Landau Levels on Noncommutative AdS_2 Surface

Usual QM has

$$[x_i, x_j] = 0 = [p_i, p_j]$$

$$[x_i, p_j] = i\delta_{ij}.$$

And the dynamics is given by the Hamiltonian $H(x, p)$:

$$\frac{dx_i}{dt} = i[H, x_i] \quad \frac{dp_i}{dt} = i[H, p_i]$$

$\Delta x \Delta p \geq \hbar/2$.
In Noncommutiative QM:

$$[x_i, x_j] = i\epsilon_{ij}\theta \Rightarrow \Delta x_1 \Delta x_2 \geq \theta(\sim \ell^2)/2$$

$$[x_i, p_j] = i\delta_{ij} \qquad [p_i, p_j] = 0.$$

$$H(x, p)$$

Have we encoutered a similar situation?

Landau Problem

A Charged particle moving in a constant Magnetic Field.

Choose Gauge so that:

$$A_i = \frac{1}{2}\epsilon_{ij}x_jB \qquad B = \epsilon_{ij}\partial_i A_j.$$

so that we have canonical momentum: $\pi_i = p_i - eA_i$. and the usual commutation relations:

$$[x_i, x_j] = 0 = [p_i, p_j] \qquad [x_i, p_j] = i\delta_{ij}$$

$$\Rightarrow [\pi_i, \pi_j] = i\epsilon_{ij}eB \qquad [x_i, \pi_j] = i\delta_{ij} \quad [x_i, x_j] = 0.$$

This is indeed NON-COMMUTATIVE (in momentum space) QM.

For $H = \pi^2/2m$ as Hamiltonian we obtain solution (Harmonic Oscillator Repn):

$$E_n = (n + \frac{1}{2})B\frac{e\hbar}{2m}.$$

The spectrum shows equal spaced Landau Levels each with infinite degeneracy.

For large B, only the lowest Landau Levels (LLL) are occupied. This is at the root of many phenomena, such as

Quantum Hall Effect
FQHE (Frac-charge Quantum Hall Effect)
Quantum Dots
Cascade Lasers(?) etc.

Landau Problem in Non-commutative QM

We start with (setting $e = 1$)

$$[x_1, x_2] = i\theta, \qquad [x_i, p_j] = i\delta_{ij} \qquad [p_1, p_2] = iB.$$

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\omega^2}{2}(x_1^2 + x_2^2)$$

There are two regions of parameter space:

$B < 1/\theta$

$\theta = \ell^2, \quad 1/\theta - B = q^2$	$\theta = \ell^2, \quad B - 1/\theta = q'^2$
$x_1 = \ell\alpha_1 \quad x_2 = \ell\beta_1$	$x_1 = \ell\alpha_1 \quad x_2 = \ell\beta_1$
$p_1 = \frac{1}{\ell}\beta_1 + q\alpha_2$	$p_1 = \frac{1}{\ell}\beta_1 + q'\alpha_2$
$p_2 = -\frac{1}{\ell}\alpha_1 - q\beta_2$	$p_2 = -\frac{1}{\ell}\alpha_1 + q'\beta_2$

with $[\alpha_i, \beta_j] = i\delta_{ij}$

$$H = \frac{1}{2}[\omega^2\ell^2 + 1/\ell^2)(\alpha_1^2 + \beta_1^2) +$$

$$+ q^2(\alpha_2^2 + \beta_2^2) + \frac{2q}{\ell}(\alpha_1\beta_2 + \alpha_2\beta_1)]; \;\; [B < 1/\theta]$$

$$H = \frac{1}{2}[\omega^2\ell^2 + 1/\ell^2)(\alpha_1^2 + \beta_1^2) +$$

$$+q'^2(\alpha_2^2 + \beta_2^2) - \frac{2q'}{\ell}(\alpha_1\beta_2 - \alpha_2\beta_1)]; [B > 1/\theta]$$

Make a Bogoliubov transformation so that $(\alpha_i, \beta_j) \to (Q_i, P_j)$

$$H = \frac{1}{2}[\Omega_+(P_1^2 + Q_1^2) + \Omega_-(P_2^2 + Q_2^2)]$$

with

$$\Omega_\pm = \frac{1}{2}\sqrt{(\omega^2\theta - B)^2 + 4\omega^2} \pm \frac{1}{2}(\omega^2\theta + B); [B < 1/\theta]$$

$$\Omega_\pm = \pm\frac{1}{2}\sqrt{(\omega^2\theta - B)^2 + 4\omega^2} + \frac{1}{2}(\omega^2\theta + B); [B > 1/\theta]$$

Further

$$\ddot{x}_i = (B + \theta\omega^2)\epsilon_{ij}\dot{x}_j - (1 - \theta B)\omega^2 x_i.$$

With $\tilde{B} = B + \theta\omega^2$ and $\tilde{\omega}^2 = (1 - \theta B)\omega^2$, we find

$$\Omega_\pm = |\pm\frac{1}{2}\sqrt{\tilde{B}^2 + 4\tilde{\omega}^2} + \frac{1}{2}\tilde{B}|$$

$\tilde{\omega} = 0$ implies PURE LANDAU LEVELS.

Landau Levels on a Sphere

Position: x_i Angular Momenta: J_i, $i = 1, 2, 3$.

Usual QM:

$[x_i, x_j] = 0$; $[J_i, J_j] = i\epsilon_{ijk}J_k$; $[J_i, x_j] = i\epsilon_{ijk}x_k$

Usual $SU(2)$ algebra.

$H = \frac{1}{2x^2}J^2$.

Two Casimirs of the algebra label the states:
(1) $x^2 = a^2$; a = radius
(2) $x \cdot \mathbf{J} = -\frac{n}{2}a$; n = monopole number.

Spectrum of states:

$$E_l = \frac{1}{2a^2}l(l+1);\; l = \frac{|n|}{2}, \frac{|n|}{2}+1, \frac{|n|}{2}+2, ...$$

degeneracy = $2l + 1$.

Further:

$$\ddot{x}_i = -\frac{1}{2}\left[\left(\frac{\mathbf{J}^2 - (\frac{n}{2})^2}{a^4}\right) x_i + x_i \left(\frac{\mathbf{J}^2 - (\frac{n}{2})^2}{a^4}\right)\right] +$$

(Centrepetal force term)

$$+ \quad \epsilon_{ijk} \dot{x}_k \frac{n}{2a^3}$$

(Lorentz force)

Non commutative Sphere

$[R_i, R_j] = i\epsilon_{ijk}R_k;$ Noncommutative sphere

$$[J_i, R_j] = i\epsilon_{ijk}R_k, \quad [J_i, J_j] = i\epsilon_{ijk}J_k$$

Notice that $K_i = J_i - R_i$ satisfies

$[K_i, K_j] = i\epsilon_{ijk}K_k$ and $[K_i, R_j] = 0$.

$\Rightarrow SU(2) \otimes SU(2)$ algebra

relevant Casimirs: $R^2 = r(r+1)$; $K^2 = k(k+1)$
We may define Magnetic Casimir as

$\frac{1}{2}(R^2 - K^2) = R.J - \frac{1}{2}J^2.$

$H = \frac{\gamma}{2a^2}J^2.$

Since $J = R + K$ and $H \propto J^2$;
with j assuming values $|r-k|$ to $(r+k)$

$E_j = \frac{\gamma}{2r(r+1)}j(j+1);$
with $j = \frac{|n|}{2}, \frac{|n|}{2}+1, ...k+r(=\frac{|n|}{2}+2r).$

LANDAU LEVELS IN NONCOMMUTATIVE AdS_2 SPACE

- An Exercise in Non-commutative Quantum Mechanics

- One more instance of relevance of Anti de Sitter Spaces

- Negative Constant curvature surface $\Rightarrow$ Non compact hyperbolic surface

- By Riemann Uniformisation theorem all compact 2 dimensional surfaces are given by one set of complex co-ordinates $z, \bar{z}$

genus	topology	Manifold
0	Sphere	$C + \{\infty\}$.
1	Torus	C/Λ.
≥ 2	bitorus etc.	H/Γ

$\Lambda: Z + \tau Z$, Im $\tau > 0$.
$\Rightarrow z = z + 1; \; z = z + \tau$
doubly periodic lattice.

Γ: a finite subgroup of $SL(2, R)$ with $2g$ generators A_i, B_i; $i = 1, 2, ..g$ and Homotopy condition

$$\prod_i [A_i, B_i] = 1$$

.

Landau Problem on AdS_2 Surface

Anti de Sitter Space in 2 dimension is a constant *negative* curvature space:

Complex co-ordinates $z, \bar{z}$ with Poinçare metric

$$ds^2 = g_{z\bar{z}} dz d\bar{z} = \frac{dx^2 + dy^2}{y^2}$$

$$\nabla = \partial + \frac{B}{z - \bar{z}} \quad \bar{\nabla} = \bar{\partial} + \frac{B}{z - \bar{z}}$$

and

$$[\nabla, \bar{\nabla}] = B/(2y^2)$$

With Hamiltonian as

$$\begin{aligned} H &= -2g^{z\bar{z}}(\nabla\bar{\nabla} + \bar{\nabla}\nabla) - B^2 \\ &= -4g^{z\bar{z}}\nabla\bar{\nabla} + B(1 - B) \\ &= -4g^{z\bar{z}}\bar{\nabla}\nabla - B(1 + B). \end{aligned}$$

Since the operators $-g^{z\bar{z}}\nabla\bar{\nabla}$ and $-g^{z\bar{z}}\bar{\nabla}\nabla$ are both semipositive definite.

$$H \geq |B|(1-|B|).$$

$$H\Psi = E\Psi.$$

The lowest eigenstate, i.e. $E = B(1-B)$, is a solution of $\bar{\nabla}\Psi_0 = 0$. given by

$$\Psi_0^{(n)} = \frac{(z-\bar{z})^B}{(i+z)^{2B}}\left(\frac{-i+z}{i+z}\right)^n,$$

corresponds to eigenvalue $B+n$ of an operator J_3, with n any nonnegative integer.

AdS_2 surface is also described as an embedding in a (2 +1) Minkowski space with constraint: $x \circ x = x_1^2 + x_2^2 - x_3^2 = -a^2$,
In this algebraic formulation

$[J_1, J_2] = -iJ_3 \ \ [J_2, J_3] = iJ_1 \ \ [J_3, J_1] = iJ_2,$
and
$[J_1, x_2] = -ix_3 \ \ [J_2, x_3] = ix_1 \ \ [J_3, x_1] = ix_2.$

AdS_2 embedded in Minkowski (flat) Mfld.

x_1, x_2, x_3 with $x_1^2 + x_2^2 - x_3^2 = -1$.

$$x_1 = i\frac{z+\bar{z}}{z-\bar{z}}, \quad x_2 = i\frac{1-z\bar{z}}{z-\bar{z}}, \quad x_3 = i\frac{1+z\bar{z}}{z-\bar{z}}.$$

Invariance Group $SO(2,1)$ Generators are:

$$J_1 = i(z\partial + \bar{z}\bar{\partial}),$$

$$J_{2,3} = -\frac{i}{2}((1 \pm z^2)\partial + (1 \pm \bar{z}^2)\bar{\partial}) \pm B(z - \bar{z}).$$

Hamiltonian is

$$H = J \circ J \equiv J_1^2 + J_2^2 - J_3^2,$$

with constraint

$$x \circ J \equiv x_1 J_1 + x_2 J_2 - x_3 J_3 = -B.$$

With the Hamiltonian as $H = J \circ J$, we verify that $x \circ J = -B$ is the Magnetic constraint. So the bound $H \geq B(1-B) \Rightarrow J \circ J + B(B-1) \geq 0$.

$SO(2,1)$ repn are:

$D_j^{\pm}$, in which $J \circ J = j(1-j)$ and $J_3 = \pm(j, j+1, .., j+n, ...)$ with $j \geq 1/2$ (DISCRETE)

C_j in which $J \circ J$ is real positive [CONTINUUM]

Our Landau levels are states D_j^{+} with $j \leq B$. Spectrum further has states C_j with $j^2 > 0$.

$\equiv J \circ J$

Noncommutative AdS_2 Landau Levels

We add to the algebra noncommutative R_i with the rules

$$[R_1, R_2] = -iR_3 \quad [R_2, R_3] = iR_1 \quad [R_3, R_1] = iR_2,$$

Two commuting $SO(2,1)$ algebra R_i and $K_i = J_i - R_i$ with Casimirs

(1) $R \circ R = r(1-r); D_r^{\pm}$, Choose D_r^{+}, so that $R_3 = r, r+1....$
(2) $K \circ K = k(1-k); k > 0$[discrete],
or $K \circ K = k^2(> 0)$[continuum]

Like in the commutative case we fix $R \circ R$ (size) and $-R \circ J = B$ (Magnetic Field)

$H = J \circ J$ as Hamiltonian gives the Landau levels in various regions of the r and B, by using SO(2,1) representation theory.

System is described by two commuting $SO(2,1)$ algebrae, $K_i = J_i - R_i$ and R_i.

The relevant decompositions are:

$$D_k^+ \otimes D_r^+ = \sum_{m=0} D_j^+, \; j = k + r + m, \; m \; integer$$

and

$$D_k^- \otimes D_r^+ = (\sum_j^{|r-k|} D_j^\pm, \; j = |r-k| \; mod(1) + n, \; n \; integer)$$

$$\oplus \int C_j.$$

with D_j^+ for $r > k$ and D_j^- for $k > r$.

Also

$$C_k \otimes D_r^+ = (\sum_{j=r+n} D_j^+, \; n = 0, 1, \ldots) \oplus \int C_j.$$

$B < -\frac{1}{2}$:

Magnetic constraint
$\Rightarrow J \circ J - K \circ K = r(r - 1 - 2B) \geq r^2$.

Given D_r^+ repr'n for R algebra.

If we have D_k^+,
$\Rightarrow D_j^+$ with $j = k + r$
Contradicts with mag. constraint
$(k - \frac{1}{2})^2 - (j - \frac{1}{2})^2 > 0$; $\Rightarrow k > j$
$\Rightarrow$ D_k^+ not allowed.

Let us have D_k^-,
Three possibilities:

(1) D_j^- with $j < k - r$
$\Rightarrow k - j = r + l$
$(k - j)(k + j - 1) = r(r - 1 - 2B)$

$\Rightarrow k + j = \frac{r(r-1-2B)}{r+l} + 1$
$\Rightarrow j = j_l = \frac{1}{2}(\frac{r(r-1-2B)}{r+l} - (r+l) + 1)$

We have
$\frac{1}{2} \leq j_l \leq \frac{1}{2}(\frac{r(r-1-2B)}{r} - r + 1) = B$.

(2) D_j^+ with $j < r - k$
$\Rightarrow j + k = r - l$
Since $(k-j)(k+j-1) = r(r-1-2B)$
$k - j = \frac{r(r-1-2B)}{r-l-1}$
$\Rightarrow j = \frac{1}{2}(r - l - \frac{r(r-1-2B)}{r-l-1})$ Not compatible in this range for B.

(3) C_j with $j^2 = r(r-1-2B) - k(k-1)$
$\Rightarrow 0 < j^2 < r(r-1-2B)$.

Finally if we have C_k, then

no discrete j, since D_j^+
needs $-j(j-1) - k^2 > r^2$, which is impossible

but allows C_j with $j^2 = r(r-1-2B) + k^2$
$\Rightarrow$cont. j spectrum with
$\infty > j^2 > r(r-1-2B)$.

In summary, when $B < -\frac{1}{2}$, we have:

Continuum spectrum
$\infty > j^2 > 0$
and Discrete Spectrum
$\frac{1}{2} \leq j_l = \frac{1}{2}(\frac{r(r-1-2B)}{r+l} - (r+l) + 1) \leq B$.

The spectrum is composed of various parts, consistent with the magnetic constraint

$$\begin{aligned} K \circ K - J \circ J &= R \circ R - 2R \circ J \\ &= -r(r-1) + 2rB. \end{aligned}$$

$B < -\frac{1}{2}$:

Discrete spectrum D_j^-, with $J \circ J = -j_l(j_l - 1)$:

$$\begin{aligned} \frac{1}{2} \leq j_l &= \left(\frac{r(r-1-2B)}{r+l} - r - l + 1\right)/2 \\ &\leq (\frac{r(r-1-2B)}{r} - r + 1)/2 \end{aligned}$$

with l an integer as $r + l \leq \sqrt{r(r-1-2B)}$.

Also entire Continuum C_j; $0 \leq j^2 \leq \infty$.

$-\frac{1}{2} \leq B \leq \frac{1}{2} - \frac{1}{2r}$:

No discrete spectrum, same continuum.

$\frac{1}{2} - \frac{1}{2r} \leq B \leq \frac{r-1}{2}$:

Discrete spectrum with D_j^+, with

$$\frac{1}{2} \leq j_l = (r - l - \frac{r(r-1-2B)}{r-1-l})/2$$
$$\leq (r - \frac{r(r-1-2B)}{r-1})/2 = \frac{r}{r-1}B.$$

with l integer as $\frac{r(r-1-2B)}{r-1-l} \leq \sqrt{r(r-1-2B)}$.

The same Continuum.

Note Commutative limit reached as $r \to \infty$,

$\frac{r(r-1-2B)}{r-1-l} \sim r-2B+l$; $\frac{r(r-1-2B)}{r+l} \sim r-1-2B-l$.

For $|B| > \frac{1}{2}$ the approximate discrete spectrum $J \circ J = -j_l(j_l - 1)$ with $j_l \sim |B| - l$.

Matches the Commutative Landau Spectrum.

We may continue with $B > \frac{r-1}{2}$ and find Landau levels, but for B in this range, when r itself is large is not very interesting.

For completeness, we note:

$\frac{r}{2} - \frac{1}{2} \leq B \leq r - \frac{3}{2} + \frac{1}{2r}$:

We get the discrete spectrum D_j^+, $J \circ J = -j_l(j_l - 1)$ with

$$\sqrt{r(2B+1-r)}+\frac{1}{2} \leq j_l \leq (\frac{r(2B+1-r)}{r-1}+r)/2,$$

with $j_l = (\frac{r(2B+1-r)}{r-1-l}+r-l)/2$ and the nonnegative integer l bounded by $l \leq r-1-\sqrt{r(2B+1-r)}$; and the Continuum $J \circ J = j^2$ for $(0 \leq j^2 \leq \infty)$.

$r - \frac{3}{2} + \frac{1}{2r} \leq B \leq r - \frac{1}{2}$:

No discrete spectrum, same continuum.

Finally for the region
$B > r - \frac{1}{2}$:

Apart from Continuum $J \circ J = j^2$ for any j, there are two parts of discrete spectrum:

(1) $J \circ J = -j_l(j_l - 1)$:

$$\sqrt{r(2B+1-r)} + \frac{1}{2} \leq j_l \leq (\frac{r(2B+1-r)}{r} + r + 1 \\ = B + 1.$$

with $j_l = (\frac{r(2B+1-r)}{r+l} + r + l + 1)/2$ and l non negative integer satisfying $r+l \leq \sqrt{r(2B+1-r)}$;

(2) another part in which

$$r \leq j_l = r + l \leq \frac{1}{2} + \sqrt{r(2B+1-r) + \frac{1}{4}}.$$

Since l is zero or integer, the two parts do not overlap and cover the range for j_l from r upto $(\frac{r(2B+1-r)}{r} + r + 1)/2 = B + 1$.

FIGURES

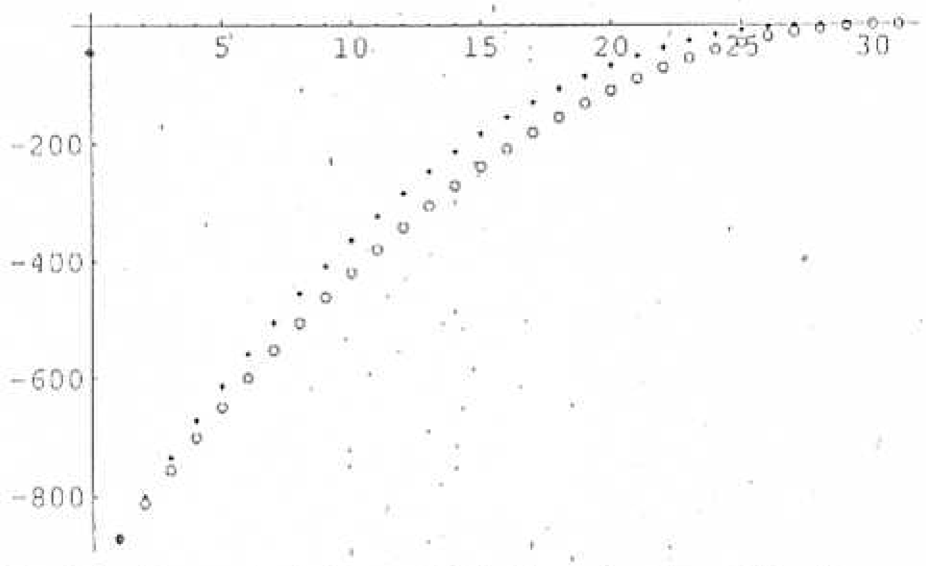

FIG. 1. Plot of the discrete levels $H_0 = -j_l(j_l - 1)$ as a function of l for the case $r = 150$, $B = -30$ (full points), see eq.3.12, compared with the levels of the commutative case (open circles).

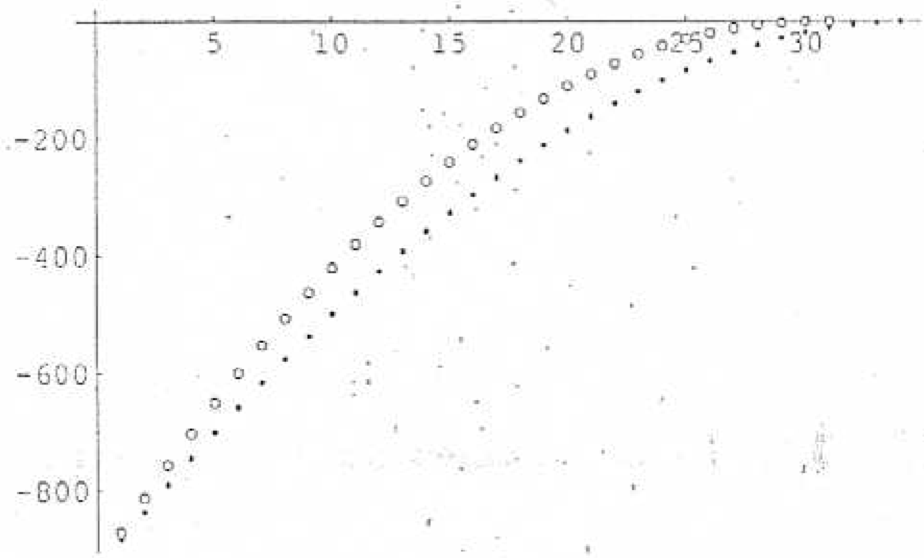

FIG. 2. Same as Fig.1 but for the case $r = 150$, $B = 30$, see eq.3.13.

A second option for Magnetic Casimir:

Choose 2 Casimirs as:

(i) Radius of Hypersphere $R \circ R = -r^2$.
(ii) Magnetic Casimir
$K \circ K - R \circ R \equiv -2R \circ J + J \circ J$.

That is just fix $K \circ K$ and $R \circ R$ and list all possible states in Spectrum:

(a) When $K \circ K > 0$:
$\Rightarrow C_k$ repns.
Continuum J spectrum C_j with non negative j and
Discrete spectrum D_j^+ with $J \circ J = j(1-j))$; $j = r, r+1, ...\infty$.
$\Rightarrow$Spectrum unbounded below.

(b) When $K \circ K < 0$,
$\Rightarrow D_k^+$ or D_k^- repns.

D_k^+ $\Rightarrow$Discrete spectrum D_j^+ with $j = r+k+m$, m any positive integer.

$\Rightarrow$ Spectrum unbounded below.

D_k^- $\Rightarrow$Continuum spectrum with $j^2 > 0$ and Discrete Spectrum $D_j^\pm$ where j assumes values $|r-k|mod(1)+n$ in the range from 0 upto $|r-k|$.

So to get an acceptable spectrum for a given value of B, we must assign magnetic casimir so that we use D_k^- repn.