Simulating the scalar field on the fuzzy sphere

(I) Discretization schemes

Lattice: 2d flat torus

Fuzzy sphere:

(II) The scalar field in 2d

on the plane

on the lattice

on the fuzzy sphere

(III) The numerical scheme

Monte-Carlo importance sampline

Implementation and tests

Results

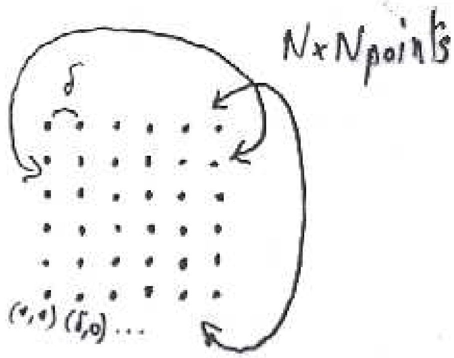
(IV) Perspectives

beyond the fuzzy sphere

beyond the scalar field

the scalar field on the Moyal plane

The discretization schemes



The lattice: discretize the space

Fields: values at each point $\varphi(i\delta, j\delta)$

Integration: Riemann sums: $\delta^2 \sum_{i,j=0}^{N-1} (\cdot)$

Derivation: finite differences

$$\text{e.g. } (\Delta\varphi)(i\delta, j\delta) = \frac{1}{\delta^2}\Big[\varphi((i+1)\delta, j\delta) + \varphi((i-1)\delta, j\delta) + \varphi(i\delta, (j+1)\delta) + \varphi(i\delta, (j-1)\delta) - 4\varphi(i\delta, j\delta)\Big]$$

Periodic boundary conditions

intuitive BUT $\begin{cases} \text{trivial topology} \\ \text{discrete symmetry group (same as a square)} \end{cases}$

The fuzzy sphere: discretize the function algebra $\mathcal{C}^\infty(S^2)$

→ Fields are matrices in some matrix algebra $Mat_{2\ell+1}$

$SU(2)$ acts naturally on $S^2 = SU(2)/U(1)$

→ introduce the $(2\ell+1)$-dimensional representation of $SU(2)$ and $L_i \in Mat_{2\ell+1}$ with $\sum_{i=1}^{3} L_i^2 = \ell(\ell+1)\mathbb{1}$

Then $\hat{x}_i = \frac{L_i}{\sqrt{\ell(\ell+1)}}$ the coordinate functions: $\begin{cases} \sum_{i=1}^{3} \hat{x}_i^2 = \mathbb{1} \\ [\hat{x}_i, \hat{x}_j] = \frac{i\,\varepsilon_{ijk} x_k}{\sqrt{\ell(\ell+1)}} \end{cases}$

Structure on Mat_{2l+1} : integration, derivation ... is derived through Coherent states ~ "points"

-North pole $\vec{n}_+$ asociated with a state $|\vec{n}_+\rangle = |l,l\rangle$

- another point $\vec{n} = \underset{SU(2)/U(1)}{\underset{\cap}{R}} \vec{n}_+$ is asociated with: $|\vec{n}\rangle = \underset{(2l+1)\text{-dim. rep.}}{\underset{\cap}{R}} |l,l\rangle$

Mapping: $\mathcal{M}: Mat_{2l+1} \longrightarrow \mathcal{C}^\infty(S^2)$

$$\varphi \longmapsto \varphi(\vec{n}) = \langle \vec{n} | \varphi | \vec{n} \rangle$$

respects the action of $SU(2)$ on both sides.

Now we can pull back the structure of $\mathcal{C}^\infty(S^2)$ onto Mat_{2l+1}

derivation: $[L_i, .] \rightarrow$ Laplacian, Dirac operator

integration: $\frac{4\pi}{2l+1} Tr(.)$

conjugation: $\varphi^\dagger$

introducing $\hat{Y}_{lm}$ the eigenvectors of the adjoint action of $SU(2)$

$\mathcal{M}(\hat{Y}_{lm}) \propto Y_{lm}$ the spherical harmonics

$\Downarrow$

$\mathcal{M}$ is into, its image is $\left\{ \sum_{s=0}^{2l} c_{lm} Y_{lm} \right\} \xrightarrow[l \to +\infty]{} \mathcal{C}^\infty(S^2)$

$*$-product: defined as the mapping of matrix products by $\mathcal{M}$

$$[\mathcal{M}(\varphi) * \mathcal{M}(\psi)](\vec{n}) = \mathcal{M}(\varphi\psi)$$

$$(\varphi * \psi)(\vec{n}) = \sum_{s=0}^{\ell} \frac{(\ell-s)!}{\ell!\, s!} \left(\partial_{A_1} \dots \partial_{A_s} \varphi\right) K^{A_1 B_1} \dots K^{A_s B_s} \left(\partial_{B_1} \dots \partial_{B_s} \psi\right)$$

$$= \varphi(\vec{n})\, \psi(\vec{n}) + \frac{1}{\ell} \left(\partial_A \varphi\right) K^{AB} \left(\partial_B \psi\right) + \mathcal{O}\left(\frac{1}{\ell^2}\right)$$

Approximation rule: to approximate an algebraïc expresion on the sphere, replace everywhere functions φ by mapped matrices $\mathcal{M}(\Phi)$, products by $*$-products, and pull back.

spheres of radius R: just scale:

derivations: $\frac{1}{R}[L_i, \cdot]$

integration: $\frac{4\pi R^2}{2\ell+1} \operatorname{Tr}(\cdot)$

coordinate functions: $\hat{x}_i = \frac{R L_i}{\sqrt{\ell(\ell+1)}}$, $[\hat{x}_i, \hat{x}_j] = \frac{iR\, \varepsilon_{ijk}}{\sqrt{\ell(\ell+1)}} \hat{x}_k$

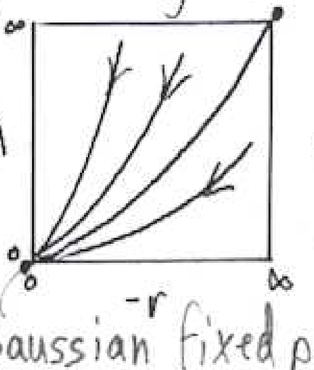
Asymptotic regimes:

- sphere: $R =$ constant, $\ell \longrightarrow +\infty$
- plane: $R = o(\ell)$, $\ell \longrightarrow +\infty$, $R \longrightarrow +\infty$. e.g. $R = \sqrt{\ell}$
- NC space: $\ell = o(R)$, $\ell \longrightarrow +\infty$
- Moyal plane: $R = \theta\sqrt{\ell}$ and focus on the North pole $\hat{x}_3 \sim R$

The scalar field in 2d

On the plane: $S(\varphi) = \int d^2\vec{n} \left[-\varphi(\Delta\varphi) + r\varphi^2 + \lambda\varphi^4 \right]$

- the field theory is entirely defined by its diagrammatic expansion
- it has only 1 divergent diagram: the tadpole: 

Critical behaviour: λ 

critical line: same universality class as 2-d Ising

On the lattice: $S(\varphi) = \sum_{i,j=1}^{N} \left[\varphi(i\delta, j\delta)[\Delta\varphi](i\delta, j\delta) + r\delta^2 \varphi^2(i\delta, j\delta) + \lambda\delta^2 \varphi^4(i\delta, j\delta) \right]$

parameters: $\begin{cases} N^2 : \text{number of points in the lattice} \\ r\delta^2, \lambda\delta^2 \text{ in the potential.} \end{cases}$

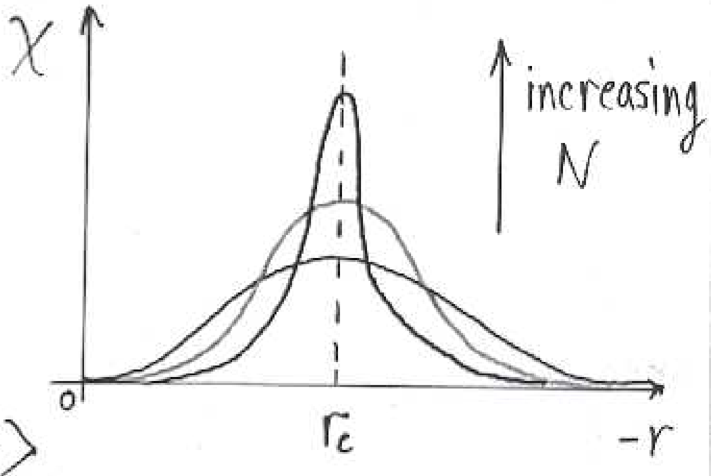
scaling limit: to recover the continuum properties, the limit must be taken at a critical point where there is scale invariance

Finding the critical point

The order parameter $s = \delta^2 \sum_{i,j=1}^{N} \varphi(i\delta, j\delta) \sim \int d^2\vec{n}\, \varphi(\vec{n})$

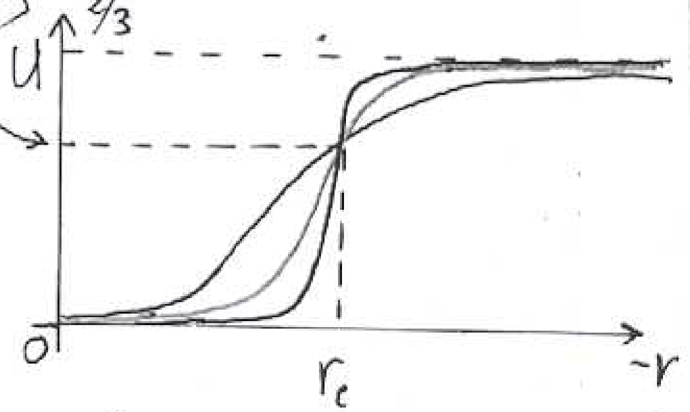
1) The susceptibility: $\chi = \langle s^2 \rangle - \langle |s| \rangle^2$

χ diverges at the critical point in the continuum limit



2) The cumulant: $U = 1 - \frac{\langle s^4 \rangle}{3\langle s^2 \rangle^2}$

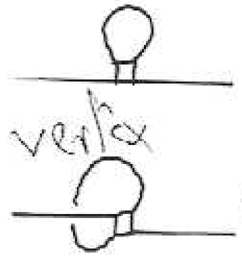
It is a universal quantity U^* at the critical point
⟶ it does not depend on N



On the fuzzy sphere: $S(\Phi) = \frac{4\pi}{2l+1} \mathrm{Tr}\left[\Phi[L_i,[L_i,\Phi]] + rR^2\Phi^2 + \lambda R^2\Phi^4\right]$

The approximation rule ensures that all finite diagrams have the right classical limit BUT

tadpole diagrams:



planar ✓

non planar ✗ UV-IR mixing

Solution: regularize the theory

$\longrightarrow$ approximation rule applies to all diagrams

1) normal order the interaction term $:\lambda\phi^4:$

does not generalize ...

2) cut down the higher momentum contribution with a regularizing term:

$$S(\Phi) = \frac{4\pi}{2l+1}\,\mathrm{Tr}\left(\Phi\left[L_i,\left[L_i,\Phi\right]\right] + rR^2\Phi^2 + \lambda R^2\Phi^4 + \frac{a}{R^2}\left[L_i,\left[L_i,\Phi\right]\right]\left[L_i,\left[L_i,\Phi\right]\right]\right)$$

$$\text{parameters:}\begin{cases} l,\ (2l+1)^2 \text{ degrees of freedom} \\ rR^2,\ \lambda R^2 \text{ in the potential} \\ a/R^2 \text{ the regularizing term} \rightarrow \text{critical behaviour} \end{cases}$$

Numerical goal: identify the critical point with the cumulant

The numerical scheme: Monte-Carlo importance sampling

goal: calculate averages of the form $\langle s^{2n}\rangle = \frac{1}{Z}\int d\varphi\,(s(\varphi))^{2n}\,e^{-S(\phi)}$

method: generate a random distribution of φ with probability distribution $e^{-S(\phi)}/Z$ partition fct.

Then: $\langle s^{2n}\rangle = \lim_{N\to+\infty}\frac{1}{N}\sum_{i=1}^{N} s^{2n}(\varphi_i)$

generating the random distribution

This is done dynamically through a Markov-process.

Given a configuration φ_t, pick a random new configuration ψ

Calculate $\Delta S = S(\varphi_t) - S(\psi)$, $\begin{cases} \text{if } \Delta S \geq 0, & \varphi_{t+1} = \psi \\ \text{if } \Delta S \leq 0, & \begin{cases} \varphi_{t+1} = \psi \text{ with probability } e^{\Delta S} \\ \varphi_{t+1} = \varphi_t \quad " \quad " \quad 1-e^{\Delta S} \end{cases} \end{cases}$

Then φ_t as probability distribution $e^{-S(\phi)}/Z$ for $t \to +\infty$

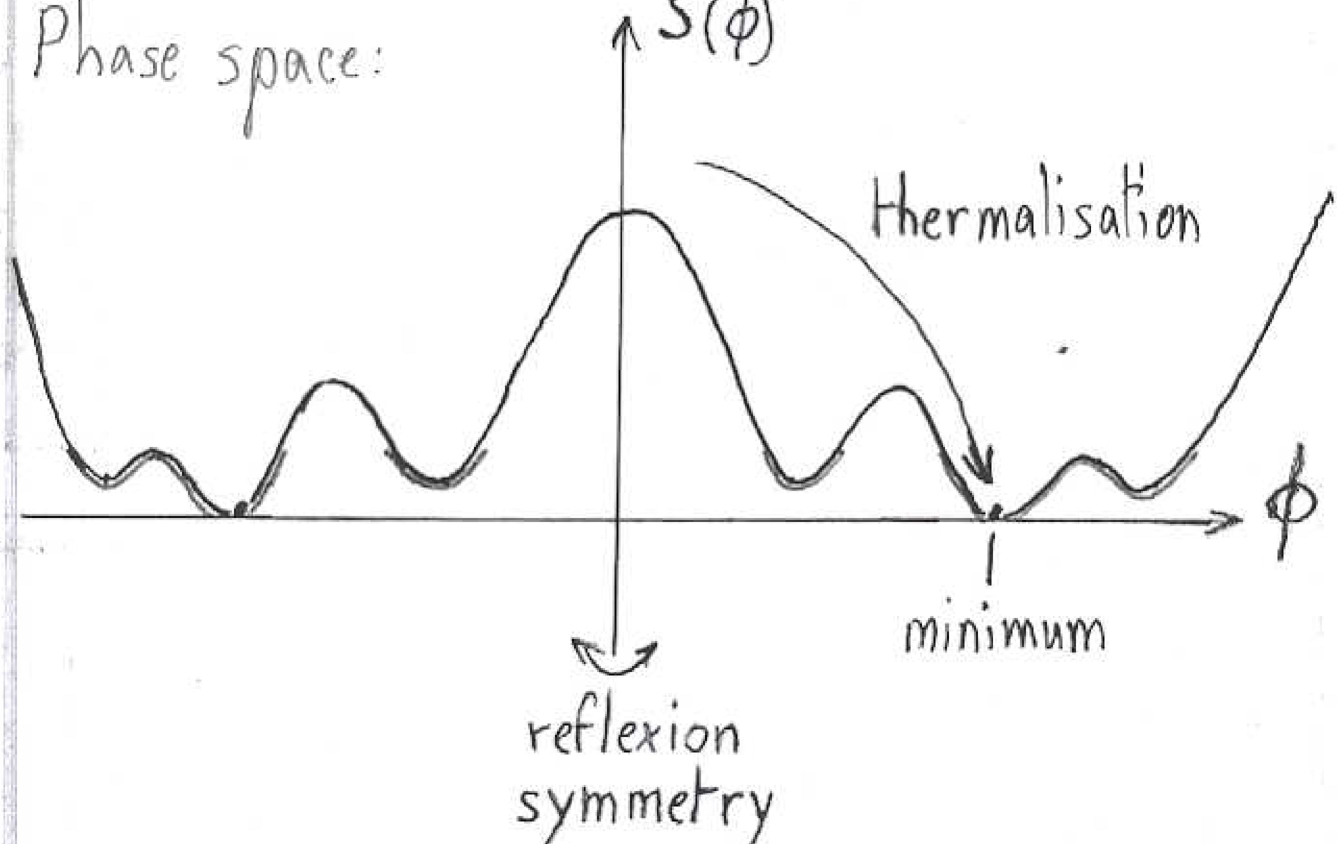
Implementation

1) Thermalisation: when does the distribution reach its asymptotic value?

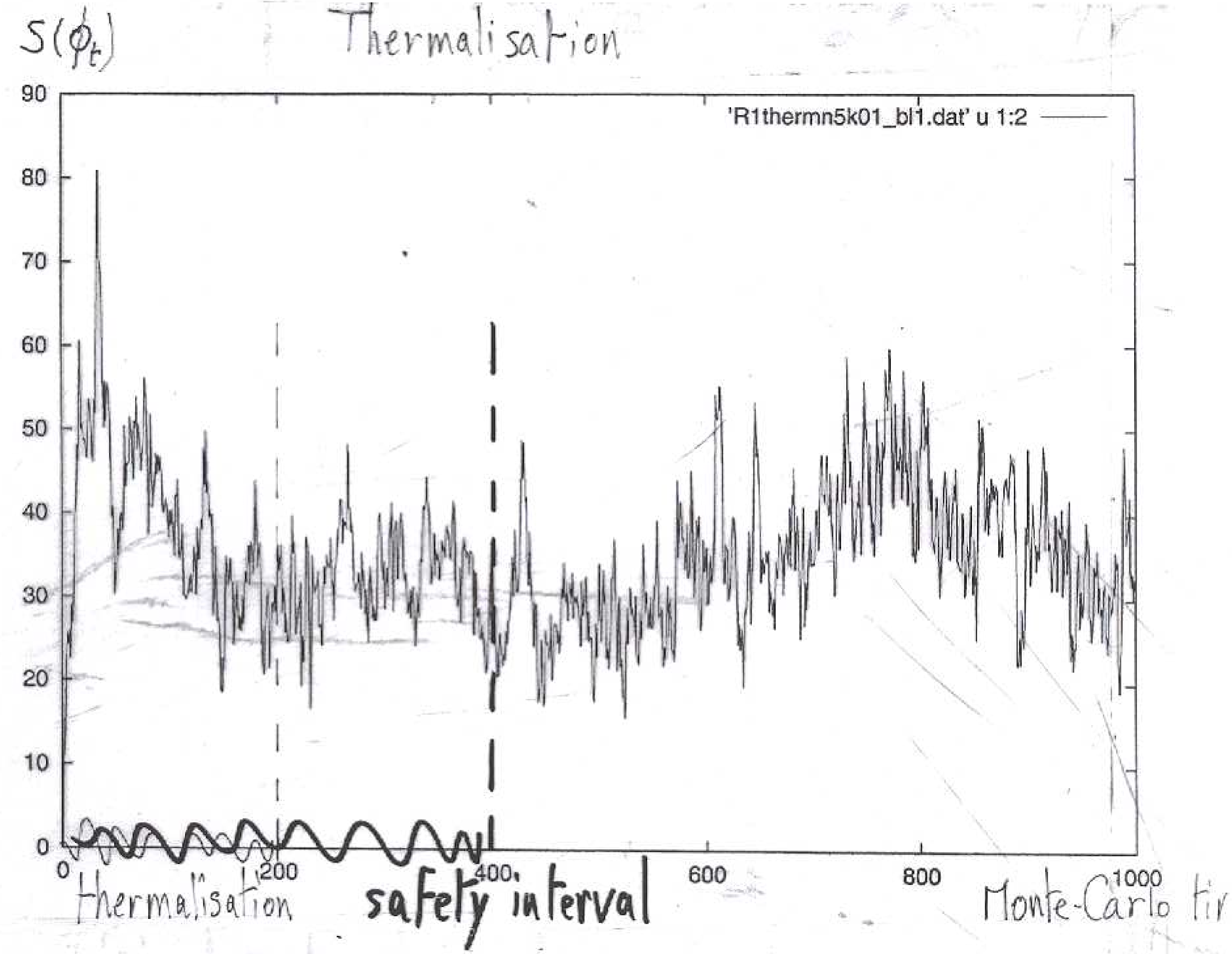
Check $S(\varphi_t)$, do it several times for safety

direct integration: $\int d\varphi \sim N^2$ integrations on p points each
$\longrightarrow p^{N^2}$ operations !!!

Phase space:



– significant contribution to $\int \frac{d\phi}{Z} e^{-S(\phi)}$



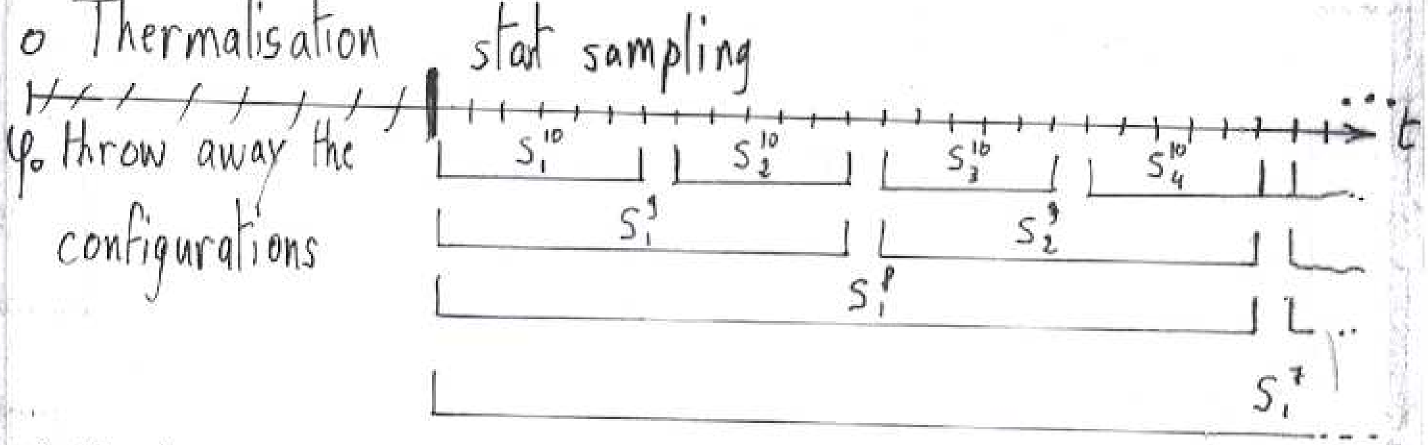
$S(\phi_t)$
Thermalisation
'R1thermn5k01_bl1.dat' u 1:2
90
80
70
60
50
40
30
20
10
0
0
200
400
600
800
1000
thermalisation
safety interval
Monte-Carlo tir

2) error control: the error goes like $\frac{\tau}{\sqrt{N}}$ with

τ the autocorrelation time $= \int_0^{+\infty} dt \int d\varphi \; \varphi(0)\varphi(t) \frac{e^{-S(\varphi)}}{Z} = \int_0^{+\infty} dt \langle \varphi(0)\varphi(t) \rangle$

autocorrelation slows the convergence.

To control this error, use binning



If the bin size $\geq \tau$ then the standard deviation does not depend on the bin size significantly

For nonlinear expressions (e.g. χ or U), use the jacknife method

3) Drawing the new configuration ψ

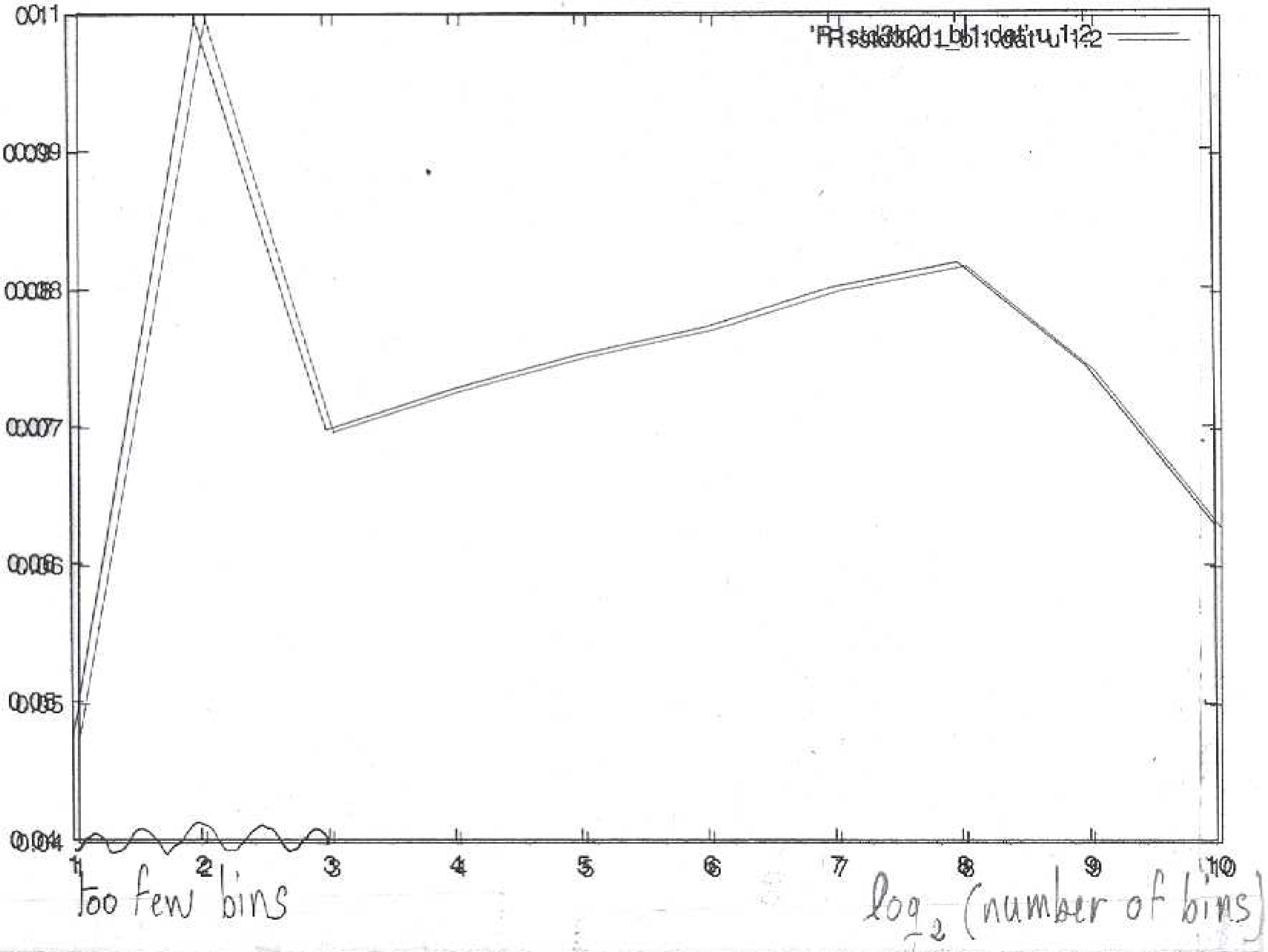
ψ is a random perturbation of the previous configuration φ_t

small perturbations φ varies slowly $\begin{cases} \text{slow thermalisation} \\ \text{large autocorrelation} \\ \text{bad exploration of the phase space} \end{cases}$

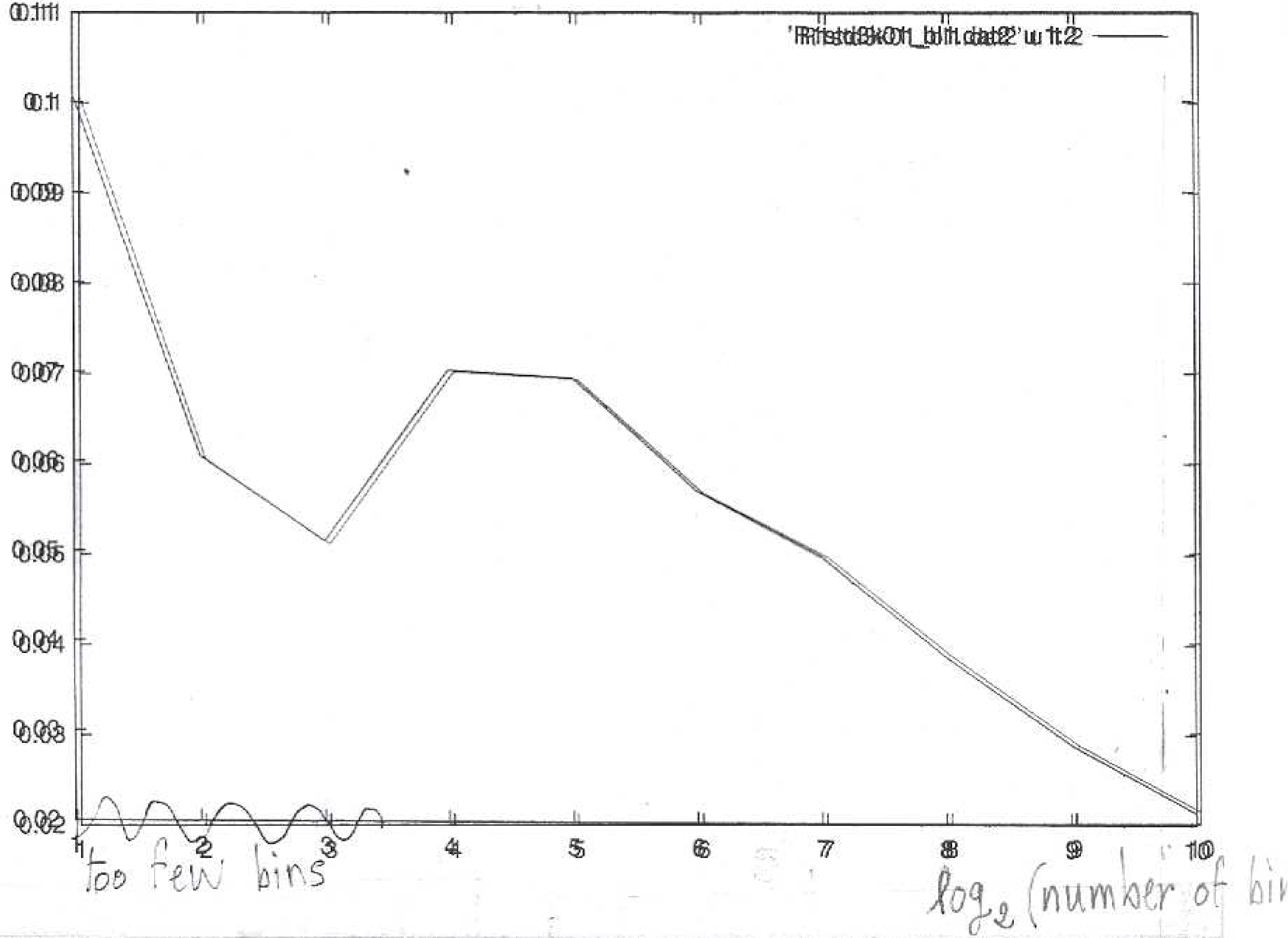
large perturbations: φ seldom varies: very large autocorrelation

intermediate: adjust acceptance rate of ψ as φ_{t+1} to ~ 15-20%

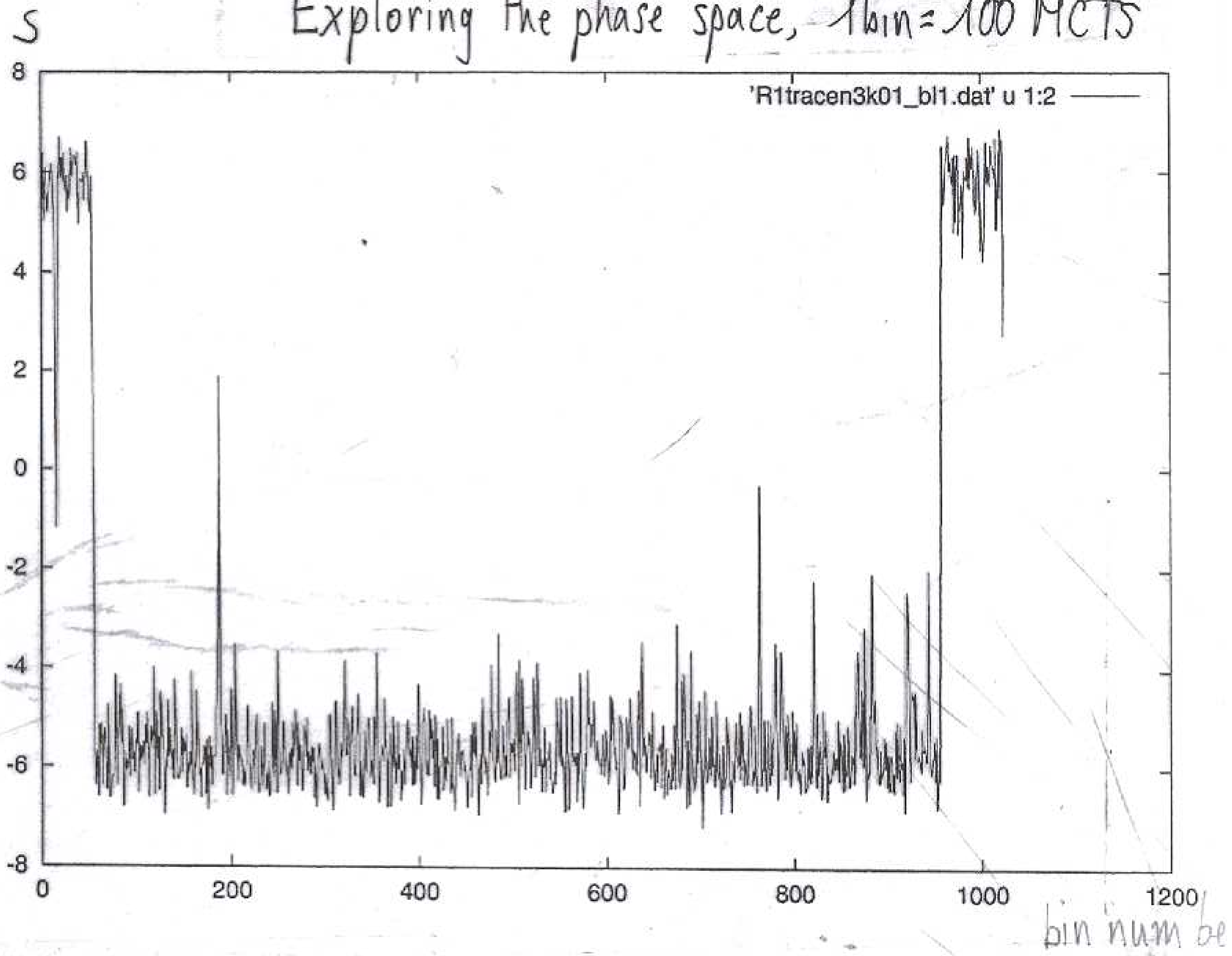
Jacknife standard deviation for χ, 1bin = 100 MCTS



Jacknife standard deviation for X, 1bin=1MCTS



Exploring the phase space, 1bin = 100 MCTS



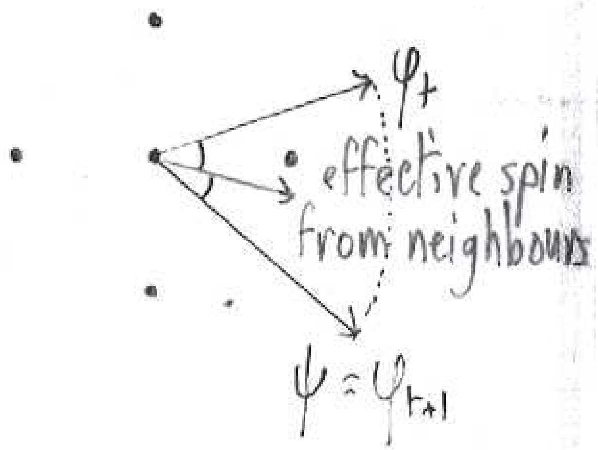
Accelerating the convergence:

Just perform a large perturbation with $\Delta S \ll 1$ or even $= 0$

Example: 2d XY Ising model (spin is $e^{i\theta}$) sophisticated

By construction, $\Delta S = 0$!!

possible in the fuzzy case?



Results:

-Lattice

-Fuzzy: in the works ☹

programs runs. Exploring the range of r possible for $l \leq 9$ in a few hours

Slower than on the lattice: time consuming calculation is ΔS

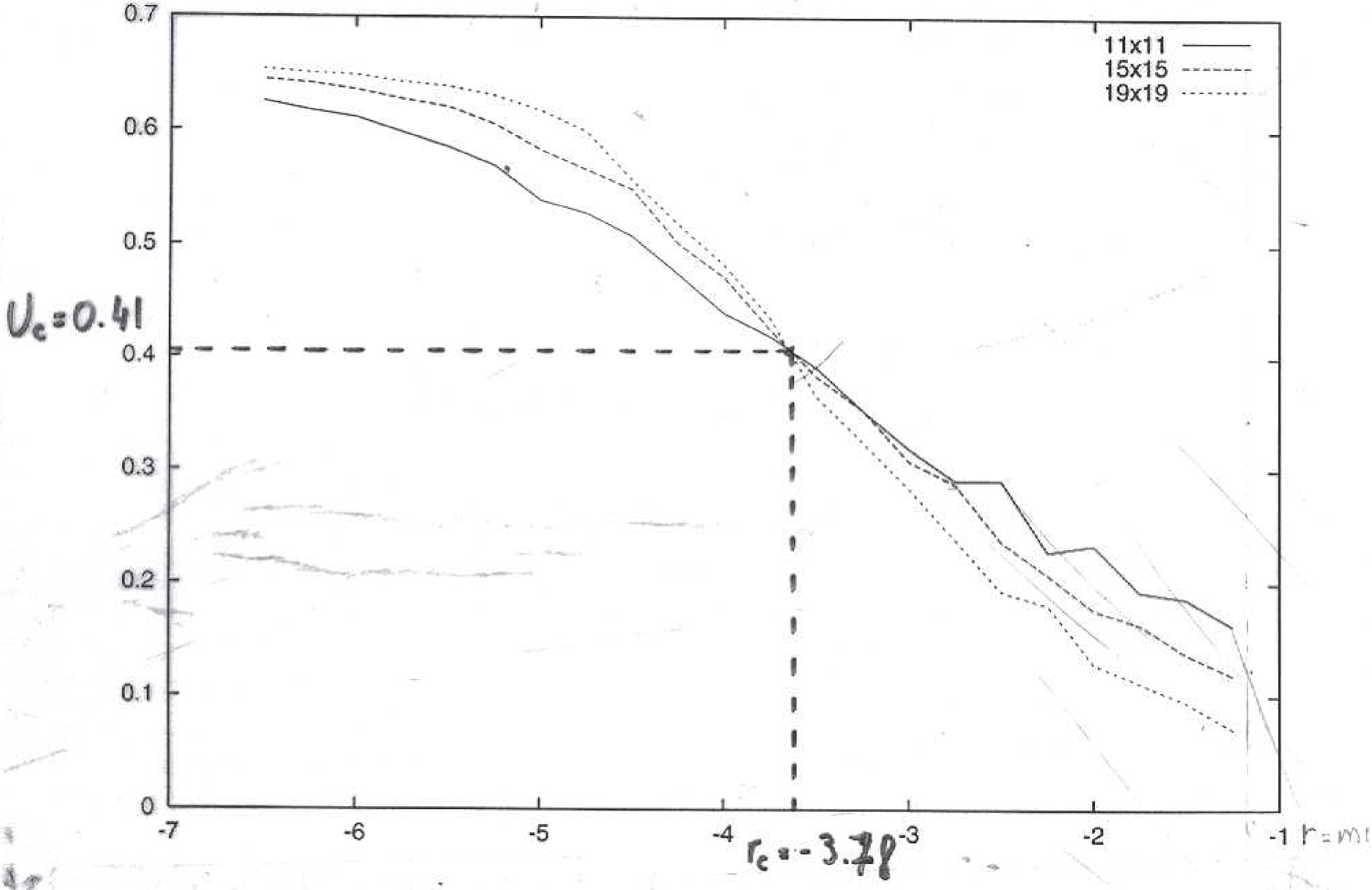
- on the lattice $\sim \mathcal{O}(1)$ operations (quasi-local)
- on the fuzzy sphere $\sim \mathcal{O}(l)$ operations (because of $\lambda\phi^4$)

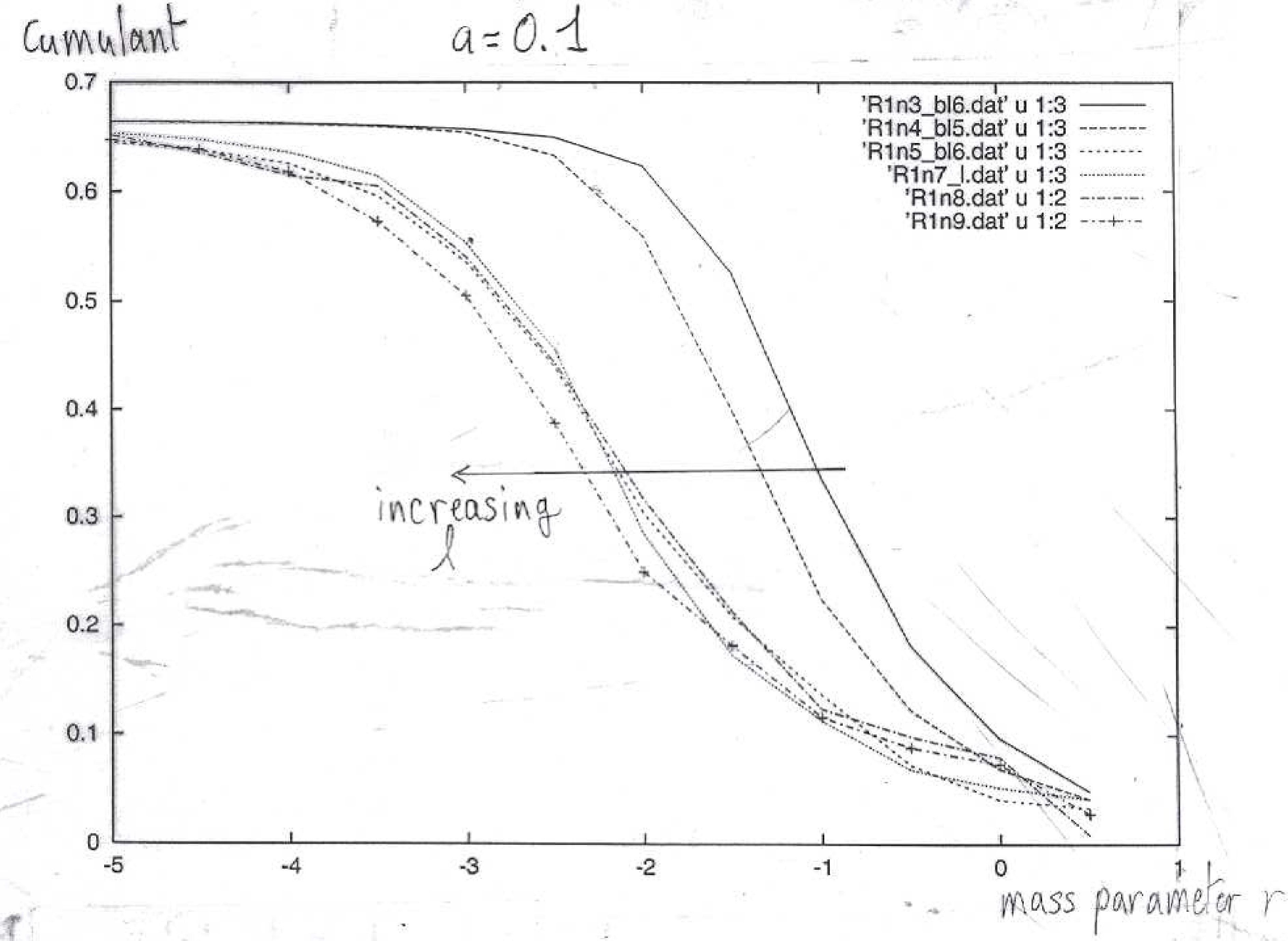
Problem: there is an extra parameter to take into account

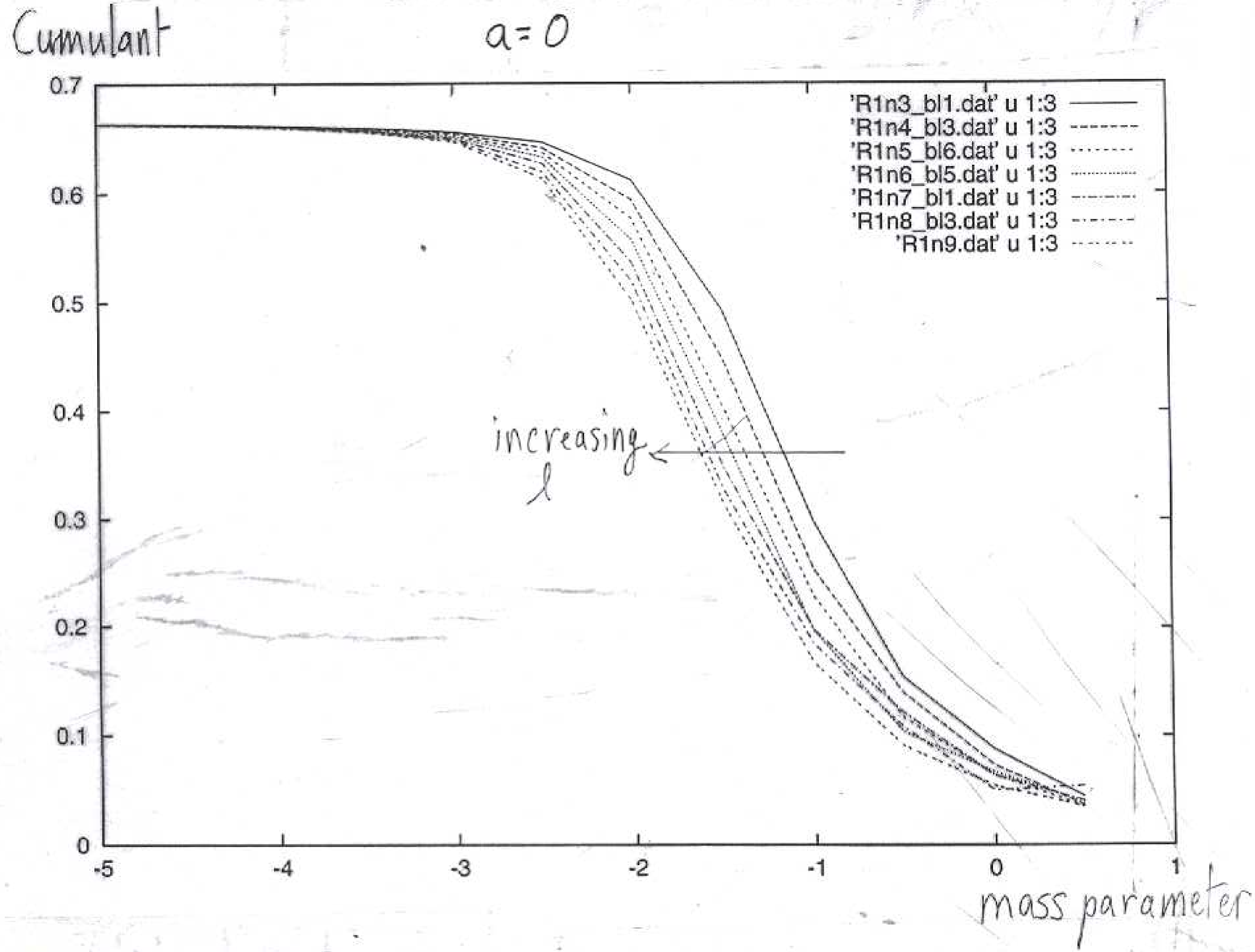
effect on the critical point?

Solution being tested: take a double limit: $\lim_{a \to 0} \lim_{l \to +\infty}$

V cumulant LATTICE $\lambda = 1, a = 0.1$




Cumulant
a=0.1
'R1n3_bl6.dat' u 1:3
'R1n4_bl5.dat' u 1:3
'R1n5_bl6.dat' u 1:3
'R1n7_l.dat' u 1:3
'R1n8.dat' u 1:2
'R1n9.dat' u 1:2
increasing l
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-5
-4
-3
-2
-1
0
1
mass parameter r


Cumulant
a=0
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-5
-4
-3
-2
-1
0
1
'R1n3_bl1.dat' u 1:3
'R1n4_bl3.dat' u 1:3
'R1n5_bl6.dat' u 1:3
'R1n6_bl5.dat' u 1:3
'R1n7_bl1.dat' u 1:3
'R1n8_bl3.dat' u 1:3
'R1n9.dat' u 1:3
increasing l
mass parameter

Perspectives

other fuzzy spaces: $\begin{cases} \mathbb{CP}^N \\ \text{product of fuzzy spaces, e.g. } S^2 \times S^2 \\ \text{some spaces imbedded in } \mathbb{CP}^N\text{, e.g. } S^4 = \mathbb{CP}^3/U(1) \end{cases}$

other field theories:

fermions: $\begin{cases} \text{chiral fermions are easy} \\ \text{fermions on lattices are also VERY non-local} \end{cases}$

gauge fields: $\begin{cases} U(N) \text{ in fundamental is easy} \\ \text{others are problematic} \\ \text{gauge groups representations can NOT be tensored up} \end{cases}$

NC scalar field on the Moyal plane:

to study the critical behaviour of a theory with UV-IR mixing

Currently: studied on the NC torus $\sim$ fuzzy lattice

$$S(\Phi) = \mathrm{Tr}\left[\Phi \underbrace{[T_i, [T_i, \Phi]]}_{\Delta} + r\Phi^2 + \lambda \Phi^4\right]$$

Δ has eigenvalues $\frac{4}{\delta^2}\left[\sin^2\left(\frac{\pi n_1}{N}\right) + \sin^2\left(\frac{\pi n_2}{N}\right)\right]$

Alternatively: - study it on the fuzzy sphere

- scale it to the Moyal plane