Higher Dimensional Fuzzy Spheres and Matrix Brane Constructions

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ABSTRACT

There is a generalization of the fuzzy two-sphere construction to general dimensions. Matrix Brane actions combined with this construction give evidence for gauge theories on non-commutative as well as non-associative geometries. Matrix theory allows a construction of spherically symmetric instantons and leads to evidence for a map between non-abelian gauge theories on the non-associative spheres and abelian theories on higher dimensional non-commutative cosets. Odd dimensional fuzzy spheres share many of these properties but also have some puzzling peculiarities.

INTRODUCTION

- For N $D0$-branes, the action low-energy action is $U(N)$ invariant and of the form

$$TR \int dt \ \ (D_o\Phi^a)^2 + \sum_{a,b} \ \ [\Phi^a, \Phi^b]^2 + \cdots$$

There are **nine scalars** Φ^a, which correspond to the **nine transverse directions** to the worldline of a particle in 10 dimensions. They transform as adjoints of the gauge group.

- The low energy **action** for **N** Dp branes takes the form

$$\int d^{p+1}x \quad TR \ \{F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}\Gamma^\mu D_\mu \Psi + [\Phi^a, \Phi^b]^2 + \bar{\Psi}\Gamma^a[\Phi_a, \Psi] \ \}$$

- The **BFSS conjecture** says that large N supersymmetric quantum mechanics with $U(N)$ gauge symmetry gives a non-perturbative formulation of string theory – first non-trivial example of **gauge-gravity duality** where stringy (i.e including gravitational) physics comes from Matrix gauge theory. Similar conjectures include the Maldacena ADS/CFT correspondence, and the BMN plane wave-Matrix Model correspondence.

- In Matrix Models of this sort and more generally in studying D-branes, it is of interest to look for non-trivial solutions of the D-brane action and analyze fluctuations around them. This leads to fuzzy geometries – which in some cases are non-commutative, in others non-associative, but all algebraic structures derived from Matrices.

- Of particular interest are highly symmetric solutions, such as spheres, which in some cases provide supersymmetric vacua (as in the BMN model).

OUTLINE

- Part I : **Fuzzy Two-sphere**

  Brane Polarization

  Physical setting for Non-commutative geometry

  Finite Realization of continuous symmetry

- Part II : **Higher Fuzzy Even Spheres**

  Non-associative geometry

  Hidden higher dimensional geometry

  Symmetric Instantons

- Part III : The **Fuzzy Odd Spheres** :

  Hidden higher dimensional geometries

  Exotic scalings of degrees of freedom

  Mixing of gauge and spatial symmetries

- Part IV : **Conclusions and Open Problems**

Part I : The fuzzy two-sphere

- It is an example of a non-commutative space (manifold).

  Recall : The **space of all functions** on a manifold $f : M \to C$ has the structure of a closed algebra, coming from point-wise multiplication.

$$f \circ g = f(x)g(x)$$

  This multiplication is commutative.

- **Non-commutative manifolds** are defined by deforming this algebra to a **non-commutative algebra**. A lot of geometry and topology can be done in this generalized setting (Connes, Madore ...) and a lot of field theory, i.e perturbation theory, solitons etc. has been developed (Balachandran, ...) in this example

- An ordinary sphere S^2 can be described by 3 cartesian coordinates x_a obeying a constraint

$$\sum_a x_a x_a = 1$$

- A basis in the space of all functions on the sphere is given by the **polynomials in** x_a (symmetric traceless polynomials to be more precise). These polynomials are just the spherical harmonics $Y_{l,m}$ in cartesian coordinates, which are used to describe wavefunctions in problems with spherical symmetry. l is an arbitrary integer, and m goes from $-l$ to l.

- Consider the algebra generated by X_a for $a = 1, 2, 3$ obeying relations :

$$[X_a, X_b] = i\epsilon_{abc}X_c$$

$$\sum_a X_a X_a = J(J+1)$$

- We know this algebra admits a finite N-dimensional representation, where $N = 2J + 1$. The X_a are $N \times N$ matrices.

- In fact the X's and their products generate the full matrix algebra $Mat(N)$. An arbitrary matrix can be expanded

$$M = C_0 + \sum_a C_a X_a + \sum_{a,b} C_{ab} X_a X_b + \cdots$$

- Defining rescaled variables $Z_a = \frac{X_a}{\sqrt{J(J+1)}}$. we have :

$$[Z_a, Z_b] = \frac{i\epsilon_{abc} Z_c}{\sqrt{J(J+1)}}$$

$$\sum_a Z_a Z_a = 1$$

- As $N \to \infty$, the Z become commuting variables defining a unit sphere. The **large** N limit of the algebra of Matrices gives the algebra of functions on the sphere, i.e the algebra of **spherical harmonics**.

- The algebra $Mat(N)$ generated by Z_a is the deformed **non-commutative algebra of functions.** The Z_a are non-commutative analogs of coordinates on the fuzzy sphere.

- The action of $SO(3)$ allows us to decompose the matrix algebra $Mat(N)$ as a direct sum of of subspaces corresponding to definite representations of $SO(3)$.

$$Mat(N) = \bigoplus_{l=0}^{2J} Mat_l(N)$$

The polynomials in X in $A_l(N)$ are tensor operators which are "fuzzy analogs of spherical harmonics."

- For example, when $J = 1/2$, $N = 2J + 1$

$$1, \quad \sigma^a$$

give a basis for 2×2 matrices and transform, respectively as $l = 0$ and $l = 1$ of $SO(3)$. This $SO(3)$ decomposition of 2×2 matrices relies on the fortunate fact

$$2^2 = 3 + 1$$

- For general $N = 2J + 1$ the analogous equation is

$$N^2 = \sum_{l=0}^{2J} (2l + 1)$$

Generalizations of these equations to higher orthogonal groups, and its consequences and D-brane interpretation, will appear in the following

Remark : This choice of non-commutative deformation guarantees the existence of an $SO(3)$ symmetry while allowing the truncation of the algebra of spherical harmonics to a finite algebra. This is an analog of "discretization" of the space, which manages to preserve the **rotational symmetry.** It also allows a **closed multiplication.**

- The **fuzzy sphere matrices** are used to solve the equations of motion of the **Matrix Model,** and describe time dependent membrane solutions. The properties agree with supergravity expectations. (Kabat and Taylor 1996)

$$\Phi_a = f_a(t) X_a$$

- More recently static solutions have been found. To get static solutions one considers $D0$-branes in a background $H^{(4)} = dA^{(3)}$ field strength

$$H_{0abc} = f \epsilon_{abc}$$

for (i, j, k) belonging to a subset of three spatial directions picked from the nine dimensions of string theory. The action of zero-branes in this background is given by

$$\int dt Tr \ \ (D_t \Phi_a)^2 + [\Phi_a, \Phi_b]^2 + f \epsilon^{abc} \Phi_a \Phi_b \Phi_c$$

- The equations of motion in this background are :

$$[\Phi_a, [\Phi_a, \Phi_b]] = f\Phi_b$$

They can be solved by an ansatz $\Phi_a = \sqrt{\frac{f}{2}} X_a$, where the X_a are the generators of the fuzzy sphere algebra.

R. Myers "Dielectric Branes" '2000

- Another context is the BMN matrix model for M-theory on pp waves where the fuzzy 2-sphere is a SUSY solution

$$\int dt Tr \ (D_t\Phi_a)^2 + [\Phi_a, \Phi_b]^2 + f\epsilon^{abc}\Phi_a\Phi_b\Phi_c + \mu\Phi_a^2$$

- Using the identification of D-brane charges in terms of Matrices, we know that the system of Matrices describes in the context of Matrix theory, bound states of zero-branes and two-branes in the shape of a sphere. In fact we have **1 2-brane** bound to N zero branes. So we expect a $U(1)$ **field theory** living on the sphere to describe the dynamics of this object.

- The number of 2-branes can be deduced in Matrix Theory since $TR[\Phi_a, \Phi_b]$ is known to be a membrane charge (Banks, Seiberg, Shenker 1997 - SUSY Ganor, Ramgoolam, Taylor 1997 - T-duality)

- Starting from the original quantum mechanics one can derive such a theory.

$$\Phi_a = X_a + A_a$$

$$\Phi_i = 0 + B_i$$

- Some of the rules that allow such a map between large N quantum mechanics and field theory are (for example Iso, Kimura et.al. 2001)

$$Z_a \rightarrow \text{ coordinates}$$

$$[X_a, \ \ ] \rightarrow \text{ derivatives}$$

$$[X_a, [X_a, \ \ ]] \rightarrow \text{ Laplacian}$$

$$\text{Traces} \rightarrow \int$$

- For example

$$\int dt \quad TR \quad (\partial_t \Phi_i)^2 \ + \ [\Phi_a, \Phi_i]^2 \ + \ \cdots$$

$$= \int dt \int sin\theta \ d\theta d\phi \ \ (\partial_t B_i(\theta, \phi))^2 \ + \ (D_a B_i)^2 \ + \ \cdots$$

Part II : Higher Fuzzy Even spheres

- This construction can be generalized to 4-spheres.

  Grosse, Klimcik, Presnjader

  "Towards Finite 4D Quantum Field Theory in Non-Commutative Geometry" 1995

  Castelino, Lee and Taylor

  "Longitudinal 5-branes as 4-spheres in Matrix Theory." 1997

- While we saw **non-commutativity** was an important ingredient for fuzzy 2-sphere.

$$\boxed{\text{Two-sphere : } f \circ g = g \circ f}$$

$$\boxed{\text{Fuzzy two-sphere : } f \circ g \neq g \circ f}$$

What about

$$(f \circ g) \circ h = f \circ (g \circ h) \qquad ?$$

We will find that

$$\boxed{\text{Fuzzy 4-sphere : } (f \circ g) \circ h \neq f \circ (g \circ h)}$$

- **Non-associativity** and **hidden extra dimensions** appear in the case of higher fuzzy spheres – related to Matrix Multiplication with **projection.**

S. Ramgoolam, "On spherical harmonics for fuzzy spheres in diverse Dimensions," Nucl. Phys. B '2001

P.M.Ho, S. Ramgoolam, "Higher dimensional geometries from Matrix Brane constructions," Nucl. Phys. B. 2002

- For the two sphere we associated with each coordinate X_a an N-dimensional matrix, where $N = 2J + 1$, and large N gave the classical limit. In the simplest case, $J = \frac{1}{2}$ we had

$$\boxed{X_a \to \sigma_a \qquad \text{a} = 1,2,3}$$

the Pauli Matrices

- Now we want,

$$\sum_a X_a X_a = R^2$$

which has $SO(5)$ symmetry, and the X_a transform as a vector of $\mathbf{SO(5)}$. Natural candidate for the smallest matrices with this property are the

$SO(5)$ Γ_a matrices $\quad a = 1, 2, 3, 4, 5$

which act on the **4-dimensional spinor representation**. We want to find a 1-parameter family of representations where the sum continues to be proportional to the identity matrix.

- R_n is obtained by symmetrization of the fundamental spinor.

$$R_n \equiv Sym(V^{\otimes n})$$

This irreducible representation has dimension $N = \frac{(n+1)(n+2)(n+3)}{6} \sim n^3$.

- The Γ matrices act the way operators are usually taken to act in **multiparticle** Hilbert spaces :

$$X_a = (\Gamma_a \otimes 1 \cdots \otimes 1 + 1 \otimes \Gamma_a \otimes 1 \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes \Gamma_a)$$

- One finds

$$\sum_a X_a X_a = n(n+4) \; \mathbf{1}$$

Indeed the sum of squares is proportional to the identity matrix.

- We have an algebra

$$[X_\mu, X_\nu] = X_{\mu\nu}$$

The $X_{\mu\nu}$ are obtained by the action of $[\Gamma_\mu, \Gamma_\nu]$ on the tensor product.

$$X_{\mu\nu} = \sum_{r=1}^{n} \rho_r([\Gamma_\mu, \Gamma_\nu])$$

After rescaling $Z_\mu = \frac{1}{n} X_\mu$ and $Z_{\mu\nu} = \frac{1}{n} X_{\mu\nu}$ the commutators vanish in the large N limit.

- Given the $SO(5)$ symmetry, we can write any matrix as a linear combination of matrices which transform according to specific irreducible tensor representations of $SO(5)$.

- The irreducible tensor representations are built from the basic vector representation using a process of symmetrization and antisymmetrization.

Symmetrization

Antisymmetrization

$Mat_N(C)$ as a direct sum of tensor operators transforming according to irreducible representations of $SO(5)$

$$\boxed{Mat(N) = \oplus_{r_1,r_2} Mat_{r_1,r_2}(N)}$$

The ranges are $n \geq r_1 \geq r_2 \geq 0$.

- **REMARK** So far everything seems to be working as for the fuzzy 2-sphere ...

BUT

This spectrum of representations is too big to describe the spherical harmonics in the large N limit, since ordinary spherical harmonics transform in **symmetric traceless representations only**, which correspond to $r_2 = 0$.

- A subspace $M_{sph}(N) \equiv M_{r_1,0}(N)$ is therefore the right space.

So we define a projection :

$$P(M_{sym} + M_{other}) = M_{sym}$$

- If we use the matrix product, this subspace does not close under multiplication, since symmetric reps. can multiply to give anstisymmetric reps.

- However we can define a **modified product** , which is obtained by first multiplying as matrices and then projecting to the desired subspace.

- This modified multiplication is **not associative**, but the non-associativity **vanishes at large N.**

as shown in S. Ramgoolam "On Spherical harmonics for fuzzy ... " NPB 2001

- A simple way to see that non-associative products can be obtained from matrices :

$$A * B = AB + BA$$

and check that

$$(A * B) * C \neq A * (B * C)$$

- The non-associative algebra of the four-sphere is a of a special form. The algebra is **commutative but not associative**.

At large N it becomes associative, and the leading $1/N$ corrections make it a Jordan algebra :

$$Z_\mu \circ Z_\nu = Z_\nu \circ Z_\mu$$

$$Z_\mu \circ (Z_\nu \circ Z_\lambda) - (Z_\mu \circ Z_\nu) \circ Z_\lambda = \frac{1}{n^2} \left[\delta_{\nu\lambda} Z_\mu - \delta_{\mu\nu} Z_\lambda \right]$$

- Jordan algebras are special commutative non-associative algebras which obey the relation

$$((A \circ A) \circ B) \circ A) = (A \circ A) \circ (B \circ A)$$

What is the geometry of the full Matrix algebra ?

- The functions on the sphere S^2 are spanned by the spherical harmonics, with $l = 0$ to ∞, each occuring once. The functions of S^4 are spanned by the symmetric traceless reps. – which are related to a projection of the Matrix algebra. Is there a manifold whose space of functions contains all representations of $SO(5)$ once ? **YES !!**

- It is $SO(5)/U(2)$, a coset like $S^4 = SO(5)/SO(4)$ but it is six-dimensional. It is in fact an S^2 bundle over S^4. Locally a product :

- At finite n the S^2 fibre is a fuzzy S^2. To see this we fix a point on the four-sphere, say the North-pole. This is done by picking states with eigenvalues $Z_5 = 1, Z_1 = Z_2 = Z_3 = Z_4 = 0$.

There is a finite number of such states, $n+1$ of them, which are transformed among each other by the operators

$$M_3 = Z_{12}$$

$$M_2 = Z_{13}$$

$$M_1 = Z_{14}$$

The Z_{ij} are self-dual. The M_i obey an $SU(2)$ algebra and can be interpreted as fuzzy 2-sphere coordinates. At each "point" we have the $(n+1) \times (n+1)$ degrees of freedom of a $U(n+1)$ gauge theory.

- The $[X_i, X_j]$ can be thought of as field strengths in the dual 4-brane field theory at the N-pole. The self-duality tells us that we have an instanton. The spherical symmetry implies that the instanton density is constant. $n+1$ is the rank of the gauge group. Instanton number $\sim \frac{n^3}{6}$ which agrees with the fact that $N \sim \frac{n^3}{6}$ is the rank of the zero-brane gauge theory.
- In a related context where 0-branes are replaced by 1-branes and the 4-branes by 5-branes, the instanton configuration was used

by Constable, Myers, Tafjord, to solve the equations of motion from the dual five-brane point of view.

Constable, Myers, Tafjord, "Non-abelian Brane Intersections" hep-th/0102080

- The obvious physical point of view is indeed that zero-brane Matrix theory around this background is indeed describing a spherical bound state with multiple 4-branes and 0-branes. Number of branes from $tr(\Phi^a\Phi^b\Phi^c\Phi^d)\epsilon_{abcd}$

- As we saw the algebra of functions on the sphere at finite n is non-associative.

- An alternative and equivalent description of the same system is suggested by the standard analysis of Matrix fluctuations. The transverse scalars are now functions on a six-dimensional space $SO(5)/U(2)$. Once we fix the expectation values of Z_i and Z_{ij}, there is no multiplicity of states. In this sense it is an abelian theory living on $SO(5)/U(2)$. Following the standard Matrix Theory manipulations described in 2-sphere case, we have

$$\delta\Phi \sim B(Z_i, Z_{ij})$$

$$TR(\partial_0\Phi)^2 \quad \sim \quad \int_{SO(5)/U(2)}(\partial_0\Phi)^2$$

In analogy to the fuzzy two sphere case we would expect that the action of X_i and X_{ij} should be expressed as differential operators acting on the the algebra generated by (Z_i, Z_{ij}) in the large N limit. This is indeed the case.

Ho, Ramgoolam, "Higher Dimensional geometries from Matrix Brane constructions," hepth/0111278, Nucl. Phys. B627

This was used to develop the action for fluctuations around a fuzzy four-sphere solution for a Matrix Action with a mass term in :

Kimura, "Noncommutative Gauge Theory on Fuzzy Four-Sphere and Matrix Model," hep-th/0204256, Nucl. Phys. B637

- Detailed description of the fluctuations as a non-abelian gauge theory on the commutative non-associative algebra and the map to the abelian non-commutative gauge theory on the higher dimensional coset remains to be found.

The story of the fuzzy four-sphere generalizes to higher spheres, i.e the relation of a higher dimensional geometry to non-abelian physics on the sphere.

As always X_μ are operators in $\mathcal{R}_n$. This is a **subspace** of $Sym(V^{\otimes n})$ which is an irreducible representation of $SO(2k+1)$, obtained by acting with the Lie algebra on $v_0^{\otimes n}$. Their commutators of X_μ are $X_{\mu\nu}$. The $X_\mu, X_{\mu\nu}$ together form the Lie algebra $SO(2k+2)$. It happens to be true that this irrep of $SO(2k+1)$ is also an irrep of $SO(2k+2)$. This means that X_μ as well as $X_{\mu\nu}$ keep us within the subspace $\mathcal{R}_n$.

$$X_\mu = \sum_r \rho_r(\Gamma_\mu)$$

- Fuzzy S^6

$$N = \frac{1}{360}(n+1)(n+2)(n+3)^2(n+4)(n+5)$$

$$N_6 = \frac{1}{6}(n+1)(n+2)(n+3)$$

- The higher dimensional geometry is $SO(7)/U(3)$, a 12-dimensional space. The spectrum of $SO(7)$ harmonics on this space agrees with the spectrum of representations of the Matrices in the large N limit.

- The spectrum of $SO(7)$ reps satisfies

$$N^2 = \sum_{r_1,r_2,r_3} Dim(r_1, r_2, r_3)$$

- The label r_1 goes from 1 to n. And we have $r_3 \le r_2 \le r_1$. This is an $SO(7)$ generalization of $2^2 = 4 = 3 + 1$ or Pauli Matrices.

Extra degrees of freedom on each point of S^6 which can be interpreted in terms of non-abelian physics by using the fact that the fuzzy $SO(6)/U(3)$ maps exactly to matrices of size N_6.

- Fuzzy S^8

$$N = \frac{1}{302400}(n+1)^2(n+2)^2(n+3)^4(n+4)^4(n+5)^4(n+6)^2(n+7)^2$$

$$N_8 = \frac{1}{360}(n+1)(n+2)(n+3)^2(n+4)(n+5)$$

- In this case the higher dimensional space is $SO(9)/U(4)$.

Part III : Higher Fuzzy Odd Spheres

- In some sense the fuzzy three-sphere is embedded in the fuzzy four-sphere. We now want matrices X_i such that

$$\sum_i X_i^2 = const.\ \mathbf{1}$$

- And we want to use the action of Γ_i on a symmetrized tensor product of spinors. The X_i and Γ_i transform as a vector of $SO(4)$. Now there are two irreducible spinor representations V_+ and V_-. The direct sum $V = V_+ \oplus V_-$ is an irreducible representation of $SO(5)$, which we worked with in discussing the fuzzy S^4.

- There we worked with $Sym(V^{\otimes n})$. This decomposes into a number of $SO(4)$ representations specified by two integers n_+ the number of positive chirality states in the n-fold tensor product, and n_- the number of negative chirality states. which obey of course $n_+ + n_- = n$.

- To construct the fuzzy three-sphere, we need n odd and we pick two irreps of $SO(4)$, which we call R_{n_+} and R_{n_-}.

- R_{n_+} has

$$n_+ = \frac{(n+1)}{2}$$

$$n_- = \frac{(n-1)}{2}$$

- R_{n_-} has

$$n_+ = \frac{(n-1)}{2}$$

$$n_- = \frac{(n+1)}{2}$$

- X_i are defined as before $\Sigma_{r=1}^{n} \rho_r(\Gamma_i)$.

- X_i is an off-diagonal operator. It maps R_{n+} to R_{n_-}, and R_{n+} to R_{n_-}

- It is useful to define X_i^+ as the matrix in the lower left block, and X_i^- as the Matrix in the upper right block. Then $X_i = X_i^+ + X_i^-$.

- X_i obeys the correct sphere equation

$$\sum_i X_i^2 = \frac{(n+1)(n+3)}{2} \; 1$$

- As before the Matrix algebra, in the large n limit contains all the $SO(4)$ representations of the sphere but in fact contains too much.

- Each of $Mat(R_+, R_+)$ and $Mat(R_-, R_-)$ have the spectrum of $S^2 \times S^2 = \frac{SO(4)}{U(1)\times U(1)}$. The off-diagonal matrices have the spectrum of sections of a homogeneous bundle over $S^2 \times S^2$, i.e a field of the form $\phi_i(\theta_1, \phi_1, \theta_2, \phi_2)$ where the i index is a vector of $SO(4)$. The spectrum of such a field can be computed using induced representations and agrees with the $SO(4)$ decomposition of the off-diagonal matrices.

- Toy Matrix Actions which admit this as a solution can be constructed. The fluctuations of a transverse scalar now give rise to a pair of scalar fields on $S^2 \times S^2$, and a pair of non-trivial fields on $S^2 \times S^2$. The action for fluctuations appears to have a $U(2) \rightarrow U(1) \times U(1)$ structure with a matter content whose gauge charges are correlated with the spatial transformation properties.

- Same construction works for higher odd spheres. For S^5 we find by the same methods, a hidden $\frac{SO(6)}{U(1)\times U(2)}$ geometry. The Matrices can be decomposed into functions and sections of a bundle over this space.

- This is not a bundle over the sphere S^5. But we have other bundle structures which may be useful.

- One possible Brane interpretation of the fuzzy five-sphere might be related to multiple three-branes in RP^5.

• **Exotic scaling of number of degrees of freedom**

In case of fuzzy four-sphere, number of degrees of freedom = $N^2 \sim n^6$. When $R = 1$, the X coordinates have eigenvalues spaced by $1/n$ so we have loosely speaking a discreteness of order $1/n$. It is a four-dimensional space so n^4 are geometrical degrees of freedom. Since $n^6 = n^4 \times n^2$, we expect n^2 to be extra dof, which agrees with the $U(n)$ gauge group (which was derived independently by fixing a point on the sphere). This simple rule works for any even sphere S^{2k} $N^2 \sim n^{(k)(k+1)}$, hence extra dof $\sim n^{k^2+k-2k} = n^{k^2-k}$. Rank of gauge group is $n^{\frac{k(k-1)}{2}}$ which agrees with independent checks.

However for three sphere, $N^2 \sim n^4$. Geometrical dof $\sim n^3$. Hence formally rank of gauge group $n^{1/2}$. Generally for S^{2k-1}, $N^2 \sim n^{k^2+k-2}$. Hence excess dof n^{k^2-k-1}. Rank formally fractional. Hence if there is a formulation as a field theory on the sphere it appears to be of an exotic sort.

Such odd exponents for the number of dof have appeared in trying to reproduce gravitational entropies, e.g N_5^3 appears for

5-branes. So there may be a way to match this with gravity — Interesting problem.

Summary and Outlook

- Constructing higher dimensional extended objects from zero branes in String/M-theory

  Worldvolumes can be **non-commutative**

  They can be **non-associative**

  And they can hide **extra dimensions.**

  In particular fuzzy spheres in diverse dimensions, provide examples of these phenomena.

- The construction of odd-spheres should allow us to describe three-brane giants and five-brane giants; the end-points of two-branes ending on five-branes perhaps blowing up into 3-spheres.

  Useful step would be to find SUSY brane actions which admit te fuzzy odd-spheres as SUSY-preserving solutions.

- Experimental probes of non-commutativity can be studied and bounds on non-commutativity found. One explores $[X_i, X_j] = \theta_{ij}$. For fixed θ_{ij} this breaks Lorentz Invariance, but if we try (as suggested by fuzzy four sphere)

$$X_i * X_j - X_j * X_i = 0$$

$$(X_i * X_j) * X_k - X_i * (X_j * X_k) = \theta \ (\ \delta_{jk} \ X_i - \delta_{ij} \ X_k)$$

then we do not superficially have a problem with Lorentz invariance.

- There are intimate connections between the 4D QHE effect and the fuzzy 4-sphere. Is there a 3D QHE related to the fuzzy three-sphere ?