# The Problème des Ménages

The problème des Ménages, or the married couples problem, is the following: How many ways $M_n$ there are to seat $n$ couples around a circular table with $2n$ chairs such that no couple sits next to each other, i.e., no husband and wife sit adjacent to each other.

Let's first fix the positions of either the wives or the husband, *say for courtesy's sake* wives. The number of ways to do this is $2 \times n!$, since the wives may either choose the even or odd numbered seats, and given this choice has been made there are $n!$ ways of seating them. Let's number the seats such that the seats on either side of the first wife is 1 and 2, on either side of the second wife 2 and 3, and so on. Now, we have to seat the husbands such that the first husband does not sit in seats numbered 1 and 2, the second husband 2 and 3 and so on the $n$th husband in seats numberd $n$ and 1. Let $u_n$ be the number of ways to do this. Then the answer to the ménage problem is $M_n = 2n!u_n$. Let's see how to get $u_n$.

We start with a recurrence for $u_n$. Let's consider a matrix representation for the husbands, where the in the $i$th row we write $h_i$ in all the places except $i-1, i \mod n$, and a $\bullet$ in these places; the modulo ensure that $h_1$ is not placed in positions 1 and $n$. The matrix looks as follows.

$$u_n := \begin{bmatrix} \bullet & h_1 & h_1 & \cdots & h_1 & \bullet \\ \bullet & \bullet & h_2 & \cdots & h_2 & h_2 \\ h_3 & \bullet & \bullet & \cdots & h_3 & h_3 \\ h_4 & h_4 & \bullet & \bullet & \cdots & h_4 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ h_n & h_n & h_n & \cdots & \bullet & \bullet \end{bmatrix}. \tag{1}$$

The diagonal and the subdiagonal entries always contain $\bullet$. What are the number of valid ways to assign the husbands? For $h_1$ we can choose any of the positions in the first row, except the "bullet" positions 1 and $n$; similarly, for $h_2$ we can choose any of the positions except 1, 2 the position for $h_1$; so on, $h_i$ has to avoid the positions $i-1, i$ and all the positions chosen earlier by $h_1, \dots, h_{i-1}$. The way we seem to make these choices suggests an resemblance to computing the "determinant" of the matrix $u_n$ where the dotted positions can be thought of as zero, the $h_i$'s as one, and we do not introduce the signs corresponding to the cofactors. Thus"dots" correspond to zero, and we put a $\star$ for $h_i$'s to show that the entry is non-zero. Moreover, this interpretation of computing $u_n$ as a determinant implies that permuting the rows and columns does not change the number of valid assignments.

Let's look at the expansion of $u_n$ along the *first row*.

The minors are $n-1$ dimensional matrices of the form

$$\chi_1(n-1) := \begin{bmatrix} \bullet & \star & \cdots & \star & \star \\ \star & \bullet & \cdots & \star & \star \\ \star & \bullet & \bullet & \cdots & \star \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \star & \star & \cdots & \bullet & \bullet \end{bmatrix}, \chi_2(n-1) := \begin{bmatrix} \bullet & \star & \cdots & \star & \star \\ \bullet & \bullet & \cdots & \star & \star \\ \star & \star & \bullet & \cdots & \star \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \star & \star & \cdots & \bullet & \bullet \end{bmatrix}, \dots, \chi_{n-2}(n-1) := \begin{bmatrix} \bullet & \star & \cdots & \star & \star \\ \bullet & \bullet & \cdots & \star & \star \\ \star & \bullet & \bullet & \cdots & \star \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \star & \star & \cdots & \star & \bullet \end{bmatrix}. \tag{2}$$

where $\chi_i$ has a $\star$ in the sub-diagonal entry corresponding to the $(i+1)$th row, i.e., the entry $(i+1, i)$. Thus

$$u_n = \chi_1(n-1) + \chi_2(n-1) + \cdots + \chi_{n-2}(n-1). \tag{3}$$

For the ease of clarity, let's carry over the definitions of $\chi_i$ to $n \times n$ matrices; thus we would have $\chi_1, \dots, \chi_{n-1}$. Now if we carefully look at $\chi_1$ and $\chi_{n-1}$ we see that expanding $\chi_1$ along the first column is the same as expanding $\chi_{n-1}$ along the last row, i.e., they have the same number of terms $\chi_1(n) = \chi_{n-1}(n)$. In general, by similar reasoning we have $\chi_i(n) = \chi_{n-i}(n)$. In addition to $\chi_i$'s, $i = 1, \ldots, n-1$, it is natural to define $\chi_0$ as the matrix that has bullets in the diagonal and *all the sub-digaonal entries*, i.e.,

$$\chi_0 := \begin{bmatrix} \bullet & \star & \cdots & \star & \star \\ \bullet & \bullet & \cdots & \star & \star \\ \star & \bullet & \bullet & \cdots & \star \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \star & \star & \cdots & \bullet & \bullet \end{bmatrix}.$$

Since $\chi_0$ has one extra zero it is natural to assume that it has fewer terms than the remaining $\chi_i$'s. This matrix plays a crucial role: we will reprent $u_n$ in terms of $\chi_0$ and $\chi_0$ in terms of $u_n$; these two relations would then give us a recurrence for $u_n$. Let's start with the latter relation. Observe that $u_n$ has an extra $\bullet$ in position $(1, n)$ when compared to $\chi_0$. Thus $\chi_0$ has all the terms in $u_n$ plus the terms appearing in the minor $(1, n)$, i.e.,

$$\chi_0(n) = u_n + \begin{bmatrix} \bullet & \bullet & \cdots & \star & \star \\ \star & \bullet & \bullet & \cdots & \star \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \star & \star & \cdots & \star & \bullet \end{bmatrix},$$

where the latter matrix has dimension $n-1$. The matrix on the RHS is almost like $u_{n-1}$ except the $\star$ in $(n-1, 1)$. Again this matrix satisfies

$$\begin{bmatrix} \bullet & \bullet & \cdots & \star & \star \\ \star & \bullet & \bullet & \cdots & \star \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \star & \star & \cdots & \star & \bullet \end{bmatrix} = u_{n-1} + \begin{bmatrix} \bullet & \cdots & \cdot & \star & \star \\ \bullet & \bullet & \cdot & \star & \star \\ . & . & . & . & . \\ \star & \star & \cdots & \bullet & \bullet \end{bmatrix} = u_{n-1} + \chi_0(n-2).$$

Thus recursilvey we obtain

$$\chi_0(n) = u_n + u_{n-1} + u_{n-2} + \cdots + u_3; \tag{4}$$

we terminate at $u_3$ since $u_2$ has zero terms.

The second important relation is of $\chi_i$'s in terms of $\chi_0$, and $\chi_{i-1}$. Consider $\chi_1(n)$. As we had noticed, it has one fewer zero than $\chi_0$, namely in the position $(2, 1)$. Thus the number of terms in $\chi_1$ are the terms in $\chi_0(n)$ and the terms in the minor corresponding to $(2, 1)$, i.e.,

$$\chi_1(n) = \chi_0(n) + \begin{bmatrix} \star & \star & \cdots & \star & \star \\ \bullet & \bullet & \star & \cdots & \star \\ \star & \bullet & \bullet & \cdots & \star \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \star & \star & \cdots & \bullet & \bullet \end{bmatrix}.$$

The matrix on the RHS is of dimension $(n-1)$ and has all the correct zeros except the star at (1,1). Thus

$$\chi_1(n) = \chi_0(n) + \chi_0(n-1) + \begin{bmatrix} \bullet & \star & \cdots & \star \\ \bullet & \bullet & \cdots & \star \\ \vdots & \cdots & \vdots & \vdots \\ \star & \cdots & \bullet & \bullet \end{bmatrix} = \chi_0(n) + \chi_0(n-1) + \chi_0(n-2).$$

Similarly, we can show

$$\chi_2(n) = \chi_0(n) + \begin{bmatrix} \bullet & \star & \cdots & \star & \star \\ \bullet & \star & \star & \cdots & \star \\ \star & \bullet & \bullet & \cdots & \star \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \star & \star & \cdots & \bullet & \bullet \end{bmatrix} = \chi_0(n) + \chi_0(n-1) + \begin{bmatrix} \bullet & \star & \cdot & \star & \star \\ \star & \bullet & \bullet & \cdot & \star \\ \vdots & \vdots & . & . & . \\ \star & \star & \cdots & \bullet & \bullet \end{bmatrix} = \chi_0(n) + \chi_0(n-1) + \chi_1(n-2).$$

In general, we have

$$\chi_i(n) = \chi_0(n) + \chi_0(n-1) + \chi_{i-1}(n-2) \tag{5}$$

and hence inductively we obtain that

$$\begin{aligned}
\chi_1(n) &= \chi_0(n) + \chi_0(n-1) + \chi_0(n-2) \\
\chi_2(n) &= \chi_0(n) + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) + \chi_0(n-4) \\
\vdots &= \vdots \\
\chi_i(n) &= \chi_0(n) + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) + \chi_0(n-4) \cdots + \chi_0(n-2i) \\
\vdots &= \vdots \\
\chi_{n-2}(n) &= \chi_0(n) + \chi_0(n-1) + \chi_0(n-2) + +\chi_0(n-3) + \chi_0(n-4) \\
\chi_{n-1}(n) &= \chi_0(n) + \chi_0(n-1) + \chi_0(n-2).
\end{aligned}$$

Substituting $n-1$ instead of $n$ and summing these equalities along with (3) we obtain

$$\begin{aligned}
u_n = {} & \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) \\
& + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) + \chi_0(n-4) + \chi_0(n-5) \\
& + \vdots \\
& + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) + \chi_0(n-4) + \chi_0(n-5) + \cdots + \chi_0(n-2i-1) \\
& + \vdots \\
& + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3) + \chi_0(n-4) + \chi_0(n-5) \\
& + \chi_0(n-1) + \chi_0(n-2) + \chi_0(n-3).
\end{aligned}$$

Substituting the expression (4) for $\chi_0(n)$ in terms of $u_i$'s we obtain

$$\begin{aligned}
u_n = {} & u_{n-1} + 2u_{n-2} + 3(u_{n-3} + u_{n-4} + \cdots + u_3) \\
& + u_{n-1} + 2u_{n-2} + 3u_{n-3} + 4u_{n-4} + 5(u_{n-5} + \cdots + u_3) \\
& + u_{n-1} + 2u_{n-2} + 3u_{n-3} + 4u_{n-4} + 5u_{n-5} + 6u_{n-6} + 7(u_{n-7} + \cdots + u_3) \\
& + \vdots \\
& + u_{n-1} + 2u_{n-2} + 3u_{n-3} + 4u_{n-4} + 5(u_{n-5} + \cdots + u_3) \\
& + u_{n-1} + 2u_{n-2} + 3(u_{n-3} + u_{n-4} + \cdots + u_3)
\end{aligned}$$

which implies

$$u_n = (n-2)u_{n-1} + 2(n-2)u_{n-2} + 3(n-2)u_{n-3} + (4n-10)u_{n-4} + (5n-14)u_{n-5} + \cdots + \frac{1-(-1)^n}{2}. \tag{6}$$

Though we have a recurrence, it is not very immediate what the general term is. Also it is quite unwieldy.

The answer above is not quite satisfying. This was the state of affairs in 1875-1888, and it wasn't until half a century later that the following direct and simple formula was given without a proof

$$u_n = n! - \frac{2n}{2n-1}\binom{2n-1}{1}(n-1)! + \frac{2n}{2n-2}\binom{2n-2}{2}(n-2)! - \cdots. \tag{7}$$

Here we present a proof based on PIE. However, a certain initial assumption still makes the proof non-trivial, which will be removed later on.

We start with the following lemmas

LEMMA 1. *The number of ways of selecting $k$ objects from $n$ objects placed in a row such that no two are consecutive is $\binom{n-k+1}{k}$.*

Let $f(n,k)$ be the number of possibilities. Consider the $n$th item: if we pick it then we have to choose the remaining $k-1$ objects from $n-2$ objects, since we cannot pick the $(n-1)$th object; if we do not pick it then we have to choose $k$ objects from $n-1$ objects, since there is the possibility of picking $(n-1)$th object. So we have the recursion

$$f(n,k) = f(n-2,k-1) + f(n-1,k),$$

where $f(n,1) = n$. Multiplying by $x^n$ and summing, we obtain

$$A_k(x) := \sum_n f(n,k)x^n = x^2 \sum_n f(n-2,k-1)x^{n-2} + x \sum_k f(n-1,k)x^{n-1},$$

which implies

$$A_k = x^2 A_{k-1} + xA_k$$

or

$$A_k = \frac{x^2}{1-x} A_{k-1} = (\frac{x^2}{1-x})^{k-1} A_1.$$

Moreover, $A_1 = \sum_n nx^n = -x/(1-x)^2$. Thus

$$A_k = \frac{-x^{2k-1}}{(1-x)^{k+1}},$$

and hence the coefficient of $x^n$ on the RHS is $(-1)^{n-2k}\binom{-k-1}{n-2k+1} = \binom{n-k+1}{k}$.

The second lemma asks the same question, but now the objects are arranged in a circle.

LEMMA 2. *The number of ways $g(n,k)$ of selecting $k$ objects from $n$ objects placed in a circle such that no two are consecutive is $g(n,k) = \binom{n-k}{k}\frac{n}{(n-k)}$.*

Given $f(n,k)$, we have to remove all the choices where both the first and the last object were picked. The number of ways of picking both is $f(n-4,k-2)$. Thus the answer is

$$g(n,k) = f(n,k) - f(n-4,k-2) = \binom{n-k+1}{k} - \binom{n-k-1}{k-2} = \binom{n-k}{k}\frac{n}{(n-k)}.$$

We now want to compute $u_n$ using PIE. Let $C_1$ be the constraint that $h_1$ is in position 1, $C_2$ is $h_2$ is in position 1, $C_3$ is $h_2$ is in position 2, so on, $C_{2n-2}$ is $h_n$ is in position $n-1$, $C_{2n-1}$ is $h_n$ is in position $n$, and $C_{2n}$ is 1 is in position $n$. Thus we have $2n$ *constraints*, and let us array them around a circle. Let $\nu_k$ be the number of ways of choosing $k$ of the constraints such that no two are adjacent; we say such a choice of constraints is **compatible**.

Then $u_n$ is the number of permutations of the husbands such that none of these $2n$ constraints are satisfied. Let $A_i$ be the event that the $i$th constraint is satisfied. Then given $I \subseteq [2n]$, $A_I$ means that all the constraints $A_j$, $j \in I$ are satisfied. How many permutations of $h_1, \ldots, h_n$ satisfy at least $k = |I|$ constraints? It is either $(n-k)!$, since the position of $k$ of the husbands is determined by $I$ are the rest can be freely permuted, or zero if the constraints are not compatible. Thus by PIE, out of the total set of $n!$ permutations of $h_i$'s the ones that *do not satisfy any of the constraints is*

$$u_n = n! - \sum_{k=1}^{n} (-1)^k \nu_k (n-k)!.$$

But from the second lemma above, we know that $\nu_k = \binom{2n-k}{k}\frac{2n}{2n-k}$, which gives us

$$u_n = n! - \sum_{k=1}^{n} (-1)^k \binom{2n-k}{k} \frac{2n}{2n-k}(n-k)!$$

(recall $M_n = 2n!u_n$). The ingenuity of the proof is in arranging the constraints around the circle, and not the husbands since permuting the husbands arbitrarily around the table may result in situations where two of them are adjacent to each other, a possibility that we are not prepared to handle.

The argument needs to be ingenious because we have *fixed the position of the wives*. If, however, we freely permute the couples, define our event as a $k$ couple being seated next to each other, and sieve from all the possible ways to seat the men and women alternatigly then the PIE gives a much straightforward argument. Removing the gender bias is the key to simplifying the argument.

Let us develop this argument further. The number of possible arrangements where the men and women alternate is $2(n!)^2$, because, as we had argued in the beginning, we can seat the women in $2n!$ ways, and for each such choice there are $n!$ ways to seat the men. Given $k$ couples, we may again ask, how many permutations $z_k$ are there such that at least these $k$ couples are seated next to each other, and the remaining men and women alternate? Again, we have to pick $k$ non-adjacent positions around the table, which can be done in in $g(2n, k)$ ways; there are two possible ways to seat the men and the women in these $k$ chairs in an alternating fashion; the $k$ couples can permute amongst each other in these $k$ positions in $k!$ ways; the remaining $(n-k)$ men can sit in men's chairs in $(n-k)!$ ways and same for the remaining $n-k$ women. Thus

$$z_k = 2\binom{2n-k}{k}\frac{2n}{(2n-k)}k!((n-k)!)^2,$$

and by the PIE we obtain

$$M_n = 2(n!)^2 - \sum_{k\geq 1} 2\binom{n}{k}\binom{2n-k}{k}\frac{2n}{(2n-k)}k!((n-k)!)^2.$$