Reciprocity in Noncommutative Gauge Theory and ADHM construction

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§1 Introduction

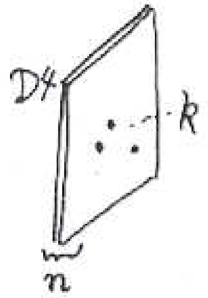
§2. ADHM and Reciprocity

§3. Inversion

§4. Discussion.

- Branes in Branes $\Rightarrow$ Soliton in effective theory.

  Static solutions in D4 - Brain

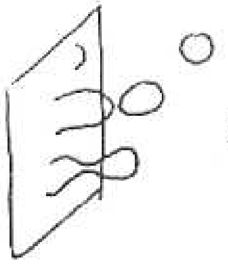


  k D0 in n D4

  $\Rightarrow$ k - instanton on $U(n)$ YM.

- Appearance of Noncommutativity

  $\Rightarrow$ Existence of Closed string: $G_{\mu\nu}$, $B_{\mu\nu}$, ϕ



Interaction $\langle B_{\mu\nu} \rangle \neq 0$

$$\omega = B_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

$\omega|_{D4}$ if $d\omega = 0$ Symplectic form on D4

$\Rightarrow$ Effective theory

NC - $U(N)$ YM theory (Witten)

Assum $\omega = B_{\mu\nu}\, dx^\mu \wedge dx^\nu \big|_{\mathbb{R}^4}$ is non degenerate

and constant. Define the Moyal product of $f, g \in C(\mathbb{R}^4)$

$$f * g = \mu\left[\exp\left\{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu\right\} f \otimes g\right]$$

$$\begin{cases} \mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} & \text{multiplication} \\ \theta = B^{-1} \end{cases}$$

Effective theory we must consider is

$$S = \frac{1}{4g^2}\int d^4x \, \left(F_{\mu\nu} F^{\mu\nu}\right)_*$$

where $(\ )_*$ means that the product is replaced by $*$-product.

Ex : $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu * A_\nu - A_\nu * A_\mu$

$\Rightarrow$ Structure of differential algebra is not deformed.

$\Rightarrow$ ASD, SD conditions are not deformed.

§ Weyl transform.

Define

$$W : C(\mathbb{R}^4) \ni f \longrightarrow \hat{f} \in \mathcal{A} = C(\mathbb{R}^4_\theta) : \left([\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \right)$$

$$\hat{f} = W[f] = \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k)\, e^{ik_\mu \hat{x}^\mu},$$

where $\tilde{f}(k) = \int d^4x\, f(x)\, e^{-ik_\mu x^\mu}$

Then.

1. $f * g = W^{-1}[W[f]W[g]]$

2. $\hat{\partial}_\mu W[f] = -i\,\theta^{-1}_{\mu\nu}[\hat{x}^\nu, \hat{f}] = W[\partial_\mu f]$

3. Trace can be defined by taking Schrödinger rep. It is proportional to $\int d^4x$ of symbol.

$$\mathrm{Tr}\, W[f] = \int d^4x\, f(x)$$

4. $W[e^{ikx}] = e^{ik\hat{x}}$

5. $$\begin{aligned} \mathrm{Tr}(W[e^{ikx}]W[e^{-ik'x}]) &= \mathrm{Tr}\, W[e^{ikx} * e^{-ik'x}] \\ &= e^{\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu} \int d^4x\, e^{i(k-k')x} \\ &= (2\pi)^4 \delta(k-k'). \end{aligned}$$

§ ADHM

Recent results based on ADHM-construction.

Mechanism behind ADHM.

T-duality : Equivalent target space in String Theory

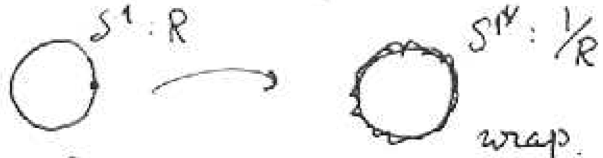
$$S^1 : \quad R \leftrightarrow \frac{1}{R}\cdot\alpha'$$
$$\vdots$$
$$\text{Torus} : \quad T^4 \leftrightarrow T^{4*}$$

dual torus

Under this transformation D-branes are :

(for S^1)

Case 1. if D-brane is $\perp$ to S^1

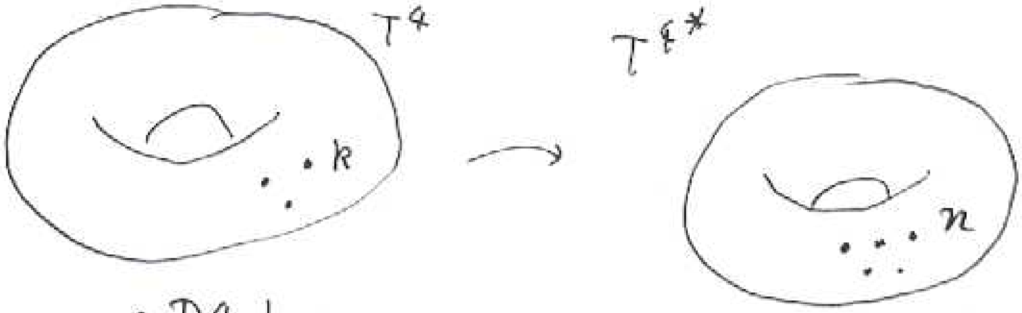


$Dp \longrightarrow Dp+1$

Case 2. if D-brane is $\parallel$ to S^1

$Dp \longrightarrow Dp-1$.

Consider D0 on D4 wrapping T^4



n D4-brane $\qquad$ k D4-brane

$$\begin{array}{c} U(n)\ \text{YM on } T^4 \\ \text{Instanto \# } = k \end{array} \iff \begin{array}{c} U(k)\ \text{YM on } T^{4*} \\ \text{Instanton \# } = n \end{array}$$

Nahm duality

ADHM.

Naively: take limit $T^4 \xrightarrow{R\to\infty} \mathbb{R}^4$ then $T^{4*} \to pt.$

$$\begin{array}{c} U(n)\ \text{YM on } \mathbb{R}^4 \\ \text{with Instanton\#} = k \end{array} \iff U(k)\ \text{YM on pt}$$

$\Downarrow$

Matrix eq.

$\xleftarrow{\quad\quad}$
ADHM const.

$\xrightarrow{\quad\quad}$
Inversion

"Reciprocity"

7.1
7

What are typical properties in NC case.

$\mathbb{R}^4_\theta \quad \longleftrightarrow \quad pt$ duality.

ADHM construction is briefly.

Instanton : ASD Gauge field $F = -*F$ (: Hodge $*$)

extremum of the Action

$$S = \frac{1}{4}\int (F_{\mu\nu})^2 d^4x \propto \text{Instanton } \#$$

⇒ Algebraic construction of ASD (SD) gauge field.

⇒ Construction of Projective module.

Define projection operator

$$P : V_x(n+2k) \to V_x(n)$$

1. For this take $(n+2k)\times 2k$ matrix Δ as

$$\Delta = a + bx$$

$$= \underbrace{\begin{pmatrix} \lambda \\ \mu \end{pmatrix}}_{2k} \begin{matrix} \}n \\ \}2k \end{matrix} + \begin{pmatrix} 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} (1_k \otimes x)$$

Quartanionic : $x = S_\mu x^\mu = x^0 - i\sigma_k x^k = \begin{pmatrix} x_0 - ix_3, & -x_2 - ix_1 \\ x_2 - ix_1, & x_0 + ix_3 \end{pmatrix}$

x^k : coordinate of R^k

- The Curvature is then

$$\Theta = P\,dP\,dP \qquad P = 1 - \Delta K \Delta^{\dagger}, \quad K = (\Delta^{\dagger}\Delta)^{-1}$$

$$= P\,dP\,(1-P)\,dP$$

$$= P\,dP\,\Delta K \Delta^{\dagger}\,dP$$

$$= P\,d\Delta\,K\,d\Delta^{\dagger}\,P \quad = \quad dx^{\mu}\wedge dx^{\nu}\ P S_{\mu} K \bar{S}_{\nu} P$$

If $[S_{\mu}, K] = 0$ ---- ($*$)

then $\quad \Theta = dx^{\mu}\wedge dx^{\nu}\ P S_{\mu}\bar{S}_{\nu} K P$

$S_{[\mu}\bar{S}_{\nu]}$ is ASD

($*$) leads to ADHM eq.

① μ is quaternionic : $\mu = \mu^{\alpha} S_{\alpha}$ $\quad \mu^{\alpha}$: $k\times k$ matrix

ADHMeq.

$$\mathrm{Tr}_2 \{ \sigma^{a}_{AB} \lambda^{\dagger}_{A}\lambda_{B} \} + i\,([\mu_{\alpha}, \mu_{\beta}] + [\chi_{\alpha}, \chi_{\beta}])\,\eta^{a}_{\alpha\beta} = 0$$

$\eta^{a}_{\alpha\beta}$: selfdual tensor

$$\eta^{a}_{\alpha\beta} = \delta_{a\alpha}\delta_{4\beta} - \delta_{4\alpha}\delta_{a\beta} + \epsilon_{\alpha\beta a4}$$

$[\mu_{\alpha}, \mu_{\beta}]$ is ASD upto source term.

~~$\lambda = (I, J^{\dagger})$~~

$$\mu = \begin{pmatrix} B_1 & -\bar{B}_2 \\ B_2 & \bar{B}_1 \end{pmatrix} \quad : \quad B_i \in k\times k \text{ matrix}$$

$I, J^{\dagger} \in k \times n$ matrix.

$U(n)$ Gauge field A_μ can be given by applying $v^\dagger$

$$v^\dagger : V_x(n+2k) \longrightarrow V_x(n)$$

$$\xi \in \Gamma = PA^{n+2k} \quad : \quad \xi(x) \in V_x(n+2k)$$

$v^\dagger \xi = \hat{\xi}$ is normal matter field.

$$v^\dagger \nabla \xi = v^\dagger d P \xi = v^\dagger d v v^\dagger \xi$$

$$= d\hat{\xi} + \underbrace{(v^\dagger d v)}_{A = -iA_\mu dx^\mu} \hat{\xi}$$

We got connection : ADHM.

Inversion (commutative case)

knowing an instanton solution A_μ

$\Rightarrow$ derive (μ, λ) : ADHM data

1. find k. zeromode of chiral Dirac eq.

$$\bar{D}\psi = \bar{\sigma}^\mu D_\mu \psi = \bar{\sigma}^\mu(\partial_\mu - iA_\mu)\psi = 0$$

where ψ is 2-component spinor ($2k \times n$ matrix)

· k-zeromodes : ψ_A is $k \times n$, $A = 1, 2$.

$$\psi_A = \left(\psi_A^{(1)} \cdots \psi_A^{(k)} \right)$$

· Normalize them

$$\int d^4x \, \psi_A^\dagger(x)\psi_A(x) = 1_k$$

Then ADHM data is given by

$$\mu_\alpha = - \int x_\alpha \psi_A^\dagger \psi_A \, d^4x$$

$$\psi_A \xrightarrow[x \to \infty]{} (\lambda^\dagger \bar{x} \epsilon) \frac{1}{4\pi x^4}$$

So we have a loop

$\boxed{\text{ASD}}$

$A_\mu = i v^\dagger \partial_\mu v$

$F = -*F$

$D^\dagger_{AB}\psi_B = 0$: 4dim Chiral Dirac eq.

$D^\dagger_{AB} = \bar{s}^\alpha_{AB}(\partial_\alpha - iA_\alpha)$

k-zeromodes ψ_A

$\int \psi_A^\dagger \psi_A = 1_k$

Inversion

$\mu_\alpha = \int x_\alpha \psi_A^\dagger \psi_A$

$\psi_A \sim g(\lambda_A^\dagger \bar{x} \epsilon)\frac{1}{\pi |x|^2}$

$\eta_{\alpha\beta}([\mu_\alpha, \mu_\beta] + [x_\alpha, x_\beta]) = i\,\mathrm{Tr}_2(\sigma^a \lambda^\dagger \lambda)$

$\boxed{\text{ASD}}$ upto source term.

0-dim Dirac eq.

$\Delta^\dagger v = 0$

$\Delta = \begin{pmatrix}\lambda \\ \mu\end{pmatrix} + \begin{pmatrix}0 \\ x\end{pmatrix}$

n-zeromodes

$v^\dagger v = 1_n$, $P = v v^\dagger$

ADHM

To prove :

Assume ASD gauge field A_μ

1. Construct zero mode ψ_A
2. Show normalization $\mathrm{Tr}\, \psi^\dagger_A \psi_A = 1_k$
3. Show that we get

   ADHM eq. for $\mu_\alpha = \mathrm{Tr}\, x_\alpha \psi^\dagger_A \psi_A$

   $\psi \longrightarrow (\lambda^\dagger \bar{x} \epsilon) \frac{1}{\pi |x|^4}$

4. And ?

1. Construct zeromodes ψ_A

Dirac equation is

$$i P \bar{\sigma}^\alpha \hat{\partial}_\alpha \psi = P \bar{\sigma}^\alpha \theta^{-1}_{\alpha\beta} [\hat{x}^\beta, \psi]$$

We see the following gives zeromodes.

$$\psi_A = \theta P b_B K \epsilon_{BA} \quad : \quad b = \begin{pmatrix} 0 & 0 \\ 1_k & \\ & 1_k \end{pmatrix} = (b_1 \; b_2)$$

$$K \text{ is } k \times k \text{ matrix} \quad : \quad K \otimes 1_2 = (\Delta^\dagger \Delta)^{-1}$$
$$P = vv^\dagger = 1 - \Delta K \Delta^\dagger$$

$\Rightarrow$ k zeromodes.

Proof

$$P [\hat{x}^\alpha, P b_B K]$$

$$P [\bar{z}_A, P b_B K] = P [\bar{z}_A, \Delta_{A'}] K \Delta^\dagger_{A'} b_B K - P b_B K [\bar{z}_A, K^{-1}] K$$

$$= \theta P b_C B^\dagger_C K \Delta^\dagger_1 b_B K^{-1} - P b_B K [\bar{z}_A, \Delta^\dagger \Delta] K$$

$$= 0$$

etc.

2. Show nomalization

for this we use the relation

$$4\,[(\theta^{-1}\hat{x})^{\alpha},[(\theta^{-1}\hat{x}^{\alpha}),K]] = K b_A^{\dagger} P b_A K = \frac{1}{\theta^2}\psi^{\dagger}{}_A \psi_A$$

To prove nomalization, we take standard form of θ

$$\theta = \begin{pmatrix} & -\theta_1 & & \\ \theta_1 & & & \\ & & & -\theta_2 \\ & & \theta_2 & \end{pmatrix}$$

and go to complex coordinate z_1, z_2 :

$$x = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}$$

Show $\mathrm{Tr}\,[z_A,[\bar{z}_A, K]] = 1 = \mathrm{Tr}\,\psi_A^{\dagger}\psi_A$

2 ways to prove

a) Use Fock space representation and show

$$\lim_{N\to\infty}\sum^{N}\sum_{n',n=0}^{N=n+n'} \langle n, n'|[z_A\,[\bar{z}_A, K]]|n n'\rangle$$

we use "stokes" theorem

$$\sum_{n=0}^{N} \langle n|[a, K]|n\rangle = \langle N+1|\,K a\,|N+1\rangle$$

$$\sum_{n=0}^{N} \langle n|[a,[a^{+},K]]|n\rangle = (N+1)\left(\langle N|K|N\rangle - \langle N+1|K|N+1\rangle\right)$$

Stokes' theorem

$$\mathrm{Tr}_U \mathcal{O} = \sum_{n \in U} \langle n | \mathcal{O} | n \rangle$$

$$U = (0, 1, \cdots, Q)$$

$$\mathrm{Tr}_U \{ \mathcal{O} \} = \sum_{n=0}^{Q} \langle n | \mathcal{O} | n \rangle$$

$$\mathrm{Tr}\{[a, \mathcal{O}]\} = \langle Q+1 | \mathcal{O} a | Q+1 \rangle$$

proof

$$\begin{aligned} \mathrm{Tr}\{[a, \mathcal{O}]\} &= \sum_{0}^{Q} \langle n | a \mathcal{O} | n \rangle - \langle n | \mathcal{O} a | n \rangle \\ &= \langle Q | a \mathcal{O} | Q \rangle - \langle Q | \mathcal{O} a | Q \rangle \\ &\quad + \langle Q-1 | a \mathcal{O} | Q-1 \rangle - \langle Q-2 | \mathcal{O} a | Q-2 \rangle \\ &\quad \vdots \\ &\quad - \langle 0 | \mathcal{O} a | 0 \rangle \\ &= \langle Q | a \mathcal{O} | Q \rangle = \langle Q+1 | \mathcal{O} a | Q+1 \rangle \end{aligned}$$

$$\circledast \quad \mathrm{Tr}_U [a [a^{\dagger} f]] = (Q+1) \{ f(Q) - f(Q+1) \}$$

↑ Boundary value.

We take then $Q \to \infty$ limit.

b) using Weyl tr. 18

$$W^{-1}\left(\frac{1}{\theta^2}[\hat{z}_A[\hat{z}_A, K]]\right) = -W^{-1}[\delta^2 K] = -\partial^2 W^{-1}[K]$$

$$\mathrm{Tr}\,\psi_A^+\psi_A = -\theta^2 N\int d^4x\ \partial^2 W^{-1}[K] \quad , \ K = (\hat{x}^2 + 2\hat{x}\mu + \mu^2)$$

$$= -\theta^2 N\int d^4x\ \partial^2\left\{\frac{1}{x^2} + O\left(\frac{1}{x^3}\right) + \cdots\right.$$

$$= 4\pi^2\theta^2 N.$$

$$= 1$$

$N = \frac{1}{\sqrt{\det 2\pi\theta}}$: trace of Schrödinger rep.

2.) ADHM - data from Instanton

Using the zeromode solution ψ_A, we prove that

$$\mu^\alpha = -\mathrm{Tr}\, \hat{x}^\alpha \psi_A^\dagger \psi_A$$

again we use the "Stokes'" theorem.

and we need higher term of expansion of K

$$K = (\hat{x}^\dagger \hat{x} + \hat{x}^\dagger \mu + \mu^\dagger \hat{x} + \mu^\dagger \mu)^{-1}$$

$$= \frac{1}{|\hat{x}|^2} - \frac{1}{\hat{x}^2} (\mu^\dagger \hat{x} + \hat{x}^\dagger \mu) \frac{1}{\hat{x}^2} + \cdots$$

where $\hat{x}^2 = \bar{z}_A z_A = \theta(N_1 + N_2 + \text{cond})$: diagonal matrix.

Then we need to evaluate

$$\mathrm{Tr}\{ z_A \psi_B^\dagger \psi_B \} = \mathrm{Tr}\{ z_A [z_B [\bar{z}_B , K]]\}$$

$$= \mathrm{Tr}[z_B, \underbrace{[\bar{z}_B , z_A K]]}_{ex.} - [\bar{z}_B , z_A] \mathrm{Tr}[z_B , K]$$

ex.

$$z_A K = \underset{\downarrow \atop 0}{z_A \frac{1}{\hat{x}^2}} + \underset{\downarrow \atop \sim B}{z_A \frac{1}{\hat{x}^2} (z_B \bar{B}_B + \bar{z}_B B_B)} \frac{1}{\hat{x}^2} + \cdots$$

$$\delta^\alpha \mu^\alpha = \begin{pmatrix} B_1 & -\bar{B}_2 \\ B_2 & \bar{B}_1 \end{pmatrix}$$

$\mathrm{Tr}[z [z , z_A K]]$

First term $\cdots \sim \langle n | z_A f(n) | n \rangle = 0$

Second term $\cdots \sim B_A$

3) Finally we must prove that

μ^α defined by ψ_A as

$$\mu^\alpha = -\mathrm{Tr}_{\mathcal{H}} \ \hat{x}^\alpha \psi_A^\dagger(x) \psi_A(x)$$

satisfies ADHM eq.

$\Rightarrow$

$$\mu^\alpha \mu^\beta = \mathrm{Tr}_{\mathcal{H}_1} \mathrm{Tr}_{\mathcal{H}_2} \left(\hat{x}^\alpha \psi_A^\dagger(x) \psi_A(x) \otimes \hat{y}^\beta \psi_A^\dagger(y) \psi_A(y) \right)$$

Note : we distinguish : 1st element of $\ldots \otimes \ldots$ by $\hat{x}^\alpha$
2nd by $\hat{y}^\alpha$

$$\Rightarrow \quad [\hat{x}^\alpha, \hat{y}^\beta] = 0$$

i.e. $\hat{x}^\alpha \sim \hat{x}^\alpha \otimes 1$
$\hat{y}^\alpha \sim 1 \otimes \hat{y}^\alpha$

To proceed the proof we need the following formula including propagator in NC theory.

$$\psi_A(x) \otimes \psi_B^\dagger(y) = P_x \hat{\delta}(\hat{x} - \hat{y}) P_y - P_x \bar{\sigma}^\alpha \partial_\alpha \hat{G} \overleftarrow{\partial}_\alpha \bar{\sigma}^\alpha P_y$$

where $\hat{\delta}$, $\hat{G}$ $\in \mathcal{A} \otimes \mathcal{A}$

$$\hat{\delta} = \int \frac{d^4k}{(2\pi)^4} \; e^{ik\hat{x}} \otimes e^{-ik\hat{y}}$$

$$\Rightarrow \quad W^{-1} \otimes W^{-1} [\hat{\delta}] = \delta^4(x-y)$$

$$\hat{G} = P_x \hat{G}_0 P_y$$

$$\hat{G}_0 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \; e^{ik\hat{x}} \otimes e^{-ik\hat{y}} \; .$$

$$\hat{\partial}_\mu \hat{\partial}_\mu \hat{G}_0 = - \hat{\delta}$$

$\Rightarrow$ Lee-Yang
hepth/0206001

It is rather complicated but

$$W^{-1} \otimes W^{-1} [\hat{G}_0] = G_0 = \frac{1}{4\pi^2 |x-y|^2}$$

Substituting this we get

$$\mu^\alpha \mu^\beta = \mathrm{Tr}_1\, \mathrm{Tr}_2 \left(x^\alpha \;\; \psi_A^\dagger (\psi_A(x) \otimes \psi_B^\dagger(y)) \psi_B y^\beta \right)$$

$$= \mathrm{Tr}_1\, \mathrm{Tr}_2 \left(x^\alpha \;\; \psi_A^\dagger \left(\delta_{AB}\, \hat{\delta}(x-y) - P\sigma\partial\, \hat{G}\, \bar{\sigma}\overleftarrow{\partial} P \right) \psi_B y^\beta \right)$$

1st term.

$$\mathrm{Tr}_1\, \mathrm{Tr}_2 \left(x^\alpha \;\; \psi_A^\dagger\, \delta(x-y)\, \psi_A\, y^\beta \right)$$

$$= \mathrm{Tr} \left(x^\alpha\, \psi_A^\dagger \psi_A\, x^\beta \right)$$

Thus

$$[\mu^\alpha, \mu^\beta] = -\mathrm{Tr}\,[x^\alpha, x^\beta]\, \psi_A^\dagger \psi_A \quad + \quad G \text{ part}$$

$$= -[x^\alpha, x^\beta] \quad + \quad G. \text{ part}.$$

To evaluate G-part. we use Weyl symbol.

and then trace is replaced by $\int d^4x \int d^4y$.

$\Rightarrow$ partial integration can be evaluate with surface

term as. $\int d^4x\, \partial_\alpha \sim \int_{R\to\infty} dS^3_\alpha$

$$\Rightarrow \lambda \;:\; v^\dagger \psi_A \to g(x_A^\dagger \bar{x} e) \frac{1}{\pi |x|^4}$$