NON COMMUTATIVE GEOMETRY & QFT.

I ?.

O(3) σ model, Hopf term & Fuzzy Sphere.

TRG

1. motivation
2. Setting, past results, review ~ O(3) σ model
3. Topological features, Soliton #, Hopf term.

4a Noncommutative Fuzzy sphere. - CP^1 formulation
4b $SU_q(2)$
5. CONCLUSIONS - FUTURE DIRECTIONS.

Work ~ done & in progress

with E Harikumar (IMSc)

hep-th / 0212 ...

1. Noncommutative Geometry.

~ good regularisation.

~ topological features and symmetries

~ alternative to lattice, nonperturbative.

~ One should develope more realistic models than toy models and achieve results.

~ 2+1 D. model $O(3)$ / CP^1

~ interesting features. topological, asymptotic free
-solitons - dim etc

~ useful in condensed matter systems

~ Ferromagnetic or Antiferromagnetic.

~ Suitable model - both as Toy Model and
- real physics.

~ Fuzzy regularisation can be used to set up the model as a matrix model and analysis can be done ~ analytically to the extent possible or numerically when needed.

Wilzcek & Zee

— Solitons acquire fractional Spin & Statistics. (1983)
(PRL, 50)

Haldane — Long wavelength fluctuations of antiferro-magnets
→ NLSM. (PRL, 50) (1983)

Bowick, Dimitrakarkbali, Wijayavardhana (1986)
— canonical frame work. — Hopf term adds a piece to angular momentum and changes Statistics.
(NPB 257)

Dzyaloshinsky, Polyakov, Wiegmann. — in the Long wavelength lt effect of Hopf term is to add a Chern Simons term in the model. (1988)
(PL, A127)

TRG, R. Shankar — With Torus boundary Conditions novel features — with "Non Abelian Stat"? (1989)
(Mod Phy L A4)

TRG, Shaji, Shankar, Sivakumar. — "equivalent" with θ ~ Hopf term ~ = $\pi/2s$ to Higher Spin Interacting theory (PRL69)(1992)

Involves ~~—~~ functional integrals — with "handwaving" arguments!? (Intutive arguments)

What's the model? What's the Hopf term? Soliton?

$\hat{n}$ $\quad n \cdot n = 1$

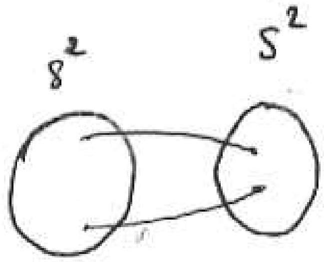
Euclidean action. (Units: $\hbar, c, \pi, 1, 2, 3 \ldots = 1$).

$$S = g^2 \int \partial_\mu n^a \partial^\mu n^a + i\theta H(n^a).$$

At any time finite energy solutions - bdry conditions

$$n \to (0,0,1)$$

compactification S^2



$$S^2 \to S^2.$$

$$Q \Rightarrow \quad \Pi_2(S^2) = \mathbb{Z}$$

$$\Pi_0(Q) =$$

$$J_\mu^{top} = \frac{1}{8\pi} \epsilon_{\mu\nu\lambda} \epsilon_{abc} n_a \partial_\nu n_b \partial_\lambda n_c$$

$$J_\mu^{top} = \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda .$$

$$H(n) = \int J_\mu^{top} A_\mu \, d^3x .$$

typical $\quad n \cdot \tau = \tau^3 \cos f(r) + \hat{x} \cdot \tau \sin f(r)$

$$f(0) = \pi, \quad f(R) = 0.$$

Time translation of a soliton generates a tube.

Hopf term in soliton sector. N,

$\propto N^2(\theta/2\pi)$!

Written in $\hat{n}$ Hopf term is Nonlocal.

CP^1 formulation of the same model renders

Hopf term is local.

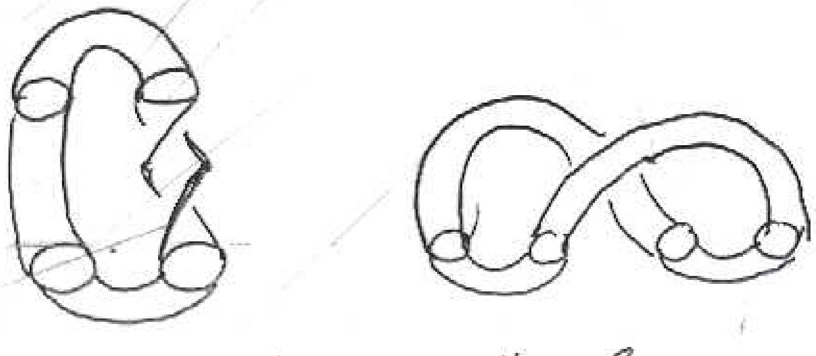
Here one uses $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$

$$\Phi^\dagger \Phi = 1.$$

$$S = \int D_\mu \phi^\dagger D_\mu \phi + \lambda(\phi^\dagger \phi - 1)$$

Hopf term $\propto \theta \int \epsilon_{\mu\nu\lambda} \phi^\dagger D_\mu \phi \; D_\nu \phi^\dagger D_\lambda \phi$ ..

Interpretation ~ Linking # of loops



Linking # Same

one can show for $\theta = \pi/2s$

$$Z = \int d\psi\, d\bar{\psi}\; e^{-\int \bar{\Psi}\left(\frac{T^\mu}{s} i\partial_\mu + M\right)\Psi + r_1 (\bar{\Psi} T^\mu \Psi)^2 + r^2 (\bar{\Psi}\Psi)^2}$$

$$M = \mathrm{Sgn}(\theta)\,(M_s - 8g^2\kappa\lambda) \qquad \kappa = \frac{\sqrt{\pi}}{\Lambda}$$

$$r_1 = \frac{1}{16g^2}, \quad r_2 = \lambda\kappa^2 \qquad \Lambda \text{ cut off.}$$

$$\langle J_\mu^{top} \rangle = 2s\,\langle J_\mu^{Noether} \rangle$$

Given this Background one can ~~analyse~~ prove establish this in the Fuzzy, Noncommutative regularisation

This is the ultimate goal for this program.

The setting up the formalism is the first aim.

This is to be done so that it is in close connection with continuum theory.

$$S_F^2 : \quad [x_i, x_j] = i\lambda\epsilon_{ijk} x_k$$

$$x^2 = R^2$$

Noncommutative analogue of Hopf Fib. $S^3 \to S^2$

$$[a, a^\dagger] = 1 = [b, b^\dagger]$$

$$[a, b] = 0.$$

$$a^\dagger a = N_1 \quad ; \quad b^\dagger b = N_2$$

$$|m\rangle = \frac{(a^\dagger \text{ or } b^\dagger)^m}{\sqrt{m!}} |0\rangle$$

$$\mathcal{L}_+ = a^\dagger \sqrt{N - a^\dagger a} + b^\dagger \sqrt{N - b^\dagger b}$$

$$\mathcal{L}_- = \sqrt{N - a^\dagger a}\, a + \sqrt{N - b^\dagger b}\, b$$

$$\mathcal{L}_3 = (a^\dagger a + b^\dagger b - N)$$

Holstein & Primakov realisation

Easy to check

$$[\mathcal{L}_+, \mathcal{L}_-] = 2\mathcal{L}_3,$$

$$[\mathcal{L}_\pm, \mathcal{L}_3] = \pm \mathcal{L}_3.$$

States $\quad |n, m\rangle \quad \begin{pmatrix} n \leq N \\ m \leq N \end{pmatrix}$

$$
\begin{array}{ccc}
 & |N, N\rangle & \\
 & |N-1, N-1\rangle, \quad |N-1, N\rangle & \\
 & \vdots & \\
|N, 1\rangle, |N-1, 2\rangle \cdots & & |2, N-1\rangle, |1, N\rangle \\
|N, 0\rangle, |N-1, 1\rangle & \cdot & |0, N\rangle \\
|N-1, 0\rangle \cdots & & |0, N-1\rangle \\
|N-2, 0\rangle & & |0, N-2\rangle \\
 & \vdots & \\
 & |0, 0\rangle &
\end{array}
\quad
\begin{array}{l}
m = N \\
\\
\\
\\
m = 0 \\
m = -1 \\
\\
\\
m = -N
\end{array}
$$

$m = 0, \pm 1, \ldots \pm N, \quad N = 2j$

For given $m = n_1 + n_2 - N$

$(2j+1)^2$ states classified into $2N+1$ sectors by the corresponding m values.

\# of states in $m = n$ and $-n$ are same

For any horizontal row there is an $SU(2)$ action

$m = -N$ to 0, $\quad J_+ = a^+ b, \quad J_3 = \frac{a^+a - b^+b}{2}$

$\ldots \; 1 \; N \quad \ldots \; a^+ \sqrt{(N-N_1)(N-N_2)} \;, \; a^+a - \ldots$

$$\phi = \sum a_{k\ell}(t)\, \mathcal{L}_+^k \mathcal{L}_-^\ell \qquad k,\ell \le N.$$

$$[\mathcal{L}_3, \Phi] = \sum (k-\ell)\, a_{k\ell}(t)\, \mathcal{L}_+^k \mathcal{L}_-^\ell .$$

$$= m\,\Phi \qquad \text{for } k-\ell = 0, \pm 1 \ldots \pm N.$$

$$\mathcal{L}_\pm \longrightarrow e^{\pm i\varphi} \mathcal{L}_\pm$$

$$\Phi \longrightarrow \Phi\, e^{im\varphi}$$

This kind of realisation ~~was~~ have been used by, many.

Presnajder etal. Chan etal ....

<u>Action</u> on $R \times S_F^2$

$$S = \frac{2}{2j+1}\, \mathrm{tr} \int_t \partial_0\phi^\dagger \partial_0\phi + \phi^\dagger\partial_0\phi\, \phi^\dagger\partial_0\phi - [\mathcal{L}_i,\varphi]^\dagger[\mathcal{L}_i,\phi] - \phi^\dagger\mathcal{L}_i\varphi\, \varphi^\dagger\mathcal{L}_i\varphi .$$

$$= \mathrm{Tr}\left(|D_0\phi|^2 - |D_i\phi|^2 - \lambda(\phi^\dagger\phi - 1)\right)$$

$$\mathrm{Tr} = \frac{2}{2j+1}\int_t \; ; \qquad D_0\phi = \partial_0\phi - i\phi A_0, \quad D_i\phi = \mathcal{L}_i\phi - i\phi A_i$$

$$A_\mu = -i\,\phi^\dagger \partial_\mu \phi$$

$$D_\mu \phi = P\,\partial_\mu \phi\,, \quad (D_\mu \phi)^\dagger = (\partial_\mu \phi)^\dagger P$$

where $P = 1 - \phi\phi^\dagger$

$$A_3 = i\,\phi^\dagger \mathcal{L}_3 \phi \quad = m\,I\,.\,!$$

$$\phi^\dagger D_\mu \phi = 0.$$

and $D_3 \phi = 0$

Define $\phi^\dagger [D_\mu, D_\nu] \phi = -i\,F_{\mu\nu}$

$$= -i\left(\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] - \epsilon_{\mu\nu 3} A_3\right)$$

$$F_{\mu\nu} \longrightarrow G^\dagger F_{\mu\nu} G$$

We also have $\epsilon_{\mu\nu\lambda} D_\mu (\phi F_{\nu\lambda}) = 0\,!$

$$\Rightarrow \epsilon_{\mu\nu\lambda} D_\mu F_{\nu\lambda} = 0$$

"Topological charge"

$$E = \frac{1}{2j+1} \operatorname{tr} |D\phi|^2$$

Using the id.

$$\frac{1}{2} \operatorname{tr} \left[(D_i\phi \pm i\epsilon_{ij} D_j\phi)^\dagger (D_i\phi \pm i\epsilon_{ij} D_j\phi) \right] + \frac{1}{2} \operatorname{tr} (D_3\phi)^\dagger (D_3\phi) \geq 0 .$$

$$E \geq |Q|$$

$$Q = -\frac{1}{2j+1} \operatorname{tr} F_{12} = -\frac{1}{2j+1} \operatorname{tr} J_0 .$$

$$J_\mu = \epsilon_{\mu\nu\lambda} F_{\nu\lambda} .$$

This current is covariantly conserved. $D_\mu J^\mu = 0$

$$Q = -\frac{i}{2j+1} \operatorname{tr} \phi^\dagger [\mathcal{L}_3, \phi] = \underline{m} .$$

<u>BPS</u> Sol<u>ns</u> $D_i\phi \pm i\epsilon_{ij} D_j\phi = 0$, $D_3\phi = \underline{0}$

$$P \mathcal{L}_+ \phi = 0 .$$

Any ~~configuration~~ satisfying $[\mathcal{L}_+, \phi] = 0$ is a BPS (anti) soliton sol<u>n</u>!

$$\phi_{\text{soliton}} = a_{m0}\, L_+^{m}.$$

$$\phi_{\text{antisoliton}} = a_{0m}\, L_-^{m}.$$

One can parametrize.

$$\phi = \mathcal{F}\,\frac{1}{\mathcal{F}^{\dagger}\mathcal{F}}$$

$$[L_\pm, \mathcal{F}] = 0.$$

Hopf term.

One can construct using covariantly conserved current J_μ. Hopf term (By analogy)

$$H = \frac{1}{2\pi}\,\mathrm{Tr}\Big[\epsilon_{\mu\nu\lambda}\Big(A_\mu F_{\nu\lambda} - \frac{2i}{3}A_\mu A_\nu A_\lambda\Big) + \epsilon_{\mu\lambda 0}\, A_\mu A_0\Big)\Big]$$

Now this Hopf term under the transformation δA_μ, of A_μ.

$$\delta H = \frac{\mathrm{Tr}}{\pi}\,\epsilon_{\mu\nu\lambda}\,\delta A_\mu F_{\nu\lambda}$$

We can express H as

$$H = -\frac{1}{\pi}\Big[\mathrm{Tr}\ \epsilon_{\mu\nu\lambda}\ \phi^\dagger \partial_\mu \phi\ \partial_\nu \phi^\dagger \partial_\lambda \phi + \mathrm{Tr}\ \phi^\dagger \partial_0 \phi\ \ \phi^\dagger [J_3, \phi]\Big]$$

$$= -\frac{1}{\pi}\Big[\mathrm{Tr}\ \epsilon_{\mu\nu\lambda}\ \phi^\dagger \partial_\mu \phi\ \partial_\nu \phi^\dagger \partial_\lambda \phi + im\ \mathrm{Tr}\ \phi^\dagger \partial_0 \phi\Big]$$

This Hopf term in Naive planar commutative ~~commutative~~ commutative lt

$$H = \frac{\epsilon_{\mu\nu\lambda}}{2\pi}\ \phi^\dagger \partial_\mu \phi\ \partial_\nu \phi^\dagger \partial_\lambda \phi \qquad \text{with } \underline{m = 0}$$

We can evaluate Hopf term in any static Soliton Configuration through $\theta = 2n\pi$ by

$$e^{i\theta L_3}\ \phi = e^{i\theta m}\ \phi,$$

$$\phi^\dagger e^{\theta L_3} = \phi^\dagger e^{i\theta m}$$

$$\cancel{\epsilon_{0ij}\ L_i \phi^\dagger L_j \phi} \rightarrow \epsilon_{0ij}\ L_i \phi^\dagger L_j \phi$$

$$\phi^\dagger \partial_0 \phi \longrightarrow i \partial_0 \theta\ m$$

Using $\int dt\, \partial_0 \theta = 2\pi \mathbb{Z}$

$$H = m^2$$

Commutative plane except $-N \leq m \leq N$!

Fuzzy Sphere & $SU_q(2)$, NLSM.

* We restrict the inf. D. Harmonic Oscillator basis states to $n \leq N$
* There is a natural way to do this.
* i.e using $SU_q(2)$ algebra
* Brief Description — Work to follow.

o q-oscillators

$$a_q a_q^{+} - q^{1/2} a_q^{+} a_q \equiv q^{-N/2}$$

$$[N, a_q] \equiv -a_q \quad [N, a_q^{+}] = a_q^{+}$$

$$|n\rangle_q = \frac{a_q^{+n}}{\sqrt{[n]!}} |0\rangle_q, \quad N|n\rangle_q = n|n\rangle_q$$

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \equiv [n]_q$$

If $q = e^{i 2\pi / n+1}$ $n+1^{th}$ root of unity.

$$[n+1]_q = 0$$

With this q one gets finite dimensional sp.

$$|0\rangle_q \quad \cdots \quad |n\rangle_q$$

With two osillators

$$a_q a_q^{+} - q^{1/2} a_q^{+} a_q = q^{-N_1/2}$$
$$b_q b_q^{+} - q^{1/2} b_q^{+} b_q = q^{-N_2/2}$$

We have Schiwinger realisation of $SU_q(2)$.

$$J_+ = a_q^{+} b_q \,, \quad J_- = b_q^{+} a_q$$

$$J_3 = 1/2 \, (N_1 - N_2)$$

$$[J_+ , J_-] = [2 J_3]_q$$

"Holstein Primakoff"

$$L_+^{(1)} = a_q^{+} \sqrt{N - N_1} \quad ; \quad L_3^{(1)} = N_1 - N/2$$

$$\mathcal{L}_+^{(q)} = L_+^{(1)} \otimes q^{L_{3}^{(2)}/2} + q^{-L_3^{(1)}/2} \otimes L_+^{(2)}$$

$$\mathcal{L}_3^{q} = L_3^{(1)} \otimes 1 + 1 \otimes L_3^{(2)}$$

These operators act exactly on the oscillator states. Here one need not fix any condition since they naturally appear

$$\phi = \sum a_{kl} (\mathcal{L}_+^{q})^k (\mathcal{L}_-^{(q)})^l .$$

Further work — in prog.

CONCLUSIONS : 1. NLSM - q - regularisation

- O(3) σ model important from Condensed Matter phys.
- Formulations analogous to continuum feasible.
- Implications — for the Soln of the Model in Numerical Work

Yet to be Done

The End.