Workshop on Noncommutative Geometry and QFT
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Equivariant cohomology and instantonic calculus

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Equivariant cohomology

of instanton moduli space

+

supermanifold techniques

=

computation of correlation functions in

topological super Yang-Mills

Equivariant Forms and the Localization Formula

M an n-dimensional manifold acted on by a Lie group G, $\mathfrak{g} = \text{Lie}(G)$. If $\xi \in \mathfrak{g}$

$$\xi^* = \left[\frac{d}{dt}\rho(-t\exp\xi)\right]_{t=0} = \xi^\alpha T^i_\alpha \frac{\partial}{\partial x^i} \in \mathcal{X}(M)$$

Grading in $\Omega(M,\mathfrak{g}) = \mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$

$$\deg(P \otimes \beta) = 2\deg(P) + \deg(\beta).$$

Action of G on $\Omega(M,\mathfrak{g})$

$$(g\cdot\alpha)(\xi) = {\rho_g}^*(\alpha(Ad_{g^{-1}}\xi))$$

$$\Omega_G(M) = (\Omega(M,\mathfrak{g}))^G = \text{equivariant forms}$$

Equivariant differential

$$d_{\mathfrak{g}}\colon \Omega(M,\mathfrak{g})^\bullet \to \Omega(M,\mathfrak{g})^{\bullet+1}$$

$$(d_{\mathfrak{g}}\alpha)(\xi) = d(\alpha(\xi)) - i_{\xi^*}\alpha(\xi)$$

${d_{\mathfrak{g}}}^2 = 0$ on $\Omega_G(M)$ since

$$({d_{\mathfrak{g}}}^2\alpha)(\xi) = -\mathcal{L}_{\xi^*}(\alpha(\xi)) = 0.$$

Example: ω symplectic form, G acts by symplectomorphisms, $\exists$ moment map $\mu\colon M \to \mathfrak{g}^*$:

$$\alpha = \mu + \omega \in \Omega^2_G(M), \qquad d_{\mathfrak{g}}\alpha = 0.$$

M, G compact ($n = \dim M$)

$M_\xi =$ [redacted] {zeroes of ξ^*} (assume it is finite)

$$\alpha \in \Omega_G(M), \qquad d_{\mathfrak{g}}\alpha = 0$$

If $p \in M_\xi$ define

$$L_p \colon T_pM \to T_pM, \qquad L_p(v) = [\xi^*, v]$$

$$\int_M \alpha(\xi) = (-2\pi)^{n/2} \sum_{p \in M_\xi} \frac{\alpha(\xi)_0(p)}{\det^{1/2} L_p}.$$

Example: if $\alpha = \mu + \omega$ we get the Duistermaat-Heckman formula:

$$\int_M e^{i\mu}\, d\beta = \sum_{p \in M_\xi} \frac{e^{i\mu(p)}}{\det^{1/2} L_p}$$

Here $d\beta$ is the symplectic volume form

Supermanifolds

Naively: given a differentiable manifold X with local coordinates $x^1, \dots, x^n$, add "fermionic" coordinates $\theta^1, \dots, \theta^n$. Superfunction:

$$
\begin{aligned}
f = {} & f_0(x) \\
& + \textstyle\sum_{\alpha=1}^{n} f_\alpha(x)\,\theta^\alpha && (\mathcal{N}/\mathcal{N}^2) \\
& + \textstyle\sum_{\alpha<\beta} f_{\alpha\beta}(x)\,\theta^\alpha\theta^\beta + \dots && \left.\right\} (\mathcal{N}^2) \quad \left.\right\} (\mathcal{N}) \\
& + f_{1\dots n}(x)\theta^1\cdots\theta^n
\end{aligned}
$$

Intrinsically: $\mathfrak{X} = (X, \mathcal{A})$, X is an m-dim. mnfd, $\mathcal{A}$ is a sheaf of $\mathbb{Z}_2$-graded commutative algebras on X, with:

1. A sheaf surjection $\sigma\colon \mathcal{A} \to C^\infty_X$
2. if $\mathcal{N}$ is the nilpotent subsheaf of $\mathcal{A}$, the quotient $\mathcal{N}/\mathcal{N}^2$ is a rank n vector bundle $\mathcal{E}$
3. $\mathcal{A} \overset{\text{loc}}{\simeq} \Lambda^\bullet\mathcal{E}$, compatibly with σ.

$$\mathbb{T}\mathfrak{X} = \mathcal{D}er_{\mathbb{R}}\mathcal{A} \simeq \mathcal{A} \otimes [TX \oplus \mathcal{E}^*]$$

$$\mathbb{T}_p\mathfrak{X} = \frac{\mathcal{D}er_{\mathbb{R}}\mathcal{A}}{\mathcal{M}_p \mathcal{D}er_{\mathbb{R}}\mathcal{A}} \simeq T_pX \oplus \mathcal{E}_p^*$$

Berezin integration

$\mathfrak{X} = (X, \mathcal{A})$ an (m, n) dim smnfd (X oriented)

$\Omega^m_{\mathfrak{X}} = m$-superforforms on $\mathfrak{X}$

$\mathcal{P}_n =$ graded diff. ops. of order n on $\mathcal{A}$.

$\Omega^m_{\mathfrak{X}}$ and $\mathcal{P}_n$ have natural graded left $\mathcal{A}$-module structure (multipl. by superfncts)

$\mathcal{P}_n$ has a (inequivalent) right $\mathcal{A}$-module structure $(D \cdot f)(g) = D(fg)$

One considers $\Omega^m_{\mathfrak{X}} \otimes_{\mathcal{A}} \mathcal{P}_n$.

Submodule $\mathcal{K} = \{\omega \mid \widetilde{\omega(f)} = \text{exact form for every compactly supported superfunction } f\}$

$$\boxed{\mathrm{Ber}\mathfrak{X} = \Omega^m_{\mathfrak{X}} \otimes_{\mathcal{A}} \mathcal{P}_n / \mathcal{K}}$$

$\mathrm{Ber}\mathfrak{X}$ is a super line bundle of rank (1,0) (resp. (0,1)) if n is even (resp. odd)

$$\int_{\mathfrak{X}} \omega = \int_X \widetilde{\lambda(1)}$$

Given a local coordinate system $(x^1, \ldots, x^m, \theta^1, \ldots, \theta^n)$,

$$\omega = \left[dx^1 \wedge \ldots \wedge dx^m \otimes \frac{\partial}{\partial \theta^1} \cdots \frac{\partial}{\partial \theta^n}\right] f$$

$$\int_{\mathfrak{X}} \omega = \int_X f_{1 \ldots n}\, dx^1 \ldots dx^m$$

Tautological smnfds & localization

If $\mathcal{A} = \Lambda^\bullet T^*X = \Omega_X$ we have a supermanifold $\mathfrak{X}$ "tautologically" associated to X*

$$T\mathfrak{X} \simeq \mathcal{A} \otimes (TX \oplus TX)$$

$$\xi^* = \xi^\alpha T^i_\alpha \frac{\partial}{\partial x^i} \quad \text{v.f. on } X$$

$$\widehat{\xi^*} = \xi^\alpha T^i_\alpha \frac{\partial}{\partial x^i} + \xi^\alpha \theta^j \frac{\partial T^i_\alpha}{\partial x^j} \frac{\partial}{\partial \theta^i} \text{ even s.v.f. on } \mathfrak{X}$$

$$Q_\xi = d + \Pi(\xi^*) = \theta^i \frac{\partial}{\partial x^i} + \xi^\alpha T^i_\alpha \frac{\partial}{\partial \theta^i} \text{ odd s.v.f. on } \mathfrak{X}$$

$$[Q_\xi, Q_\xi]_+ = 2\widehat{\xi^*}$$

Under the isomorphism $\tau\colon \Omega_X \xrightarrow{\sim} \mathcal{A}$:

1. the equivariant differential $d_{\mathfrak{g}}$ corresponds to the action of Q_ξ as a derivation, i.e.,

$$\tau((d_g\alpha)(\xi)) = Q_\xi(\tau(f)(\xi));$$

2. the Lie derivative $\mathcal{L}_{\xi^*}$ corresponds to the action of $-\widehat{\xi^*}$ as a derivation, i.e.,

$$\tau(\mathcal{L}_{\xi^*}(\alpha(\xi))) = -\widehat{\xi^*}(\tau(\alpha)(\xi)).$$

*Already used in connection with TQFT in U.B., F. Fucito et al., Nucl. Phys. B **611** (2001) 205

Considering the induced action of G on $\mathcal{A}$, *

$$\Omega_G \Leftrightarrow \mathcal{A}_G = \ker(\widehat{\xi^*}), \qquad d_{\mathfrak{g}} \Leftrightarrow Q_\xi$$

The Berezinian bundle of a 'tautological' supermanifold $\mathfrak{X}$ has a canonical global section

$$\Theta = \left[dx^1 \wedge \ldots \wedge dx^n \otimes \frac{\partial}{\partial\theta^1} \cdots \frac{\partial}{\partial\theta^n}\right]$$

X, G compact, F a G-equivariant element in $\mathcal{A}_G$, $Q_\xi(F(\xi)) = 0$, ξ^* only isolated zeroes.

$$\boxed{\int_{\mathfrak{X}} \Theta\, F(\xi) = \sum_{p \in X_\xi} \frac{\widetilde{F(\xi)(p)}}{|\mathrm{Sdet}(L_p \circ \Pi)|}}$$

$$\Pi \colon T\mathfrak{X} \to T\mathfrak{X} \qquad \text{(change of parity)}$$

$$L_p \colon T_p\mathfrak{X} \to T_p\mathfrak{X}, \qquad L_p(v) = \widetilde{[Q_\xi, v]};$$

$$\text{if} \quad v = v_0^i \frac{\partial}{\partial x^i} + v_1^j \frac{\partial}{\partial \theta^j},$$

$$L_p(v) = -v_1^j \frac{\partial}{\partial x^j} - \xi^\alpha\, v_0^i \left(\frac{\partial T_\alpha^j}{\partial x^i}\right)_p \frac{\partial}{\partial \theta^j}.$$

* $\hat{\xi}^*$ acting on $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}$ as
$\hat{\xi}^*(f)(\xi) = \hat{\xi}^*(f(\xi))$

If a submanifold $Y \subset X$ is locally given by equations

$$f_1 = \ldots = f_r = 0$$

then $\mathfrak{Y} = (Y, \mathcal{B} = \Omega_Y)$ is a sub-supermanifold of $\mathfrak{X} = (X, \mathcal{A} = \Omega_X)$ locally given by equations

$$f_1 = \ldots = f_r = 0, \quad g_1 = \ldots = g_r = 0$$

with

$$g_j(x, \theta) = \sum_{i=1}^{\dim X} \frac{\partial f_j}{\partial x^i} \theta^i$$

this provides a recipe to 'supersymmetrize' a constrained submanifolds; the constraints on the 'fermionic' variables are the linearizations of the 'bosonic' constraints.

Instanton moduli space and ADHM data

Instantons: anti-self-dual $SU(N)$ connections on $\mathbb{R}^4$ with a fixed framing at infinity.

Moduli space of instantons modulo gauge equivalence is a noncompact, singular manifold of dimension $4kN$, with k = second Chern class (instanton number).

Smooth part has a hyperkähler structure (induced by the standard hyperkähler structure of $\mathbb{R}^4$).

ADHM description: $B_1, B_2 \in \mathrm{Mat}_{\mathbb{C}}(k, k)$, $I \in \mathrm{Mat}_{\mathbb{C}}(N, k)$, $J \in \mathrm{Mat}_{\mathbb{C}}(k, N)$

Constraints:

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0$$

$$[B_1, B_2] + IJ = 0$$

$$\mathcal{M}_0 = \{B_1, B_2, I, J \,|\, \text{constraints}\}/\mathrm{U}(k)$$

$\mathcal{M}_0^{\mathrm{reg}}$ is obtained by taking only the data with trivial stabilizer

There is a natural resolution of the singularities (which can be described by a (hyperkähler) quotient construction).

In supersymmetric gauge theories one supplements the ADHM data by fermionic moduli provided by the zero modes of the gaugino field.

For $\mathcal{N} = 2$ fermionic moduli can be identified with differential forms on the bosonic moduli space $\Rightarrow$ *Tautological supermanifold associated with the bosonic moduli space.*

The constraints on the fermionic data are obtained by linearizing the bosonic constraints.

The multi-instanton action is obtained by plugging into the SYM action the bosonic and fermionic zero modes in terms of the (unconstrained) ADHM data and imposing the ADHM constraints via Lagrangian multipliers.

The resulting action turns out to be **BRST-exact**

$\mathcal{N} = 2$ Supersymmetric Theories with Group $SU(N)$

Field content: B_1, B_2, I, J

+ fermionic partners M_1, M_2, μ_I, μ_J
+ Lagrange multipliers $H_{\mathbb{R}}$, $H_{\mathbb{C}}$
+ their fermionic partners $\chi_{\mathbb{R}}$, $\chi_{\mathbb{C}}$
+ aux. bosonic field $\bar{\phi}$ with its partner η

If we consider only the action of the group $U(k) \times SU(N)$ the fixed points are not isolated $\Rightarrow$ (Nakajima) consider additional T^2 action

$$(B_1, B_2, I, J) \quad \mapsto \quad (\epsilon_1 B_1,\ \epsilon_2 B_2,\ (\epsilon_1 + \epsilon_2) I,\ (\epsilon_1 + \epsilon_2) J)$$

$$\xi^* = (\phi I - Ia)\frac{\partial}{\partial I} + (-J\phi + aJ + \epsilon J)\frac{\partial}{\partial J} + ([\phi, B_\ell] + \epsilon_\ell)\frac{\partial}{\partial B_\ell}$$

$$
\begin{aligned}
Q_{\xi} &= \mu_I \frac{\partial}{\partial I} + \mu_J \frac{\partial}{\partial J} + M_\ell \frac{\partial}{\partial B_\ell} + [\phi, \chi_{\mathbb{R}}] \frac{\partial}{\partial H_{\mathbb{R}}} \\
&+ ([\phi, \chi_{\mathbb{C}}] + \epsilon \chi_{\mathbb{C}}) \frac{\partial}{\partial H_{\mathbb{C}}} + \eta \frac{\partial}{\partial \bar{\phi}} \\
&+ (\phi I - I a) \frac{\partial}{\partial \mu_I} + (-J\phi + aJ + \epsilon J) \frac{\partial}{\partial \mu_J} \\
&+ ([\phi, M_\ell] + \epsilon_\ell M_\ell) \frac{\partial}{\partial M_\ell} \\
&+ H_{\mathbb{R}} \frac{\partial}{\partial \chi_{\mathbb{R}}} + H_{\mathbb{C}} \frac{\partial}{\partial \chi_{\mathbb{C}}} + [\phi, \bar{\phi}] \frac{\partial}{\partial \eta}.
\end{aligned}
$$

$(\epsilon = \epsilon_1 + \epsilon_2)$

= infinitesimal generator of the BRST transformations!

CRITICAL POINTS - CASE N=1

Instantons with instanton charge k are replaced by configurations of k points in $\mathbb{C}^2$ (unordered)

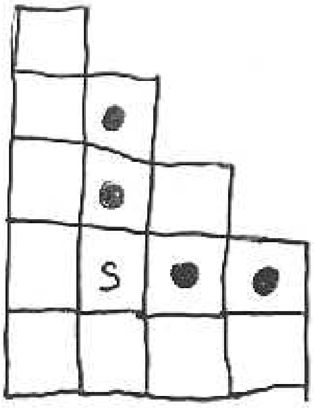
$(\mathbb{C}^2)^{[k]}$ desingularization of $S^k(\mathbb{C}^2)$

$$(\mathbb{C}^2)^{[k]} \simeq \left\{ B_1, B_2, I \;\middle|\; \begin{array}{l} [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + I I^\dagger = 0 \\ [B_1, B_2] = 0 \end{array} \right\} \Big/ U(k)$$

+ stability condition

Here $G = U(k) \times T^2$

To each critical point of ξ^* one associates a tableau describing the irreducible decomposition
Tang of the tangent space to moduli space at the critical point



$l(s)$ = number of ●

$a(s)$ = number of •

$$\det{}^{1/2} L_p = \mathrm{Sdet}(\hat{L}_p \cdot \Pi) = \prod_{s \in D} \varepsilon_1 (l(s) + a(s) + 1) \cdot \varepsilon_2 (-l(s) + a(s) + 1)$$

Critical points:

$$\begin{aligned}(\varphi_{IJ}+\epsilon_\ell)\,B^\ell_{IJ} &= 0\\ (\varphi_I - a_\lambda)\,I_{I\lambda} &= 0\\ (-\varphi_I + a_\lambda + \epsilon)\,J_{I\lambda} &= 0\end{aligned}$$

To each critical point one associates a set of N Young Tableaux $(Y_1,\ldots Y_N)$ with $k = \sum_\lambda k_\lambda$ boxes distributed between the Y_λ's. The boxes in adiagram Y_λ are labelled either by the instanton index $I_\lambda = 1,\ldots,k_\lambda$ or by the pair of integers i_λ, j_λ denoting the vertical and horizontal position in the Young diagram.

$$\varphi_{I_\lambda} = \varphi_{i_\lambda j_\lambda} = a_\lambda - (j_\lambda - 1)\epsilon_1 - (i_\lambda - 1)\epsilon_2$$

$J = B_\ell = I = 0$ except for the components $B_{1(i_\lambda j_\lambda),(i_\lambda j_\lambda+1)}$, $B_{2(i_\lambda j_\lambda),(i_\lambda+1 j_\lambda)}$, $I_{\lambda,(i_\lambda=j_\lambda=1)}$.

These values are fixed by the ADHM constraints.

We are now ready to apply the localization formula

$$\begin{aligned}\mathcal{Z}_k &= \int \frac{\mathcal{D}\phi}{U(k)}\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{F}e^{-S}\\ &= \int \prod_I d\varphi_I \frac{\prod_{I\neq J}\varphi_{IJ}}{\text{Sdet}\,\mathcal{L}} = \sum_{x_0}\frac{1}{\text{Sdet}(\Pi\circ L_{x_0})}\end{aligned}$$

$$
\begin{aligned}
\mathcal{Z}_k &= \sum_{x_0} \frac{1}{\mathrm{Sdet}(\Pi \circ L_{x_0})} \\
&= \sum_{\{Y_\lambda\}} \prod_{\lambda,\tilde{\lambda}}^{N} \prod_{s \in Y_\lambda} \frac{1}{E(s)(E(s)-\epsilon)}
\end{aligned}
$$

$$E(s) = a_{\lambda\tilde{\lambda}} - \epsilon_1 h(s) + \epsilon_2 (v(s)+1)$$

$$h(s) = \nu_i - j \qquad v(s) = \tilde{\nu}'_j - i$$

$\nu_{i_\lambda}, \nu'_{j_\lambda}$ denote the length of the i_λ-th columm and j_λ-th row respectively

Notice that $\tilde{\nu}'_{j_\lambda}$ is defined only for $j_\lambda \leq \tilde{\nu}_{1_\lambda}$. For $j_\lambda > \tilde{\nu}_{1_\lambda}$ we take $\tilde{\nu}'_{j_\lambda} = 0$. $h(s)$ $(v(s))$ is the number of black (white) circles in the following picture.