundary of the jth pentagon

The proof proceeds by an intermediate step. Let $E(n)$ be the number of partitions of n into even number of distinct parts, and $O(n)$ the number of partitions of n into odd number of distinct parts. Then we will show that both series in the equation in Theorem 1 are equal to

$$\prod_{n\geq 1}(E(n) - O(n))x^n. \tag{4}$$

We start with the easy step:

$$\prod_{k\geq 1}(1-x^k) = \prod_{n\geq 1}(E(n) - O(n))x^n. \tag{5}$$

Consider the coefficient of x^n on the LHS. It is formed as the sum of coefficients of the form $(-1)^j x^{i_1+i_2+\cdots+i_j}$, where $i_1 + i_2 + \cdots + i_j = n$ and $i_1, \ldots, i_j$ are all distinct. Each solution to this equation forms a partiton of n and contributes a factor $(-1)^j$ to the coefficient of x^n. If j is even then the contribution from the solution is 1, i.e., each partition of n that has an even number of distinct parts contributes a 1; similarly, if j is odd then each partition of n that has an odd number of distinct parts contributes a -1. Thus the coefficient of x^n on the LHS is $E(n) - O(n)$.

To show the second claim, we have to show that

$$E(n) - O(n) = \begin{cases} 1 & \text{if } n = f(\pm j),\ j \text{ even;} \\ -1 & \text{if } n = f(\pm j),\ j \text{ odd;} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

We present a combinatorial proof by Franklin from 1881: if n is not a pentagonal number, then we will give a bijection between partitions of n containing even and odd number of distinct parts; the bijection fails when n is a pentagonal number, and in these cases there is exactly one unmatched partition.

Given a partition of n into distinct parts, we want to form a new partition of n that maintains distinctness but changes the parity. The most natural step is to take the dots in the last row and distribute it along the remaining rows by adding one dot to each of the rows. However, as shown in Figure 3, the map as described is not bijective and doesn't always work. To overcome these issues, we introduce some definitions.

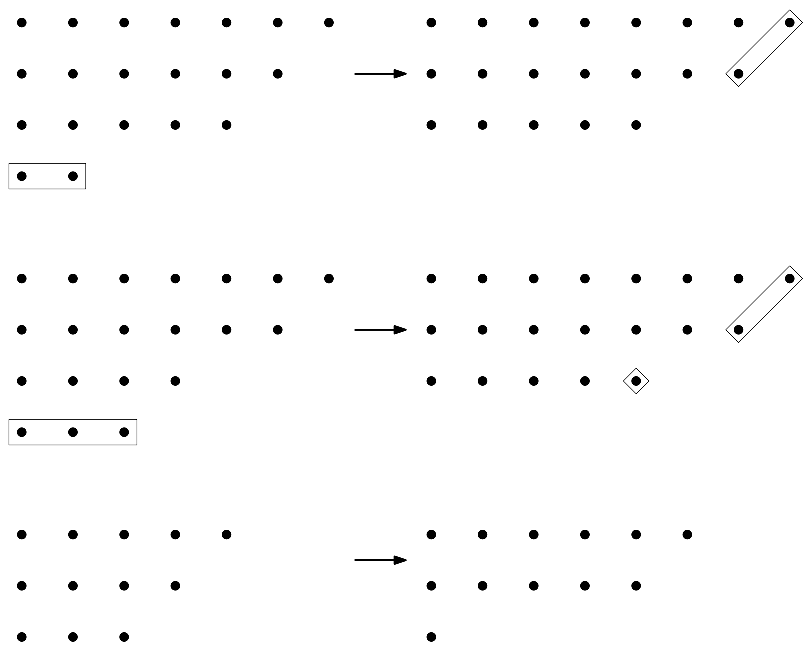


Figure 3: Transforming a partition to change parity and maintain distinctness

Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$. Define **base**, b, of λ as the *number of dots* in λ_k, and **slope**, s, of λ as the number of dots in the rightmost diagonal of the Ferrers diagram starting from the last dot in λ_1; see Figure 4 for an illustration. The base and the slope may or may not intersect; in Figure 4 (b), (d) they do not intersect, whereas in (c), (e) they do.

If a partition λ is such that $b \le s$ then we "move the base up", i.e., move the dots in the base to the topmost b rows. This results in a new partition λ' whose parity is different from λ, whose rows are distinct, and whose slope b is *smaller* than its base, which is at least $b+1$; e.g., see Figure 5(a), (b). Conversely, if λ is such that $b > s$ then we "move the slope down", i.e., remove the dots in s to form a new row with s dots. This results in a partition λ' whose base s is not larger than its slope, which is at least $s+1$, and the rows are distinct except in certain cases; e.g., see Figure 5(c),(d)

However, both the transitions do not work under certain circumstances. The first does not work when $b = s$ *and the base and slope intersect*, because moving the base up does not decrease the parity as desired; note when the the base and slope do not intersect then it works perfectly, as illustrated in Figure 5(b). The converse does not work when $b = s+1$ *and the base and slope intersect*, because moving the slope down results in two partitions of the same size s; again when the base and slope do not intersect, then the transition works as shown in Figure 5(d). Thus barring these

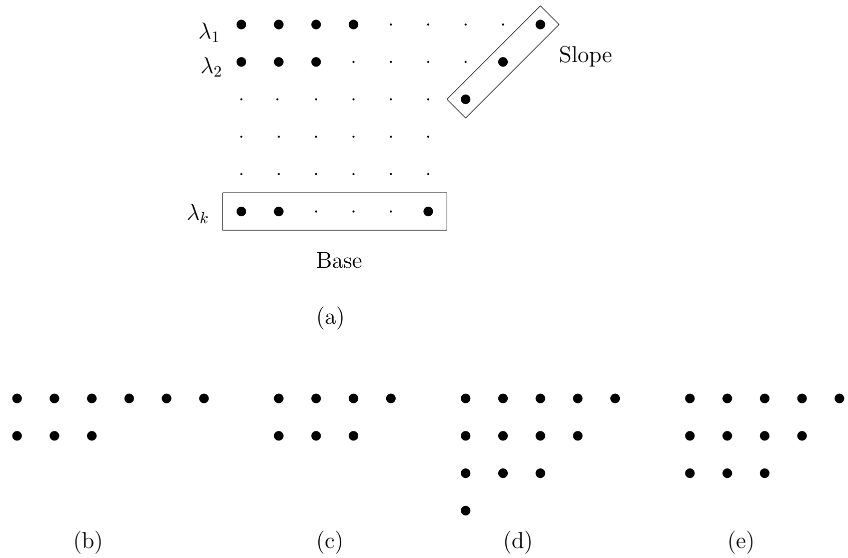


Figure 4: (a) Illustration of base and slope for a partition; (b) $b = 3$, $s = 1$; (c) $b = 3$, $s = 2$; (d) $b = 1$, $s = 3$; and (e) $b = 3$, $s = 3$

two casees, the transition works in both directions as described. So we can break the partitions of n into distinct parts into three classes:

Class 1 : $b < s$, or $b = s$ and base and slope do not intersect.

Class 2 : $b > s + 1$, or $b = s + 1$ and base and slope do not intersect.

Class 3 : $b = s$, or $b = s + 1$ and the base and slope intersect.

The transition that move the base up takes a partition in Class 1 to a partition in Class 2, and the inverse transition of moving the slope down takes a Class 2 partition to a Class 1 partition. Thus a partition with even (resp. odd) parts in Class 1 is mapped to a partition with odd (resp. even) parts in Class 2. This can be done for all partitions of n, except when n has a partiton in Class 3. But in this case what is n? It is a pentagonal number! Since when $b = s$ and the base and slope intersect, as in Figure 3(e), we have $n = s + (s + 1) + (s + 2) + \cdots + 2s - 1 = s^2 + s(s - 1)/2 = s(3s - 1)/2 = f(s)$; when $b = s + 1$ and the base and slope intersect, as in Figure 3(c), we have $n = (s + 1) + (s + 2) + \cdots + 2s = s^2 + s(s + 1)/2 = f(-s)$. Thus n has a partition in Class 3 iff it is a pentagonal number. Moreover, when s is even $E(n) = O(n) + 1$, and when s is odd we have $O(n) = E(n) + 1$. This completes the proof of the second claim in (6), and hence of Theorem 1.

To conclude our argument describing the recurrence for $p(n)$, we have that

$$\prod_{k\geq 1}(1 - x^k) = 1 + \sum_{j\geq 1}(-1)^j(x^{f(j)} + x^{f(-j)}).$$

Thus using the recurrence formula (2) we get

$$p(n) = \sum_{j\geq 1}(-1)^j(p(n-f(j))+p(n-f(j))) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+p(n-15)-\cdots . \tag{7}$$

The summation extends to the largest pentagonal number smaller than n, i.e., to a j such that $f(-j) \leq n$. This implies that $j < (-1 + \sqrt{1 + 24n}/6$, which is $O(\sqrt{n})$. Thus there are $O(\sqrt{n})$ terms in the summation, and each term takes at most n operations, since $p(n)$ depends upon $p(n - 1)$ we might have to compute all $p(j)$, $1 \leq j \leq n$. Thus a straightforward implementation of the algorithm based upon the recurrence takes $\Theta(n^{3/2})$ arithmetic operations.

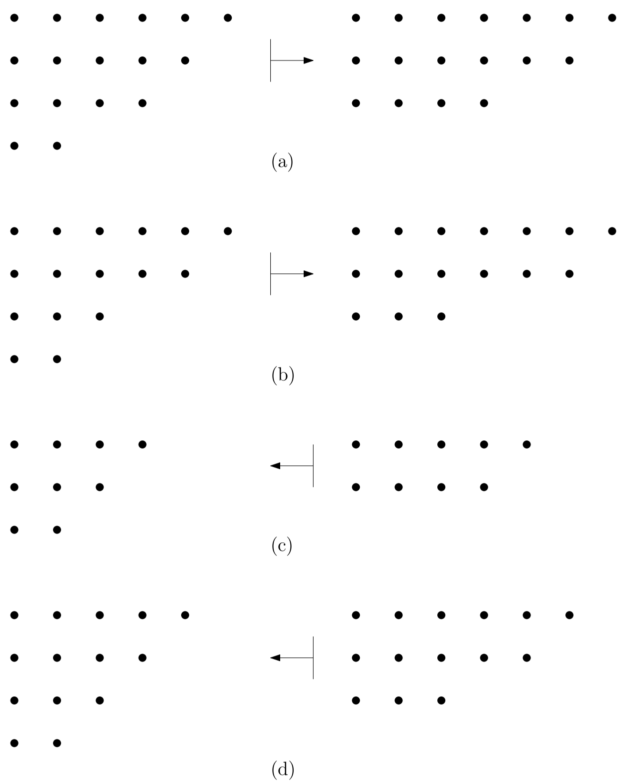


Figure 5: The transitions that take Class 1 partitions to Class 2 partitions and vice versa