NEW SCALING LIMIT FOR FUZZY SPHERES.

· S. V. (hep-th/0102212)

· S. V. & B. Ydri (hep-th/0209131).

· S. V. & B. Ydri (in preparation).

- The fuzzy sphere S_F^2 $\leftrightarrow$ matrix algebra.
- Field theory on S_F^2 $\longleftrightarrow$ Matrix model.
- Different scaling limits of the matrix model
- UV-IR mixing (or absence of).
- Physical interpretation of a new scaling limit.

Quantum properties of nc field theories are subtle (and often quite different) from ordinary field theories.

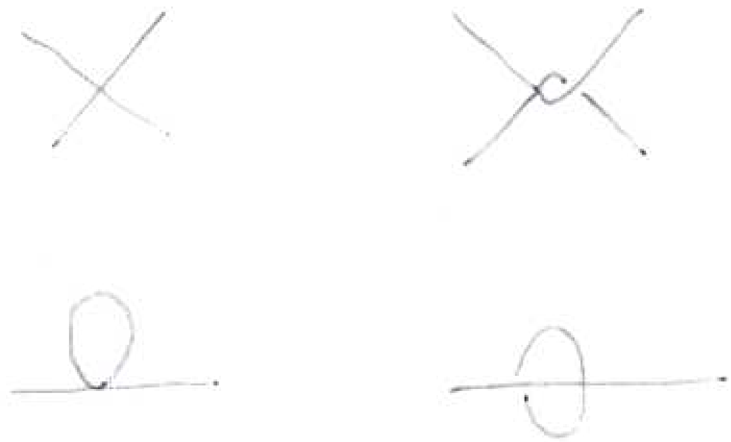
Eg Scalar field theory on nc $\mathbb{R}^4$ has "UV-IR mixing" (Minwalla et al hep-th/9912072).

n-point correlation functions are singular f^ns of external momenta.

(Meaning of UV-IR mixing is different, depending on whether the UV cut-off Λ is finite or infinite !!).

Eg nc Φ^4 theory:

$\Gamma^{(2)}$ has contributions from planar & non planar diagrams



$$\Gamma^{(2)}_{np}(p) \sim \frac{m}{\left(p^2\theta^2 + \frac{4}{\Lambda^2}\right)^{1/2}} K_1\left(m\sqrt{p^2\theta^2 + \frac{4}{\Lambda^2}}\right).$$

If $\Lambda \neq \infty$, then $\Gamma^{(2)}$ is a smooth f^n of p.

$$\lim_{\Lambda\to\infty} \Gamma^{(2)}_{np}(p) \sim \frac{m^2}{4\pi^2(\theta p)^2} + \frac{m^2}{8\pi^2}\ln(\theta p) + \cdots$$

In either case, low energy Wilsonian action is not defined.

Obvious question: would it help to put the system in a box? (i.e. also put in an IR cut-off).

Strategy: Use a UV cut-off Λ, an IR cut-off $1/L$, see how correlation functions change when a thin shell is integrated out.

(Wilsonian approach to renormalization).

Theories of fuzzy spaces are a perfect setting for realizing the above.

For fuzzy S^2, there is a IR cut off $1/R$, a UV cut-off $2l/R$.

Added bonus: Fields are simply matrices, so there are only finite no. of degrees of freedom. (Easy to simulate on a computer).

The fuzzy sphere is described by 3 matrices X_i

$$[X_i, X_j] = i\theta \epsilon_{ijk} X_k$$

Representations of $SU(2)$ generators satisfy this, with $X_i = \theta J_i$.

Choosing J_i of a U.I.R. gives the interpretation of a "2-sphere":

$$X_i X_i = \theta^2 j(j+1) \mathbb{1} \equiv R^2 \mathbb{1}.$$

Any $(2j+1) \times (2j+1)$ matrix Φ can be expanded in terms of $\mathbb{1}$, X_i, $X_i X_j X_k$, $X_i X_j$ etc.

$$\Phi = \varphi_0 \mathbb{1} + \varphi_i X_i + \varphi_{ij} X_i X_j + \cdots\cdots$$

This is very much like expanding a function on an ordinary sphere, EXCEPT that there only a finite no. of terms in this expansion.
($\because$ Φ is finite-dim, the basis for expansion is also finite dim).

Equivalently

$$\Phi = \sum_{l=0}^{2j} \sum_{m=-l}^{l} f_{lm} \Psi_{lm}$$

compare

$$\Phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \varphi_{lm} Y_{lm}(\Omega)$$

on ordinary S^2.

$\Psi_{lm} \equiv$ 'fuzzy spherical harmonics'.

S^2		S_F^2
• ~~Matrices~~ Functions ~~(hermitian)~~ (real)	$\longleftrightarrow$	• ~~Functions~~ Matrices ~~(real)~~ (hermitian)
• Pointwise multiplication	$\longleftrightarrow$	• Matrix multiplication
• ∞-dim' commutative algebra	$\longleftrightarrow$	• Finite dim' non commutative algebra
• 'Derivatives' $\mathcal{L}_i$ act as $\mathcal{L}_i \Phi$	$\longleftrightarrow$	• Derivatives L_i act as $[L_i, \Phi]$
• $\int d\Omega$	$\longleftrightarrow$	• Trace.
• Scalar product $(f, g) = \int d\Omega\, f^* g$	$\longleftrightarrow$	Scalar prod $(f, g)_j = \frac{1}{2j+1} \mathrm{Tr}\, f^\dagger g$.

Fuzzy sphere is a simple new (i.e. non lattice like) discretization of ordinary S^2.

Fuzzification remembers underlying (SO(3)) symmetry.

Stringy interest :

→ Myers Dielectric Effect

Giant gravitons.

→ Projectors interpreted as nc solitons

(Other fuzzy surfaces like $\mathbb{C}P^2$, $\dfrac{SU(3)}{U(1) \times U(1)}$, ...).

Our main interest is in actions of the form

$$S_j[\Phi] = \frac{1}{2j+1}\, \mathrm{Tr} \left[[L_i, \Phi]^\dagger [L_i, \Phi] + \mu_j^2 \Phi^2 + \frac{\lambda}{4!}\Phi^4 \right]$$

This model has the correct continuum $(j \to \infty)$ limit classically:

$$S_\infty = \int d\Omega \left[-(\mathcal{L}_i\Phi)(\mathcal{L}_i\Phi) + \mu^2\Phi^2 + \frac{\lambda}{4!}\Phi^4 \right]$$

where $\mathcal{L}_i\Phi = -i\epsilon_{ijk}\hat{x}_j\partial_k\Phi$ etc.

Quantization is simplest via path integrals:

Write $\Phi = \sum \varphi_{lm}\Psi_{lm}$

Path integral measure $D\Phi = \prod_{l=0}^{2j}\prod_{m=-l}^{l}\left(\frac{d\varphi_{lm}\, d\bar{\varphi}_{lm}}{2\pi i}\right)$

Partition function

$$Z = \int D\Phi\, e^{-S}$$

Correlation functions $\langle \varphi_{l_1 m_1} \cdots \varphi_{l_n m_n} \rangle$ can be calculated by adding source term $\mathrm{Tr}\, J\Phi$ to the action, and differentiating wrt J.

Free propagator $\langle \varphi_{l_1 m_1} \varphi_{l_2 m_2} \rangle = \dfrac{\delta_{l_1 l_2}\, \delta_{m_1+m_2,0}\, (-1)^{m_1}}{l(l+1) + \mu_j^2}$

Interactions : $S_{int} = \frac{1}{(2j+1)}\frac{\lambda}{4!} \, Tr\, \Phi^4$

$$= \frac{\lambda}{4!} \varphi_{l_1 m_1} \cdots \varphi_{l_4 m_4} \underbrace{Tr\,(\Psi_{l_1 m_1} \cdots \Psi_{l_4 m_4})}_{V(l_i, m_i;\, j).}$$

Notice that V is (only) cyclic in l_i, as opposed to the commutative case (where $\Psi_{lm} \leftrightarrow Y_{lm}$).

This has consequences for quantization (UV-IR mixing).

Wilsonian Matrix Renormalization:

With $\Phi = \varphi_{lm} \Psi_{lm}$,

$$S_0 = \sum_{l,m} \left(l(l+1) |\varphi_{lm}|^2 + \mu_j^2 |\varphi_{lm}|^2 \right)$$

"Energy" of the mode φ_{lm} is $l(l+1)$. So modes can naturally be arranged in energy SHELLS labelled by l.

$$\Phi = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ - & \cdot & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}_{(2j+1)\times(2j+1)} \quad \text{matrix.}$$

Integrate out the modes $\varphi_{2j,m}$ in the path integral.

Remaining modes $\varphi_{00}, \varphi_{1m}, \dots \varphi_{2j-1,m}$ can be assembled into a $(2j)\times(2j)$ matrix Φ'

Rescale $\left[\begin{array}{c|c} \Phi' & 0 \\ \hline 0 & 0 \end{array}\right]$ and re-express in terms of original Ψ_{lm}

To make original & rescaled action look the same, we have to redefine

$$\mu_j^2 \longrightarrow \mu_j'^2$$

$$\lambda_j \longrightarrow \lambda_j' \qquad \text{etc.}$$

Rate of change of the coupling gives us RG eq^{n}s:

$$\frac{d\mu_j^2}{dt} \equiv M(\mu_j^2, \lambda_j)$$

$$\frac{d\lambda_j}{dt} \equiv \beta(\lambda_j) \qquad \left(\frac{2j-1}{2j} \equiv e^{dt} \right)$$

For quartic theory, we find

$$\mu_j'^2 = \frac{2j+1}{2j}\left[\mu_j^2 + \frac{2\lambda_j}{4!} A(2j)\left(2 + (-1)^\ell B(\ell, j)\right)\right]$$

on integrating out a single shell.

$$A = \frac{2(2j)+1}{2j(2j+1) + \mu_j^2}, \quad B = \frac{(2j+1)!\,(2j)!}{(2j-\ell)!\,(2j+\ell+1)!}$$

$$\frac{d\mu_j^2}{dt} = \mu_j^2 + \frac{\lambda_j}{4} \qquad (\ell = 0)$$

$$\frac{d\mu_j^2}{dt} = \mu_j^2 + \frac{\lambda_j}{12} \qquad (\ell = 1)$$

EXTREME SENSITIVITY TO WHETHER ℓ IS EVEN / ODD.

With conventional renormalization (integrate out all internal momenta, not just a shell), UV-IR mixing manifests itself differently: (Chu, Madore Steinacker 0106205)

$$\Delta\mu^2_{pl} + \Delta\mu^2_{n-p} \equiv \Delta\mu^2_{commutative} + \Delta$$

One can show that $\Delta \neq 0$, and does not vanish even when $J \to \infty$. (Chu et al).

Having an IR cut-off does not really help to get rid of UV-IR mixing!

Interestingly, there is a "stereographic" projection to the nc plane:

$$y_+^F = 2R\, x_+^F \frac{1}{R - x_3^F}, \qquad y_-^F = 2R \frac{1}{R - x_3^F}\, x_-^F$$

$$(x_i^F = \theta J_i)$$

Take $R, j \to \infty$, with $\theta'^2 = \dfrac{R^2}{\sqrt{j(j+1)}}$ fixed.

Then $[y_+^F, y_-^F] = -2\theta'^2 \longrightarrow$ nc $\mathbb{R}^2$.

To get nc $\mathbb{R}^4$, start with $S_F^2 \times S_F^2$, flatten using above prescription.

In this prescription,

$$\theta \simeq \frac{\theta'}{\sqrt{j}} \to 0 \quad \text{and}$$

$$\text{Max. energy} = \frac{2j(2j+1)}{R^2} \sim \frac{4j}{\theta'^2} \longrightarrow \infty.$$

Applying this to scalar field theory on S_F^2, we get the theory on nc $\mathbb{R}^2$, with UV-IR mixing (not surprising).

A new planar limit:

Take $J, R \to \infty$, keeping $\theta \sim \dfrac{R}{J}$ fixed.

Co-ords on nc plane are

$$x_\pm^{nc} = \frac{\theta}{\sqrt{\gamma}} J_\pm , \quad \text{satisfying}$$

$$[x_1^{nc}, x_2^{nc}] = i\theta^2$$

In terms of old co-ords $y_\pm$, this relation looks like

$$[y_+^{nc}, y_-^{nc}] = -2R\theta$$

So this limit corresponds (in old scaling) to maximal (or infinite) noncommutativity.

To map the field theory to that on the plane, we use spectrum matching:

$$\frac{\ell(\ell+1)}{R^2} + \frac{k(k+1)}{R^2} \equiv \vec{p}_\ell^{\,2} + \vec{p}_k^{\,2}$$

$$S_F^2 \times S_F^2 \qquad\qquad \mathbb{R}^2 \times \mathbb{R}^2$$

Free action scales as

$$\sum_{\ell,k} \sum_{m_\ell, m_k} \left[R^2 \ell(\ell+1) + R^2 k(k+1) + R^4 M^2\right] |\varphi^{\ell k m_\ell m_k}|^2 \simeq$$

$$\int d^2p_\ell \, d^2p_k \left[\vec{p}_\ell^{\,2} + \vec{p}_k^{\,2} + M^2 \right] |\varphi_{nc}|^2$$

$$\left(\varphi_{nc} = R^4 \varphi^{\ell k m_\ell m_k} \right)$$

Planar contribution to mass

$$(\delta M^2)^{\text{planar}} \sim \int_0^\Lambda \int_0^\Lambda \frac{d^2p_k \, d^2p_\ell}{p_k^2 + p_\ell^2 + M^2_f} \sim \int^\Lambda \frac{d^4k}{k^2 + M^2_f}$$

Usual Quadratic divergence.

Non planar contribution :

$$(\delta M^2)^{\text{non-pl}}_{(p)} \sim \int^\Lambda \frac{d^4k}{k^2 + M^2_f} \, e^{iR\theta p \cdot B \cdot k}$$

$$\to 0 \quad \text{as} \quad f, R \to \infty .$$

Thus, new scaling limit gives us contributions only from planar graphs!!

This is related to the fact that we are working in the limit of maximal (infinite) noncommutativity.

(Minwalla et al).

Since all non-planar diagrams vanish in the new scaling limit. it is an overkill if we want to study effects for whom UV-IR mixing is important.

There are claims that there is a <u>first</u> order phase transition ('stripes' phase).

To access / study this, we need to be able to keep the non-planar diagrams.

$$\Lambda = 2/\theta$$

Planar diagram : $\sim \displaystyle\int_{|k|\leq\Lambda} \frac{d^4k}{k^2+\frac{m^2}{g}} = \frac{1}{g}\int_{|p|\leq\sqrt{g}\Lambda} \frac{d^4p}{p^2+m^2}$

Non planar : $\sim \displaystyle\int_{|k|\leq\Lambda} \frac{d^4k}{k^2+\frac{m^2}{g}}\, e^{i\theta_g^2 p\cdot B\cdot k} \sim \frac{1}{g}\int_{|k|\leq\sqrt{g}\Lambda} \frac{d^4k}{k^2+m^2}\, e^{i\theta^2 p\cdot B\cdot k}$

The overall factor of $\frac{1}{J}$ can be re-absorbed in field redefinition.

But ~~to~~ now the UV cut-off diverges as

$$\sqrt{J}\,\Lambda \equiv \sqrt{J}\,\frac{2}{\theta}.$$

The remedy is obvious: work with Polchinski's strategy for renormalization.

Introduce an intermediate scale cutoff Λ_{int} ;

On fuzzy $S^2 \times S^2$, ~~cut~~ supress all modes with quantum # ~~momenta~~ larger that $\sqrt{J}$.

As a result, all integrals like

$$\int_{|p| \leq \sqrt{J}\Lambda} d^4p \quad \text{collapse to} \quad \int_{|p| \leq \Lambda} d^4p .$$

This precisely gives us the (usual) known result for the 2-pt vertex:

$$\Gamma^{(2)} = p^2 + m^2 + \frac{2\lambda}{\#}\int^{\Lambda} \frac{d^4k}{k^2+m^2} + \frac{\lambda}{\#}\int^{\Lambda} \frac{d^4k \; e^{ip\wedge k}}{k^2+m^2}$$

One loop contribution to the 4pt f^n is (much) harder to calculate.

But it can be done:

$$\delta\lambda_4 = \lambda_4^2 \int_{\Lambda} \frac{d^4p}{(2\pi)^4} \frac{\Phi(p_i, p)}{(p^2+m^2)(\vec{p}_1^{\,2}+\vec{p}_2^{\,2}+m^2)}, \quad \text{where}$$

$$\begin{aligned}
\Phi = {} & 4\cos(\theta^2 \vec{p}_1 \wedge p_2)\cos(\theta^2 \vec{p}_3 \wedge \vec{p}_4)\cos^2(\theta^2 \vec{p} \wedge \vec{k}) \\
& + 2\cos(\theta^2 \vec{p}_1 \wedge \vec{p}_2)\cos(\theta^2 \vec{p}_? \wedge \vec{k})\cos(\theta^2 \vec{p}_3 \wedge \vec{p}_4 - \theta^2(\vec{p}_3 - \vec{p}_4)\wedge \vec{p}) \\
& + 2\cos(\theta^2 \vec{p}_1 \wedge \vec{p}_2 + \theta^2(\vec{p}_1 - \vec{p}_2)\wedge \vec{p})\cos(\theta^2 \vec{p} \wedge \vec{k})\cos(\theta^2 \vec{p}_3 \wedge \vec{p}_4) \\
& + \cos(\theta^2 \vec{p}_1 \wedge p_2 + \theta^2(\vec{p}_1 - \vec{p}_2)\wedge p)\cos(\theta^2 \vec{p}_3 \wedge \vec{p}_4 - \theta^2(\vec{p}_3 - \vec{p}_4)\wedge \vec{p})
\end{aligned}$$

where $\vec{k} \equiv -(\vec{p} + \vec{p}_1 + \vec{p}_2)$

Remarkably, this is <u>exactly</u> the result in literature, except that the integrals is with UV cut-off (Arefeva et al 9912075)

Thus using an intermediate cut-off $\sqrt{\tau}$ allows us to get the conventional UV cut-off theory.

Why is the intermediate cutoff value $\sqrt{J}$ special?

Hint from matrix product of fuzzy spherical harmonics:

$$\Psi_{L_1M_1}\Psi_{L_2M_2} = \sum_{L,M} (-1)^{2J+L}\sqrt{(2L_1+1)(2L_2+1)} \begin{Bmatrix} L_1 & L_2 & L \\ J & J & J \end{Bmatrix} C^{L_1 L_2 L}_{M_1 M_2 M}\, \Psi_{LM}$$

When L_1, L_2 are fixed and $J \to \infty$, we get the usual (commutative) multiplication rule for Y_{lm}'s.

Thus $L_i \ll J \Rightarrow$ (almost) commutative.

If L_i are comparable to J, deviation from commutativity is large.

Such Ψ_{LM} lie deep inside the nc regime.

Clearly, there is a place where the system 'trembles' between noncommutativity & noncommutativity.

· Asymptotics (of CG's 6-j's) suggests that $L_i = O(\sqrt{j})$ is that place!

· Thus if we cut-off (à la Polchinski) at momenta $l_i = O(j^{\alpha})$, $\alpha < 1/2$ we would get a commutative theory..

For $\alpha > 1/2$, we would get a strongly noncommutative (only planar diagrams contribute) theory.

As $j \to \infty$, Φ is an ∞-dim^{l} matrix.

The three situations discussed here are

3 different 'corners' of this matrix model.

- Take ~~the~~ ~~field~~ all the modes of Φ (φ_{lm}, $l = 0, \cdots 2j$)

  scale $R, j \to \infty$ with $\theta' = \frac{R}{\sqrt{j}}$ fixed.

- Take all the modes, scale $R, j \to \infty$

  with $\theta = \frac{R}{j}$ fixed.

- Take a low energy sector (φ_{lm} with $l = 0, \cdots \sqrt{j}$)

  scale $R, j \to \infty$ with $\theta = \frac{R}{j}$ fixed.

- Field theory on S_F^2 or $S_F^2 \times S_F^2$ is a matrix model.

  We get classical theory on S^2 (or $S^2 \times S^2$) with $J \to \infty$

- Quantum theory shows UV-IR mixing

- Scaling rates ($\frac{R}{J^\alpha}$, $\alpha = 1/2$ or 1) sensitive to the kind of theory obtained on the nc plane

- In particular, allows us to explore the interesting edge between commutativity & noncommutativity.